Recent applications of and trends in model theory.

Anand Pillay

University of Leeds

February 22, 2013

Logic I

- Modern mathematical logic developed at the end of the 19th and beginning of 20th centuries with the so-called foundational crisis or crises.
- There was a greater interest in mathematical rigour, and a concern whether reasoning involving certain infinite quantities was sound.
- In addition to logicians such as Cantor, Frege, Russell, major mathematicians of the time such as Hilbert and Poincaré participated in these developments.
- Out of all of this came the beginnings of mathematical accounts of higher level or "metamathematical" notions such as set, truth, proof, and algorithm (or effective procedure).

Logic II

- These four notions are still at the base of the main areas of mathematical logic: set theory, model theory, proof theory, and recursion theory, respectively.
- Classical foundational issues are still present in modern mathematical logic, especially set theory.
- But various relations between logic and other areas have developed: set theory has close connections to analysis, proof theory to computer science, category theory and recently homotopy theory.
- And we will discuss the case of model theory. Early developments include Malcev's applications to group theory, Tarski's analysis of "definability " in the field of real numbers, and Robinson's rigurous account of infinitesimals (nonstandard analysis).

Model theory I

- What is model theory?
- Some may think of it as a collection of techniques and notions (compactness, quantifier elimination, o-minimality,..) which come to life in applications.
- But there *is* a "model theory for its own sake" which I would tentatively define as the classification of first order theories.
- A first order theory T is at the naive level simply a collection of "first order sentences" in some vocabulary L with relation, function and constant symbols as well as the usual logical connectives "and", "or", "not", and quantifiers "there exist", "for all".
- "First order" refers to the quantifiers ranging over elements or individuals rather than sets.

Model theory II

- A *model* of T is simply a first order structure M consisting of an underlying set or universe M together with relations (subsets of M^n), functions $M^n \to M$ and "constants corresponding to the symbols of L, in which the sentences of T are true. It is natural to allow several universes (many-sorted framework)
- There is a tautological aspect here: the set of axioms for groups is a first order theory in an appropriate language, and a model of T is just a group.
- On the other hand, the axioms for topological spaces, and topological spaces themselves have on the face of it a "second order" character. (A set X is given the structure of a topological space by specifying a collection of *subsets* of X satisfying various properties..)

Model theory III

- Another key notion is that of a definable set.
- If $(G, \cdot)$ is a group, and $a \in G$ then the collection of elements of G which commute with a is the solution set of an "equation", $x \cdot a = a \cdot x$.
- However $Z(G)$, the centre of G, which is the collection of elements of G which commute with every element of G, is "defined by" the first order formula $\forall y(x \cdot y = y \cdot x)$.
- In the structure $(\mathbb{R}, +, \cdot, -)$ the ordering $x \leq y$ is defined by the first order formula $\exists z(y - x = z^2)$.
- Our familiar number systems already provide quite different behaviour or features of definable sets.

Model theory IV

- In the structure $(\mathbb{N}, +, \times, 0)$, subsets of $\mathbb{N}$ definable by formulas $\phi(x)$ which begin with a sequence of quantifiers $\exists y_1 \forall y_2 \exists y_3 ... \forall y_n$ get more complicated as n increases.
- The collection of definable subsets of $\mathbb{N}$ is called the arithmetic hierarchy, and already with one existential quantifier we can define "noncomputable" sets.
- Whereas in the structure $(\mathbb{R}, +, \cdot)$, the hierarchy collapses, one only needs one block of existential quantifiers to define definable sets. Moreover the definable sets have a geometric feature: they are the so-called semialgebraic sets.
- Namely finite unions of subsets of $\mathbb{R}^n$ of form $\{\bar{x} : f(\bar{x}) = 0 \wedge \bigwedge_{i=1,..k} g_i(\bar{x}) > 0\}$ where f and the g_i are polynomials with coefficients from $\mathbb{R}$.

Model theory V

- In the case of the structure $(\mathbb{C}, +, \times)$ it is even better: the hierarchy collapses to sets defined without any quantifiers. The definable sets are precisely the constructible sets: finite Boolean combinations of algebraic varieties. (Chevalley's theorem.)
- We will restrict our attention to *complete* theories T, namely theories which decide every sentence of the vocabulary.
- For example ACF_0, the axioms for algebraically closed fields of characteristic 0.
- So attached to a first order theory there are at least two categories, $Mod(T)$ the category of models of T, and $Def(T)$ the category of definable sets, where the latter can be identified with $Def(M)$, the category of definable sets in a "big" model M of T.

Stability I

- Model theory in the 1960's and 70's had a very "set-theoretic" character (influenced by Tarski among others) and the original questions which led to the development of the subject as something for its own sake have this form.
- For example the *spectrum problem*: given a (complete) theory, we have the function $I(-T)$ from (infinite) cardinals to cardinals, where $I(\kappa, T)$ is the number of models of T of cardinality κ, up to isomorphism. What are the possible such functions, as T varies?
- Shelah solved the problem for countable theories, in the process identifying the class of *stable* first order theories. In hindsight these are the "logically perfect" first order theories.

Stability II

- The spectrum problem is "about" $Mod(T)$, but the proof gave an enormous amount of information about and tools for understanding $Def(T)$ when T is stable.
- The (or a) definition of stability is not particularly enlightening but is a good example of a "model-theoretic" property: T is stable if there is no model M of T definable relation $R(x, y)$ and $a_i, b_i \in M$ for $i = 1, 2, ..$ such that $R(a_i, b_j)$ if $i < j$.
- ACF_0 is the canonical example of a stable theory. Another (complete) example is the theory of vector spaces over a fixed division ring.
- More recently it was discovered (Sela) that the first order theory of the free group $(F_2, \cdot)$ is stable, yielding new connections between model theory and geometric group theory.

Finite rank stable theories I

- Among the more tractable classes of stable theories are those of "finite rank", i.e. where all definable sets X have finite rank/dimension, in a sense that we describe now:
- The relevant dimension notion is traditionally called "Morley rank" and is simply Cantor-Bendixon rank on the Boolean algebra of definable subsets of X:
- X has Morley rank 0 if X is finite, in which case the multiplicity of X is its cardinality.
- X has Morley rank $n+1$ and multiplicity 1 if it has Morley rank $> n$ and cannot be partitioned into two such definable subsets.
- The building blocks of all definable sets (in a geometric sense that I will not make precise) are what I will call the *minimal* definable sets.

Finite rank stable theories II

- Loosely speaking X is minimal if generically it cannot be partitioned into 2 infinite definable sets.
- A special case is strongly minimal meaning precisely Morley rank and multiplicity 1, namely cannot be partitioned into 2 infinite definable sets.
- There is a natural equivalence relation on minimal sets: $=X \sim Y$ if there is a definable $Z \subseteq X \times Y$ projecting generically finite-to-one on each of X, Y (a definable correspondence).
- A basic and very influential conjecture of Boris Zilber (from early 80's??) was that that any minimal set is of three possible (mutually exclusive) types:
- (a) "field like": up to $\sim$, X has definably the structure of an algebraically closed field

Finite rank stable theories III

- (b) "vector space like": up to $\sim X$ has a definable commutative group structure such that moreover any definable subset of $X \times ..X$ is up to finite Boolean combination and translation, a definable subgroup.
- (c) X is "trivial": there is no infinite definable family of self correspondences of X.
- A counterexample was found by Hrushovski in the late 80's, and the methods for constructing such examples have become a subarea of model theory.
- However the conjecture has been proved for some very rich finite rank stable theories (originally via Zariski geometries, but other proofs were found later).

DCF_0 and CCM I

- DCF_0 is the theory of differentially closed fields of characteristic 0, the theory of a "universal" differential field $(\mathcal{U}, +, \times, \partial)$.
- It is stable (not finite rank) but the collection of finite rank definable sets is rich.
- CCM is the many sorted structure of compact complex manifolds, where the distinguished relations (on finite Cartesian products of manifolds) are the analytic subvarieties. It is a finite rank stable structure (theory).

DCF_0 and CCM II

- Both these structures contain algebraic geometry (i.e. ACF_0). For $\mathcal{U}$ algebraic geometry lives on the field of constants $\mathcal{C}$. And in CCM it lives on the sort $P^1(\mathbb{C})$.
- The Zilber conjecture is valid for both these theories/structures (as well as a related theory of difference fields) and in the case of DCF_0 gave rise to new proofs of function field diophantine geometric theorems.
- In both cases minimal sets of kind (a) live in the algebraic geometry part of the structure. Minimal sets of kind (b) are connected with abelian varieties/complex tori, and there is a growing interest in identifying minimal sets of type (c).
- In DCF_0 the property of being of type (c) has implications for algebraic independence of solutions. As an example.

Theorem 0.1

(with Nagloo.) Consider the Painlevé II family of second order ODE's: $y'' = 2y^3 + ty + \alpha$ *where* $\alpha \in \mathbb{C}$. *Then the solution set* Y_α *of the relevant equation is strongly minimal iff* $\alpha \notin \mathbb{Z} + 1/2$, *and moreover for all such* α, Y_α *is of type (c) (trivial).*
Moreover any "generic" equation in each of the Painlevé families I -VI, is strongly minimal and strongly trivial in the sense that if $y_1, .., y_n$ *are distinct solutions, then the* $y_1, y_1', ..., y_n, y_n'$ *are algebraically independent over* $\mathbb{C}(t)$.

DCF_0 and CCM IV

- Finally I discuss some results, as well as problems/conjectures (some of which I hear from F. Campana, about minimal sets in CCM, sometimes assumed Kaehler.
- First the complex geometers have a name for minimal ccm's. They call them simple (although sometimes reserved for $dim > 1$.
- A general problem is to classify simple ccm's up to bimeromorphic equivalence.
- When X, Y are strongly minimal ccm's then they are bimeromorphic iff biholomorphic.
- When X is simple and of kind (a) then X is a smooth projective curve, and in cases (b), (c), $dim(X) > 1$.
- If X is strongly minimal (meaning no proper subvarieties, except finite ones), and X is of type (b), then X IS a (simple) complex torus.

DCF_0 and CCM V

- We conjecture that if X is strongly minimal, Kahler of kind (c) then X is hyperkaehler (up to finite cover).
- It is a theorem that if X is strongly minimal, $dim(X) > 1$ then X is of kind (b) iff $H^0(X, T_X) \neq 0$.
- We conjecture that again for X strongly minimal of $dim > 1$, X is of type (b) iff X has infinite fundamental group....