ON THE ISOTRIVIALITY OF PROJECTIVE ITERATIVE ∂-VARIETIES

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Abstract. We study algebraic varieties X over a universal iterative differential field (K, ∂) (typically of positive characteristic), together with an extension of ∂ to an iterative derivation D of the structure sheaf of X. Our work is motivated by the conjecture that if X is projective then the pair (X, D) is isotrivial, namely isomorphic over K to a pair (Y, D_0) where Y is defined over the constants C of (K, ∂) and D_0 is the lifting to K of the trivial iterative derivation on Y_C. We prove that up to isomorphism there is at most one such D on X extending ∂, thus answering the question when X is defined over C. Other special cases are also proved, including abelian varieties, and smooth curves.

1. Introduction

In this paper we attempt to generalize results of Buium (see [2]) on projective ∂-varieties over differential fields of characteristic 0, to the positive characteristic case. In the characteristic 0-case, the ground field K is equipped with a derivation ∂ such that (K, ∂) is differentially closed. A ∂-structure on a variety X defined over K is an extension of ∂ to a derivation D of the structure sheaf of X. Giving X a ∂-structure is equivalent to equipping X with a regular section $s : X \to T_\partial(X)$ (defined over K) of a certain twisted version $T_\partial(X)$ of the tangent bundle of X. The pair (X, D) or (X, s) is called a *∂-variety* over K.

If X is defined over the field of constants C of K, then the structure sheaf of X over C can be equipped with the 0-derivation, which can be tensored with ∂ over K, to get a derivation D_0 of the structure sheaf of X. This corresponds to the 0-section of the tangent bundle of X. We call such a pair (X, D_0) a trivial ∂-variety.

There is a natural notion of *morphism* of ∂-varieties, and (X, D) is said to be isotrivial if it is isomorphic to a trivial ∂-variety.

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In [2], Buium proves that (in this characteristic 0 context), any ∂-variety (X, D) over K such that X is projective, is isotrivial.

The work presented here is an attempt to generalize Buium's theorem to a suitable positive characteristic context. The descent part of this problem is related to [3, Question 1].

In characteristic 0 if we equip a function field $K = C(t)$ with the derivation d/dt then the field of constants is C. However in characteristic $p > 0$, the field of constants of K is $C(t^p)$ rather than K. The situation can be remedied by replacing the single derivation d/dt by a suitable sequence of maps (a Hasse-Schmidt derivation) whose common field of constants *will be* C.

So we will work with such generalized derivations.

Definition 1.1. Let R be a ring. Then
(i) A sequence $\partial = (\partial_n : n < \omega)$ of additive maps from R to itself is called a Hasse-Schmidt derivation if ∂_0 is the identity, and for all $n > 0$ and $x, y \in R$,

$$\partial_n(xy) = \sum_{i+j=n} \partial_i(x)\partial_j(y).$$

(ii) We call the sequence ∂ an iterative Hasse-Schmidt derivation on R if in addition to (i) we have for all i, j

$$\partial_i \circ \partial_j = \binom{i+j}{i} \partial_{i+j}.$$

We will sometimes use the expression "iterative derivation" for "iterative Hasse-Schmidt derivation". (K, ∂) will usually denote a field K of characteristic $p > 0$ equipped with an iterative derivation ∂. Ziegler [13] identified a complete first order theory $SCH_{p,1}$ (the theory of separably closed iterative fields of characteristic p and Ershov invariant 1) whose models are appropriate to work over in our context. In fact it will usually be appropriate to take (K, ∂) to be a "universal domain" namely a saturated model of $SCH_{p,1}$.

The field of (absolute) constants C of K consists of those $x \in K$ such that $\partial_n(x) = 0$ for all $n > 0$, which coincides with the intersection of all the K^{p^n}.

In section 2 we define iterative ∂-schemes over K in terms of group scheme actions. For now, an iterative ∂-variety over K is a variety X over K together with an extension D of ∂ to an iterative derivation of the structure sheaf of X. If (X, D_1) and (Y, D_2) are such then we have the obvious notion of a morphism from (X, D_1) to (X, D_2): namely a morphism $f : X \to Y$ (defined over K) of varieties such that

$f^*D_2 = D_1 f^*$ where f^* is the induced map from the structure sheaf of Y to the structure sheaf of X.

As in the characteristic zero case we have the notion of a trivial ∂-variety over K. Isotrivial means again isomorphic over K to a trivial object. To have "enough" isomorphisms we need here to assume that (K, ∂) is a universal domain.

Our first result, proved in section 3, is:

$(*)$ if X is a proper iterative ∂-variety over K, then X has at most one structure of an iterative ∂-variety over K. Namely if D_1, D_2 are iterative ∂-structures on X over K, then (X, D_1) and (X, D_2) are isomorphic.

This result should also hold in the context of a "Hasse-Schmidt system extending to fields" from [10].

We also prove (section 4) the full analogue of Buium's result in special cases, such as when X has ample canonical or anticanonical divisor. This proof does not use $(*)$.

We will also mention the work of Benoist [1] which is very relevant to our main conjecture. Benoist proves that if the algebraic variety X (over a model (K, ∂) of $SCH_{p,1}$) can be equipped with the structure of an iterative ∂-variety over K, then K descends to K^{p^n} for all n. If X belongs to a family with a good "moduli space" one can conclude that X descends to C. By this means we can also, using $(*)$, conclude that the conjecture holds when for example X is an abelian variety.

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2. Iterative ∂-schemes

Let us fix an algebraically closed field C and assume R is a C-algebra. A *Hasse-Schmidt C-derivation* on R is a Hasse-Schmidt derivation on R which vanishes on C. It is the same as a C-algebra map $\partial : R \to R[\![X]\!]$ which is a section of the projection map $R[\![X]\!] \to R$. We will abbreviate iterative Hasse-Schmidt C-derivation by *iterative C-derivation*. We define *iterative C-algebras (fields)* in the obvious way. If ∂ is an iterative C-derivation on R and ∂' an iterative C-derivation on R' then an *iterative C-homomorphism* is a C-algebra map $R \to R'$ which commutes with each ∂_n, ∂'_n.

We will give now an interpretation of the iterativity condition in terms of Hopf algebras (co-)actions, which will be useful in the sequel (see Section 27 in [7]). For each $n \in \mathbb{N}$, consider the C-algebra $C[X]/(X^{p^n})$. It is a Hopf algebra with the coaddition map c_n given by $X \mapsto X \otimes 1 + 1 \otimes X$ and the counit map u_n being the projection onto C. Let α_n denote the corresponding group scheme (this group scheme is usually denoted α_{p^n} but to ease the notation we prefer to denote it α_n). Then α_n is the (group-scheme) kernel of the n-th power of the Frobenius endomorphism on the additive group scheme $\mathbb{G}_{\mathrm{a}}$, and the limit of the direct system (α_n) coincides with the formal group $\widehat{\mathbb{G}}_{\mathrm{a}}$ (see [6, Lemma 1.1]). Let us fix a C-algebra R. For a C-algebra map $\partial : R \to R[\![X]\!]$, let ∂_n denote the following composite map:

$$R \xrightarrow{\partial} R[\![X]\!] \longrightarrow R[X]/(X^{p^n}) \xrightarrow{\cong} R \otimes_C C[X]/(X^{p^n}).$$

Then ∂ is an iterative C-derivation if and only if for each n, ∂_n is a section of the map

$$\mathrm{id}_R \otimes u_n : R \otimes_C C[X]/(X^{p^n}) \to R \otimes_C C = R,$$

and the following diagram is commutative:

$$\begin{array}{ccc} R & \xrightarrow{\partial_n} & R \otimes_C C[X]/(X^{p^n}) \\ \downarrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle \mathrm{id} \otimes_C c_n} \\ R \otimes_C C[X]/(X^{p^n}) & \xrightarrow{\partial_n \otimes_C \mathrm{id}} & R \otimes_C C[X]/(X^{p^n}) \otimes_C C[X]/(X^{p^n}). \end{array}$$

Therefore, an iterative C-derivation is the same as a sequence $(\partial_n)_{n \in \mathbb{N}}$ of compatible group actions of α_n on $\mathrm{Spec}(R)$ over C. It is easy to see that an iterative C-homomorphism is the same as an equivariant C-morphism with respect to the given actions.

Remark 2.1. More generally (see Section 4 in [3]), if the characteristic of R is arbitrary then an iterative C-derivation on R is the same as a *formal* group action of $\widehat{\mathbb{G}}_{\mathrm{a}}$ on $\mathrm{Spec}(R)$ over C.

We can easily extend the notion of an iterative C-derivation to arbitrary C-schemes (see Section 4 in [3]).

Definition 2.2. An *iterative C-scheme* is a pair (X, ∂) consisting of a C-scheme X and a sequence $\partial = (\partial_n)$ of compatible group scheme actions $\partial_n : \alpha_n \times_C X \to X$ over C.
An *iterative C-morphism* of iterative C-schemes $(X, D), (X', D')$ is an equivariant C-morphism $f : X \to X'$, i.e. for each $n \in \mathbb{N}$ the following

diagram is commutative:

$$\begin{array}{ccc} \alpha_n \times_C X & \xrightarrow{D_n} & X \\ {\scriptstyle \mathrm{id} \times_C f}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \alpha_n \times_C X' & \xrightarrow{D'_n} & X' \end{array}$$

Remark 2.3. If R is an iterative C-algebra, then $\mathrm{Spec}(R)$ becomes an iterative C-scheme. Iterative C-homomorphisms correspond to iterative C-morphisms.

Proposition 2.4. *Let (X, ∂) be an iterative C-scheme and $U \to X$ an open C-immersion. Then there is a unique iterative C-scheme structure ∂' on U such that $(U, \partial') \to (X, \partial)$ is an iterative C-morphism.*

Proof. For each $n \in \mathbb{N}$, consider the following commutative diagram:

$$\begin{array}{ccc} U & \longrightarrow & X \\ {\scriptstyle =}\big\uparrow & & \big\uparrow{\scriptstyle \partial_{U,n}} \\ U & \xrightarrow{(1,\mathrm{id})} & \alpha_n \times_C U, \end{array}$$

where $\partial_{U,n}$ is the composition of $\alpha_n \times_C U \to \alpha_n \times_C X$ with ∂_n and $1 : U \to \alpha_n$ comes from the neutral element morphism of the group scheme α_n. Note that U may be represented as $\mathrm{Spec}(\mathcal{O}_{\alpha_n \times U}/I)$ for a nilpotent ideal sheaf I on $\alpha_n \times U$ and then the map

$$(1, \mathrm{id}) : U \to \alpha_n \times_C U$$

corresponds to the quotient morphism. Since the open immersion $U \to X$ is étale, we have a unique C-morphism $\partial'_{U,n}$ such that the following diagram is commutative:

$$\begin{array}{ccc} U & \longrightarrow & X \\ {\scriptstyle =}\big\uparrow & \nwarrow^{\partial'_{U,n}} & \big\uparrow{\scriptstyle \partial_{U,n}} \\ U & \xrightarrow{(1,\mathrm{id})} & \alpha_n \times_C U, \end{array}$$

giving us the required group scheme action. □

Let us fix now an iterative C-field (K, ∂). We consider the category of ∂-algebras and ∂-maps: a *∂-algebra* is a K-algebra $K \to R$ which is also an iterative C-algebra such that the map $K \to R$ is an iterative C-homomorphism; ∂-homomorphisms are defined in a natural way.
We want to extend the definition of ∂-algebra to the context of schemes. Let $S = \mathrm{Spec}(K)$, it is an iterative C-scheme.

Definition 2.5. (1) An *iterative ∂-scheme* is a K-scheme $X \to S$ with an iterative C-scheme structure such that the morphism $X \to S$ is C-iterative.
(2) A *∂-morphism* of ∂-schemes $(X, D), (X', D')$ is a K-morphism which is C-iterative.
(3) A *∂-point* of an iterative ∂-scheme X is a ∂-morphism $x : S \to X$ (so it is a K-rational point of X).
(4) The set of all ∂-points of X is denoted by $X^\sharp$.

It is clear from the definitions that for a K-algebra R and $X = \mathrm{Spec}(R)$, giving X a ∂-scheme structure is the same as finding an extension of ∂ to R. So, we can think of iterative ∂-schemes structures on a scheme X as "extensions" of ∂ to X. Note that the definitions of ∂-algebras/morphisms make sense on the level of sheaves of K-algebras. Using 2.4, we get:

Proposition 2.6. *Let X be a K-scheme. Giving X a ∂-scheme structure is equivalent to finding a ∂-sheaf structure on $\mathcal{O}_X$.*

Proof. Having a ∂-sheaf structure on $\mathcal{O}_X$, we can define the required group scheme actions on X using an open affine cover of X and 2.3. Having a ∂-scheme structure on X, we define a ∂-structure on $\mathcal{O}_X$ using 2.4 and 2.3 again. □

We will need an obvious lemma about ∂-points.

Lemma 2.7. *Let $f : X \to Y$ be a ∂-morphism. Then $f(X^\sharp) \subseteq Y^\sharp$.*

Proof. It is enough to notice that the composition of ∂-morphisms is a ∂-morphism. □

As in the characteristic 0 case, if $X = S \times_C X_C$ for a C-scheme X_C, then there is a natural ∂-structure on X, since we can trivially extend the group scheme actions on S to X.
We call iterative ∂-schemes as above *trivial* and ones ∂-isomorphic to them *isotrivial.*

Lemma 2.8. *Let $S \times_C X_C$ be a trivial ∂-scheme. Then $X^\sharp = X_C(C)$, where $X_C(C)$ is naturally embedded into $X(K)$.*

Proof. Without loss $X = \mathrm{Spec}(R)$, where $R = K \otimes_C R_C$ for a C-algebra R_C, and the ∂-algebra structure on R is trivial on R_C (i.e. the operators ∂_n vanish on R_C for $n > 0$). It is enough to notice now that a K-algebra homomorphism $R \to K$ is a ∂-homomorphism if and only if it maps R_C into C. □

We want to prove an analogue of Hilbert's Nullstellensatz in this context. To this end we need the base ∂-field K to have enough solutions of systems of ∂-equations. Such an iterative field is called "rich" in [10], where it is shown [10, 3.15] that each iterative field extends to a rich one. In our case we can describe such fields in a more explicit way. From now on we will assume that (K, ∂) is a "universal domain" which in this context means the following:

- (K, ∂) is separably closed, $[K : K^p] = p$ and $C = K^{p^\infty}$;
- (K, ∂) is *ω-compact*: in this context we can just assume that each countable descending chain of non-empty solution sets of systems of ∂-equations has a non-empty intersection.

By results of Ziegler [13] such universal domains exist and have properties analogous to properties of algebraically closed fields.

Proposition 2.9. *If X is a ∂-scheme which is locally of finite type over K, then $X^\sharp$ is Zariski dense in X.*

Proof. By 2.4, we can replace X with any open subscheme, so it is enough to show that $X^\sharp$ is non-empty and we can assume that X is affine, irreducible and of finite type over K. Thus $X = \mathrm{Spec}(R)$, where R is a finitely generated K-algebra such that $R_{\mathrm{red}}(=R/\sqrt{(0)})$ has no zero-divisors. By 2.7, there is an iterative C-derivation D on R extending ∂. Our aim is to find a ∂-homomorphism $R \to K$.
Let a be a finite tuple such that $R = K[a]$. Let g be a finite tuple of K-polynomials generating the ideal of a over K and each f_i be a tuple of K-polynomials such that

$$D_i(a) = f_i(a).$$

By [12, Proposition 2.1], R_{red} has a ∂-algebra structure such that the quotient map $R \to R_{\mathrm{red}}$ is a ∂-homomorphism. The system of ∂-equations

$$g(x) = 0, \quad \partial_1(x) = f_1(x), \quad \partial_2(x) = f_2(x), \quad \dots$$

has a solution in R, so it has also a solution in the fraction field of R_{red} (it has a natural ∂-structure by [12, Proposition 2.3]). Since (K, ∂) is existentially closed and ω-compact, this system has also a solution $b \subseteq K$ (see [13]). Then $a \mapsto b$ extends to a ∂-homomorphism from R to K. □

3. The automorphism group functor and the first isotriviality theorem

In this section we construct a certain ∂-structure on the automorphism group of a projective variety which is needed for the proof of

Theorem 3.2. This construction works in any category with fiber products. Our strategy is the following: we perform this construction in the category of sets, point out what needs to be done to extend this construction to any category with fiber products and state the conclusion we need (Proposition 3.1) for the category of schemes.

Let $\mathcal{C}$ be a category with fiber products and $X \to Y$ be a morphism in $\mathcal{C}$. Let $\mathcal{C}_Y$ denote the category of morphisms $A \to Y$ in $\mathcal{C}$. It is possible to extend the group of automorphisms of X over Y to the following contravariant functor (see page 11 in [8]):

$$A_{X/Y} : \mathcal{C}_Y^{\mathrm{op}} \to \mathbf{Groups}, \quad A_{X/Y}(Z) = \mathrm{Aut}_Z(X \times_Y Z).$$

Let us work now in the category of sets keeping the notation from Section 2, i.e. we have a fixed set S, a fixed group α and a group action $\partial : \alpha \times S \to S$ (we focus on a single group action). Assume we have also a function $X \to S$ and group actions $D_1, D_2 : \alpha \times X \to X$ such that $X \to S$ is equivariant with respect to both D_1 and D_2. Then we can define a group action

$$\alpha \times \mathrm{Aut}_S(X) \to \mathrm{Aut}_S(X), \quad g \cdot \phi := D_2(g)\phi D_1(g)^{-1}.$$

However, we want to define a group action on the domain of the function into S representing $A_{X/S}$ which is

$$\bigcup_{s\in S} \mathrm{Aut}(X_s) \to S,$$

where for each $s \in S$, X_s denotes the fibre of $X \to S$ over s. Therefore, we need to restrict the above group action to each fiber:

$$g \cdot \phi_s := D_2(g)_s \phi_s D_1(g^{-1})_{g\cdot s}.$$

Let us take now a section (an "S-point") $g : S \to \bigcup_{s\in S} \mathrm{Aut}(X_s)$. Such a section corresponds to an element of $\mathrm{Aut}_S(X)$. Clearly, g is an equivariant section (a "$\sharp$-point") if and only if g is α-invariant as an element of $\mathrm{Aut}_S(X)$ which is in turn equivalent to g being an equivariant map between (X, D_1) and (X, D_2). Replacing points with morphisms in the category $\mathcal{C}$ and using Yoneda's Lemma we get the corresponding result in any category with fiber products replacing the category of sets. We formulate it below for the category of schemes.

Proposition 3.1. *Let X be an S-scheme and assume D_1, D_2 are ∂-structures on X. Assume also that $A_{X/S}$ is representable by a group scheme G over S. Then there is a ∂-structure on G such that any $\phi \in G^\sharp$ is a ∂-isomorphism between (X, D_1) and (X, D_2).*

We can prove now our first isotriviality theorem.

Theorem 3.2. *Let X be a proper scheme over K, and suppose D_1 and D_2 are ∂-structures on X. Then (X, D_1) is ∂-isomorphic to (X, D_2). In particular, if X is defined over C, then any iterative ∂-structure on X is isotrivial.*

Proof. By [8, 3.7], the functor $A_{X/S}$ is representable by a group scheme G which is locally of finite type over K. Let us consider the ∂-scheme structure on G given by 3.1. By 2.9, $G^\sharp$ is nonempty. By 3.1, any $\phi \in G^\sharp$ is a ∂-isomorphism between (X, D_1) and (X, D_2).
For the final clause, we take for D_2 the trivial ∂-structure on X. □

We need the properness assumption in 3.2, only for the representability of the automorphism group functor. Therefore 3.2 holds for any scheme over K such that its automorphism group functor is representable by a group scheme which is locally of finite type over K.
We would like to point out that even if we start from a projective variety over K, then the resulting automorphism group scheme need not to be reduced (see Example 4 in [8]). Thus we have to go beyond the category of algebraic varieties and consider schemes.

4. ∂-sheaves and the second isotriviality theorem

In this section we focus more on ∂-structures on sheaves (see 2.6). Suppose (R, ∂) is an iterative differential ring. By a ∂-module over R, or a ∂-R-module, we mean an R-module V together with a sequence $D = (D_n)_{n\in\mathbb{N}}$ of endomorphisms of the abelian group V, such that D_0 is the identity and for $n > 0$, $r \in R$ and $x \in V$ we have

$$D_n(rx) = \sum_{i+j=n} \delta_i(r) D_j(x),$$

as well as the iterativity property

$$D_i \circ D_j = \binom{i+j}{i} D_{i+j}.$$

The theory of such modules (called *ID-modules* there) over iterative fields and the related Picard-Vessiot theory was developed in [9]. The theory of Picard-Vessiot extensions for iterative fields originates from a paper of Okugawa [12].
The notion of a ∂-homomorphism between ∂-modules is clear (see [12]) and for any iterative C-scheme (X, ∂) we define the notion of a ∂-(pre)sheaf of ∂-$\mathcal{O}_X$-modules (∂-(pre)sheaf for short) in the obvious way. When we say that a sheaf of $\mathcal{O}_X$-modules is a ∂-sheaf, we mean that there is a natural ∂-structure on each module of sections such that the restriction maps are ∂-maps.

Fact 4.1. *Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves on an iterative C-scheme (X, ∂). Then:*
(i) If $\mathcal{F}$ is a ∂-presheaf, then $\mathcal{F}^+$ (the sheafification of $\mathcal{F}$) is a ∂-sheaf and the natural morphism $\mathcal{F} \to \mathcal{F}^+$ is a morphism of ∂-presheaves.
(ii) If $\mathcal{F}, \mathcal{G}$ are ∂-sheaves, then $\mathcal{F} \otimes \mathcal{G}$, $\mathcal{F} \oplus \mathcal{G}$, $\mathcal{F}^*$, $\bigwedge^n \mathcal{F}$ are ∂-sheaves.
(iii) Let $(U_i)_{i \in I}$ be an open basis of X, $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules and assume that for each U_i there is an iterative derivation ∂_i on $\mathcal{F}(U_i)$ such that the restriction maps are ∂-maps. Then the ∂_i's extend to make $\mathcal{F}$ a ∂-sheaf.

Proof. (i) Since the direct limit of ∂-algebras is a ∂-algebra, each stalk $\mathcal{F}_x$ is a ∂-algebra and $\mathcal{F}^+(U)$ (the module of the sections of $\bigcup_{x \in U} \mathcal{F}_x \to U$) has a ∂-module structure.
(ii) It follows from the corresponding properties of ∂-algebras which are easy to establish (see Section 2.2 in [9]).
(iii) It is a standard exercise on sheaves of sets. □

Let us assume now that (K, ∂) is an iterative field and a universal domain (see Section 2). By 2.6, for every ∂-scheme X, $\mathcal{O}_X$ is a ∂-sheaf. If (V, D) is a ∂-module, then we define

$$V^\sharp := \{x \in V \mid D_1(x) = 0, D_2(x) = 0, \ldots\}.$$

We easily obtain:

Fact 4.2. *If X is an iterative ∂-scheme, then $\mathcal{O}_X(X)^\sharp$ corresponds to the sheaf of ∂-morphisms from X to the trivial ∂-variety $\mathbb{A}^1$.*

We will use the assumption that (K, ∂) is an iterative field to see that $V^\sharp$ is large in a ∂-module V.

Lemma 4.3. *Let V be a ∂-module of finite dimension over K. Then $V^\sharp$ contains a basis of V.*

Proof. Since K is existentially closed, it contains a Picard-Vessiot field of V, so $V^\sharp$ contains a basis of V (see 3.3 and 3.4 in [9]). □

Remark 4.4. The above lemma also follows from the more general Proposition 2.9. Let $\{x_1, \ldots, x_n\}$ be a basis of V and $R := K[x_1, \ldots, x_n]$ be the polynomial algebra. The ∂-module structure ∂_V on V induces the unique ∂-algebra structure ∂_R on R extending ∂_V. Let us consider the ∂-structure on $\mathbb{A}^n$ corresponding to ∂_R^{-1} (the set of iterative derivations on R has a group structure which is usually non-commutative). It can be checked that $(\mathbb{A}^n)^\sharp = V^\sharp$ after the identification of $\mathbb{A}^n(K)$ and V given by the basis $\{x_1, \ldots, x_n\}$. By 2.9, $(\mathbb{A}^n)^\sharp$ is Zariski dense in $\mathbb{A}^n(K)$, in particular $V^\sharp$ contains a basis of V.

We extend the notion of a locally trivial sheaf, to the ∂-sheaves context:

Definition 4.5. Let X be a ∂-scheme.

(1) A ∂-sheaf on X is *∂-trivial*, if it is ∂-isomorphic to $\mathcal{O}_X^{\oplus n}$ for some $n \in \mathbb{N}$.
(2) A ∂-sheaf $\mathcal{F}$ on X is *locally ∂-trivial* if there is an open cover $X = \bigcup_{i \in I} U_i$ such that each $\mathcal{F}|_{U_i}$ is ∂-trivial.

Proposition 4.6. *If $\mathcal{F}$ is an invertible ∂-sheaf without base points on a projective iterative ∂-variety X, then:*

(1) *$\mathcal{F}$ is locally ∂-trivial,*
(2) *the morphism into the ∂-trivial projective space defined by a basis of $\mathcal{F}(X)^\sharp$ is a ∂-morphism.*

Proof. (1) Let $X = \bigcup_{i \in I} U_i$ be an open cover of X such that for each $i \in I$, there is an isomorphism of sheaves of $\mathcal{O}_{U_i}$-modules

$$f_i : \mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i}.$$

Since X is projective, $\mathcal{F}(X)$ is a ∂-module of finite dimension over K. By 4.3, there is $\{s_0, \ldots, s_n\}$, a basis of $\mathcal{F}(X)$ contained in $\mathcal{F}(X)^\sharp$. For each $i \in I$ and $j \in \{0, \ldots, n\}$, let:

$$U_{ij} := U_i \setminus Z(f_i(s_j|_{U_i})), \quad f_{ij} := f_i|_{U_{ij}}, \quad s_{ij} := s_i|_{U_{ij}}.$$

Since $\mathcal{F}$ has no base points, $\{U_{ij}\}_{i,j}$ is a cover of X. We will rescale each f_{ij} to make it a ∂-isomorphism. For a linear map between ∂-modules, it is enough to check the ∂-homomorphism condition on a given basis. Thus multiplying f_{ij} with $f_{ij}(s_{ij})^{-1}$ makes it a ∂-isomorphism, since the new map takes $s_{ij} \in \mathcal{F}(U_{ij})^\sharp$ to $1 \in \mathcal{O}_X(U_{ij})^\sharp$.

(2) Let $X = \bigcup_{i \in I} U_i$ be an open cover of X such that for each $i \in I$, there is an isomorphism of ∂-sheaves

$$f_i : \mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i}$$

(such a cover exists by (i)).
If $s \in \mathcal{F}(X)^\sharp$, then $f_i(s|_{U_i}) \in \mathcal{O}_X(U_i)^\sharp$. By 4.2,

$$f_i(s|_{U_i}) : U_i \to \mathbb{A}^1$$

is a ∂-map, where $\mathbb{A}^1$ has the trivial ∂-variety structure.
Let $B = \{s_0, \ldots, s_n\}$ be a basis of $\mathcal{F}(X)^\sharp$. Then the map

$$f_B : X \to \mathbb{P}^n$$

is a ∂-map, where $\mathbb{P}^n$ has the trivial ∂-variety structure. □

Fact 4.7. *If $A \to B$ is a ∂-map between iterative ∂-rings and $b \in B$, then:*

(1) $\Omega_{B/A}$ *is naturally a* ∂*-module,*
(2) B_b *has a* ∂*-*B*-algebra structure,*
(3) *the map* $\Omega_{B/A} \to \Omega_{B_b/A}$ *is a* ∂*-map.*

Proof. (1) By [5, II.8.1A.], $\Omega_{B/A}$ is isomorphic to I/I^2, where

$$I = \ker(B \otimes_A B \to B).$$

The ideal I has clearly the ∂-module structure, so has I^2, hence I/I^2 gets the quotient ∂-module structure.
(2) See the equation (2.1) in [12].
(3) Since the localization map $B \to B_b$ is a ∂-map, its tensor square is a ∂-map as well, and it clearly preserves the kernel of multiplication, so the result follows. □

Proposition 4.8. *If* X *is an iterative* ∂*-scheme, then* Ω_X *is a* ∂*-sheaf.*

Proof. We will use 4.1(iii). Take $(U_i)_i$ the open base of X consisting of open affine subvarieties. By the 4.7(i), each $\Omega_X(U_i)$ has a natural ∂-module structure. We need to check that the restriction maps preserve the ∂-module structure. Since any affine variety has an open basis consisting of subsets corresponding to localizations, it is enough to use 4.7(ii). □

Corollary 4.9. *The canonical and anticanonical sheaves are locally* ∂*-trivial invertible* ∂*-sheaves.*

Proof. By the previous proposition, 4.1(ii) and 4.6(i). □

We can prove now our second isotriviality theorem.

Theorem 4.10. *If* V *is a projective iterative* ∂*-variety and the canonical or the anticanonical divisor of* V *is ample, then* V *is* ∂*-isotrivial.*

Proof. By 4.3, 4.6 and 4.9 (after taking a suitable tensor power), we obtain a closed ∂-immersion $f : V \to \mathbb{P}^n$ (a suitable $n \in \mathbb{N}$), where $\mathbb{P}^n$ has the trivial ∂-structure. By 2.7, 2.8 and 2.9, the set $\mathbb{P}^n(C) \cap f(V)$ is Zariski dense in $f(V)$. By an automorphism argument, $f(V)$ is defined over C, so f is a ∂-isomorphism between V and $f(V)$ with the trivial ∂-structure. □

Corollary 4.11. *If* V *is a smooth projective* ∂*-curve, then* V *is* ∂*-isotrivial.*

Proof. By [5, IV.3.3] a divisor X on V is ample if and only if $\deg(X) > 0$. Hence, by 4.10, we are done in the cases when the degree of the canonical divisor is non-zero, i.e. when V is not an elliptic curve. But the case of an elliptic curve is solved in [1]. □

5. Further remarks

In [1] Benoist proves the following:

Proposition 5.1. *Let X be an algebraic variety over K (where (K, ∂) is a universal iterative differential field). Then X has an iterative ∂-structure if and only if for each n there is an isomorphism f_n between X and a variety X_n defined over K^{p^n} such that for each n the isomorphism $f_n \circ f_{n+1}^{-1}$ between X_{n+1} and X_n is defined over K^{p^n}.*

So we ask here whether any projective variety X over K satisfying the conditions of the proposition descends to $C = \bigcap_n K^{p^n}$.

On the other hand, if X belongs to a family with a fine moduli space, then simply the fact that X is (isomorphic to something) defined over each K^{p^n} implies that X descends to the intersection. See [4] for a discussion of fine moduli spaces and the related issues.

In particular the class of "principally polarized abelian varieties of dimension g with level $n \geqslant 3$ structure" has a fine moduli space (see [11]). Together with Theorem 3.2 we obtain:

Corollary 5.2. *Let (X, D) be an iterative ∂-variety over K, where X is an abelian variety. Then (X, D) is ∂-isotrivial.*

We finish with giving yet another characterization of ∂-schemes, which correspond to the twisted tangent space way of defining ∂-varieties in the case of characteristic 0 (see Introduction). This definition also appears in [10]. We will use Buium's prolongations ∇_n [2] for $n \in \mathbb{N}$. Each ∇_n is a right adjoint functor to the functor $X \mapsto \alpha_n \times_\partial X$. By adjointness, having a group action $\alpha_n \times X \to X$ such that the structure morphism $X \to S$ is equivariant is equivalent to having a section $s_n : X \to \nabla_n(X)$ such that the following diagram is commutative

$$\begin{array}{ccc} \nabla_n(\nabla_n X) & \xleftarrow{\nabla_n(s_n)} & \nabla_n X \\ {\scriptstyle c_n^\dagger}\uparrow & & \uparrow{\scriptstyle s_n} \\ \nabla_n X & \xleftarrow[s_n]{} & X, \end{array}$$

where $c_n^\dagger : \nabla_n X \to \nabla_n(\nabla_n X)$ is adjoint to the multiplication morphism on α_n.

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