Differential schemes

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Abstract

We start with the definition of differential scheme using the treatment of Hartshorne. Immediately we find that there are problems: the global section functor of an affine differential scheme does not recover the original ring. We give some examples to show what goes wrong. However, reduced differential schemes are more more tractable. We shall discuss what is known and not known.

Notation

All rings are commutative and unitary. We fix a set of commuting derivations $\Delta = \{\delta_1, \ldots, \delta_m\}$. We use the prefix Δ instead of the word "differential". R is a Δ-ring (arbitrary characteristic), K is a Δ-field. If S is a subset of R then $[\![S]\!]$ is the smallest radical Δ-ideal of R that contains S. If $\mathbb{Q} \subset R$ (i.e. R is a Ritt algebra) then $[\![S]\!] = \sqrt{[S]}$.

Definition

$X = \operatorname{Diffspec} R$ is the set of prime Δ-ideals of R.

If $\mathfrak{a}$ is a Δ-ideal then

$$V(\mathfrak{a}) = \{\mathfrak{p} \in X \mid \mathfrak{a} \subset \mathfrak{p}\}.$$

These sets are the closed sets in the *Kolchin* topology.
If $a \in R$ then

$$D(a) = \{\mathfrak{p} \in X \mid a \notin \mathfrak{p}\}.$$

These sets are the basic opens.

Structure sheaf

The structure sheaf is defined exactly as in Hartshorne.

Theorem

If $s \in \mathcal{O}_X(U)$, then, for $\mathfrak{p} \in U$,

$$s(\mathfrak{p}) = \begin{cases} \frac{a_1}{b_1} & \text{for } \mathfrak{p} \in D(b_1) \\ \vdots & \vdots \\ \frac{a_n}{b_n} & \text{for } \mathfrak{p} \in D(b_n) \end{cases}$$

and $D(b_1) \cup \ldots \cup D(b_n) = U$ (equivalently $1 \in [\![b_1, \ldots, b_n]\!]$).

Theorem

The stalk $\mathcal{O}_{X,\mathfrak{p}}$ is Δ-isomorphic to the local ring $R_\mathfrak{p}$.

Other definitions

Buium defines a Δ-scheme as a scheme whose sheaf consists of Δ-rings. Umemura calls this "a scheme with derivations".

Carra Ferro changes the definition of the sheaf. If $X = \operatorname{Diffspec} R$ and $Y = \operatorname{Spec} R$ then $\mathcal{O}_X(U)$ is defined to be $\mathcal{O}_Y(V)$ where V is the largest open subset of Y with $V \cap X = U$.

Hrushovski has a paper in the Archives about difference schemes.

Global sections

Definition

$\widehat{R} = \mathcal{O}_X(X) = \Gamma(X, \mathcal{O}_X)$ is the ring of global sections of X.

Theorem

There is a canonical mapping

$$\iota\colon R \to \widehat{R}, \qquad \iota(a)(\mathfrak{p}) = \frac{a}{1} \in R_{\mathfrak{p}}.$$

However ι is neither injective nor surjective in general.

Closure

Theorem

$\iota_{\widehat{R}} \colon \widehat{R} \to \widehat{\widehat{R}}$ is a Δ-isomorphism.

Thus "taking hat" is a kind of closure operation.
Write $\widehat{X} = \operatorname{Diffspec} \widehat{R}$. We have an induced morphism

$$ {}^{a}\iota \colon \widehat{X} \to X, \qquad {}^{a}\iota(\mathfrak{q}) = \iota^{-1}\mathfrak{q}. $$

We also have one in the other direction.

$$ H \colon X \to \widehat{X} \qquad H(\mathfrak{p}) = \widehat{\mathfrak{p}} = \rho_{\mathfrak{p}}^{-1}(\mathfrak{m}_{\mathfrak{p}}). $$

Here $\mathfrak{m}_p \subset R_{\mathfrak{p}}$ is the maximal ideal and $\rho_{\mathfrak{p}} \colon \mathcal{O}_X(X) \to \mathcal{O}_{X,\mathfrak{p}}$ is the restriction.

Theorem

${}^{a}\iota \circ H \colon X \to \widehat{X} \to X$ is the identity.

In general, I do not know about the other composition. A priori, there may be ideals of $\widehat{R}$ that are not of the form $\widehat{\mathfrak{p}}$.

Differential zeros

In a ring, the condition $1 \in \operatorname{Ann}(z)$ is equivalent to the condition that $z = 0$.

Definition

$z \in R$ is a *Δ-zero* if $1 \in [\![\operatorname{Ann}(z)]\!]$. The set of Δ-zeros of R is denoted by $\mathfrak{Z} = \mathfrak{Z}(R)$.

Theorem

$\mathfrak{Z}$ is a Δ-ideal of R.

Theorem

$R/\mathfrak{Z}$ has no non-zero Δ-zero. $\mathfrak{Z}$ is the smallest Δ-ideal with that property.

Theorem

$a \in R$ is a Δ-zero if and only if it goes to 0 in $R_{\mathfrak{p}}$ for every Δ-prime $\mathfrak{p}$.

Differential zeros and global sections

Theorem

The kernel of $\iota\colon R \to \widehat{R}$ *is* $\mathfrak{Z}$.

Theorem

$\operatorname{Diffspec} R \approx \operatorname{Diffspec}(R/\mathfrak{Z})$.

So we simply replace R by $R/\mathfrak{Z}$ and everything is hunky-dory.

NOT!

Differential zeros - problems

We would like the absence of Δ-zeros to be local, i.e. $\mathfrak{Z}(R) = (0)$ if and only if $\mathfrak{Z}(R_\mathfrak{p}) = (0)$ for every prime Δ-ideal $\mathfrak{p}$. This is false.

We would like it to behave well with respect to basic opens. It doesn't. It may happen that $\mathfrak{Z}(R) = 0$ but $\mathfrak{Z}(R_a) \neq 0$ for some $a \in R$.

Definition

A Δ-ring R is AAD (<u>A</u>nnihilators <u>A</u>re <u>D</u>ifferential) if for every $r \in R$, $\mathrm{Ann}(r)$ is a Δ-ideal.

Any ring with trivial derivations is AAD. Thus the rings of classical Δ-algebraic geometry are AAD. The corresponding condition for difference rings is called “well-mixed”.

Properties of AAD rings

Theorem

R is AAD if and only if $R_{\mathfrak{p}}$ is AAD for every $\mathfrak{p} \in X$.

Theorem

If R is AAD then $R\Sigma^{-1}$ is AAD for every multiplicative set Σ.

Theorem

If R is AAD, then the canonical mappping $\iota \colon R \to \widehat{R}$ is injective.

The AAD-ical

Theorem

For any Δ*-ring* R *there is a unique smallest* Δ*-ideal* $\mathfrak{A} = \mathfrak{A}(R)$ *such that* $R/\mathfrak{A}$ *is AAD.*

Theorem

If $\mathfrak{N}$ *is the nil radical of* R *then* $\mathfrak{Z} \subset \mathfrak{A} \subset \mathfrak{N}$*. A reduced ring is AAD.*

Theorem

$R \to R/\mathfrak{A}$ *induces a homeomorphism from* $\operatorname{Diffspec}(R/\mathfrak{A})$ *onto* X*.*

But not an isomorphism of schemes. (Neither does $R \to R/\mathfrak{N}$.)

AAD schemes

Definition

A Δ-scheme $(X, \mathcal{O}_X)$ is AAD if for every open set $U \subset X$, the Δ-ring $\mathcal{O}_X(U)$ is AAD.

Theorem

$(X, \mathcal{O}_X)$ is AAD if and only if each stalk $\mathcal{O}_{X,x}$ is AAD.

Theorem

Let $X = \operatorname{Diffspec} R$. If R is AAD then so is X. Conversely if X is AAD then there exists an AAD Δ-ring S with $\operatorname{Diffspec} S$ isomorphic to X.

But R itself need not be AAD.

Reduced schemes

Suppose that $X = \operatorname{Diffspec} R$ is reduced (the stalks are all reduced). Is R reduced? No.

Theorem

If X is reduced then $\widehat{R}$ is reduced and $X \approx \widehat{X} = \operatorname{Diffspec} \widehat{R}$.

Thus we can always choose a reduced ring S with $X \approx \operatorname{Diffspec} S$.

Δ-units

In a ring, units are elements u with $1 \in (u)$.

Definition

$u \in R$ is a Δ-*unit* if $1 \in [\![u]\!]$. The set of Δ-units of R is denoted by $\mathcal{U} = \mathcal{U}(R)$.

Example

Every non-zero element of $\mathbb{Q}[x]$ is a Δ-unit.

Facts about Δ-units

Theorem

$\mathcal{U}$ is a multiplicative set.

Theorem

Every Δ-unit of $R\mathcal{U}^{-1}$ is a unit.

Theorem

Every Δ-unit of $\widehat{R}$ is a unit.

Theorem

If R is AAD, then $R \to R\mathcal{U}^{-1}$ and $R\mathcal{U}^{-1} \to \widehat{R}$ are injective.

I.e. $R \subset R\mathcal{U}^{-1} \subset \widehat{R}$.
But these inclusions may be proper.

Integral domains

For an integral domain R we know that

$$R = \bigcap_{P \in \operatorname{Spec} R} R_P$$

where the intersection is over all prime ideals.

Theorem

Suppose that R is a Δ-ring that is a domain. Then

$$R\mathcal{U}^{-1} \subset \bigcap_{\mathfrak{p} \in \operatorname{Diffspec} R} R_{\mathfrak{p}}$$

However the inclusion may be proper.

Δ-K-schemes

Let K be a Δ-field and $X = \operatorname{Diffspec} R$. Then X is a Δ-scheme over $\operatorname{Diffspec} K$ if there is a morphism $X \to \operatorname{Diffspec} K$. This does not imply that R is a K-algebra.

Example

Let $K = \mathbb{Q}(x)$ and $R = \mathbb{Q}[x]$. Then $X = \operatorname{Diffspec} R$ is a Δ-scheme over $Y = \operatorname{Diffspec} K$ (in fact they are isomorphic). However R is not a K-algebra.

This affects the existence of products. If $X = \operatorname{Diffspec} R$ and $Z = \operatorname{Diffspec} S$ are Δ-schemes over $Y = \operatorname{Diffspec} K$ then

$$X \times_Y Z \stackrel{?}{=} \operatorname{Diffspec} R \otimes_K S.$$

But the right-hand side need not make sense!

Subschemes

Suppose that $Y \subset X = \operatorname{Diffspec} R$ is a closed subscheme. Does there exist an ideal $\mathfrak{a} \subset R$ such that $Y \approx \operatorname{Diffspec}(R/\mathfrak{a})$? I don't know.

Theorem

If $Y \subset X$ are both reduced then there is a radical Δ-ideal $\mathfrak{a} \subset R$ such that $Y \approx \operatorname{Diffspec}(R/\mathfrak{a})$.

This theorem is probably true with "reduced" replaced by "AAD", but I do not have a proof.

Adjunction

In algebraic geometry there is an adjunction between Spec and the global sections functor. Here too.

Theorem

Suppose that $X = \operatorname{Diffspec} R$ *and* $Y = \operatorname{Diffspec} S$. *Then there is a bijection*

$$\operatorname{Mor}(Y, X) \approx \operatorname{Hom}(R, \widehat{S})$$

This is not symmetric in R and S!

Theorem

If R *and* S *are AAD then*

$$\operatorname{Mor}(Y, X) \approx \operatorname{Hom}(R, \widehat{S}) \approx \operatorname{Hom}(\widehat{R}, \widehat{S}).$$

It's not pretty but at least it is symmetric.

Rittian Δ-rings

Definition

A Δ-ring R is *Rittian* if every strictly increasing chain of radical Δ-ideals is finite.

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Theorem

If R is finitely Δ-generated over K then R is Rittian.

Theorem

R is Rittian if and only if $X = \operatorname{Diffspec} R$ is a Noetherian topological space.

This is actually better than the algebraic version!

Reduced Rittian Ritt algebras

Suppose that R is a Ritt algebra (contains $\mathbb{Q}$) that is reduced and Rittian.

Theorem

R has a finite number of minimal prime ideals and they are Δ-ideals.

Theorem

The complete ring of quotients of R is a finite product of Δ-fields.

Theorem

R has a finite number of minimal idempotents.

Using these we get the usual theorems about connected and irreducible components of X.

Rational functions

If R is reduced then the canonical mapping $\iota\colon R \to \widehat{R}$ is injective and we identify R with a subring of $\widehat{R}$.

Theorem

If R is a domain then so is $\widehat{R}$ and $\mathrm{qf}(R) = \mathrm{qf}(\widehat{R})$.

Hence the field of rational functions of an irreducible variety is what you expect.

If R is not a domain, the ring of rational functions classically is $Q(R)$, the complete of fractions of R, i.e. $R\Sigma^{-1}$ where Σ is the multiplicative set of elements that are not zero divisors in R.

Rational functions of a reducible Δ-scheme

Recall that a global section has the form

$$s(\mathfrak{p}) = \begin{cases} \dfrac{a_1}{b_1} & \text{for } \mathfrak{p} \in D(b_1) \\ \vdots & \vdots \\ \dfrac{a_n}{b_1} & \text{for } \mathfrak{p} \in D(b_n) \end{cases}$$

It may happen that every b_i is a zero divisor.

Theorem

If R is a reduced Rittian Ritt algebra then there is an injective homomorphism

$$\widehat{R} \longrightarrow Q(R).$$

If we identify $\widehat{R}$ with its image then $\widehat{R}$ is the subring of $Q(R)$ consisting all everywhere defined functions (in the sense of Cassidy)

Dimensions

Suppose that R is a Δ-K-algebra that is finitely Δ-generated over K. I assume K has characteristic 0 to avoid problems with separability. We choose a set of generators

$$R = K\{x_1, \ldots, x_n\}.$$

For $s \in \mathbb{N}$, let be R_s the ring (not Δ-ring) generated by θx_i where θ is a "higher derivation", $\theta = \delta_1^{d_1} \cdots \delta_m^{d_m}$, of order bounded by s.

Theorem

There is a numerical polynomial Φ_{Krull} such that

$$\Phi_{Krull}(s) = \dim R_s \qquad \textit{(Krull dimension)}$$

for $s >> 0$.

If we let $X_s = \operatorname{Spec} R_s$ then $\Phi_{\mathrm{Krull}}(s)$ is the topological dimension of X_s. Unfortunately the schemes X_s depend on the choice of Δ-generators of R.

Local ring

For notions like "simple point" we are interested in finding the dimension of a stalk, $R_{\mathfrak{p}}$. For simplicity I assume that we are working with a rational point, i.e. $R_{\mathfrak{p}}/\mathfrak{m} = K$.
For each $s \in \mathbb{N}$, we have a homomorphism

$$\alpha_x \colon (R_s)_{\mathfrak{p}_s} \to R_{\mathfrak{p}}, \qquad \frac{a}{b} \mapsto \frac{a}{b}.$$

The image is $(R_{\mathfrak{p}})_s$. I do not know if α_s is always injective or not.
The picture is

$$R \xrightarrow{\text{cut down}} R_s \xrightarrow{\text{localize}} (R_s)_{\mathfrak{p}_s}$$

versus

$$R \xrightarrow{\text{localize}} R_{\mathfrak{p}} \xrightarrow{\text{cut down}} (R_{\mathfrak{p}})_s$$

But it doesn't matter since it turns out that, for $s >> 0$, every element of $\ker \alpha_s$ is nilpotent, so the Krull dimensions of these rings are the same.

Theorem

There is a numerical polynomial Φ_{Krull} such that

$$\Phi_{Krull}(s) = \dim(R_{\mathfrak{p}})_s$$

for $s >> 0$.

$\mathfrak{m}/\mathfrak{m}^2$ is a Δ-vector space over $K = R_{\mathfrak{p}}/\mathfrak{m}$.

Theorem

There is a numerical polynomial Φ_{vs} such that

$$\Phi_{vs}(s) = \dim_K(\mathfrak{m}/\mathfrak{m}^2)_s, \qquad \textit{(vector space dimension).}$$

As before there is a mapping

$$\beta_s \colon \mathfrak{m}_s/\mathfrak{m}_s^2 \to (\mathfrak{m}/\mathfrak{m}^2)_s$$

which is surjective but not necessarily injective.

But this time we have a counterexample!

Example

$R = K[\overline{y}] = K[y]/[y' - yy'']$. Then localize at the ideal $\mathfrak{p} = [\overline{y}]$.

Krull intersection theorem

This example also illustrates that the Krull intersection theorem does not always hold.

$$\bigcap_{d \in \mathbb{N}} \mathfrak{m}^d \neq (0).$$

Indeed $\overline{y}' = \overline{yy}''$ and $\mathfrak{m} = [\overline{y}]$ so

$$\begin{aligned}
\overline{y}' &= \overline{yy}'' \in \mathfrak{m}^2, \\
\overline{y}' &= \overline{y}(\overline{y}')' = \overline{y}(\overline{yy}'')' \in \mathfrak{m}^3, \\
\overline{y}' &= \overline{y}(\overline{y}(\overline{y}')')' = \overline{y}(\overline{y}(\overline{yy}'')')' \in \mathfrak{m}^4, \\
&\vdots
\end{aligned}$$

An inequality

Because β_s is surjective we have an inequality.

Theorem

for $s >> 0$

$$\Phi_{vs}(s) \leq \dim_K(\mathfrak{m}_s/\mathfrak{m}_s^2).$$

There is also the standard inequality for rings

$$\dim_K(\mathfrak{m}_s/\mathfrak{m}_s^2) \geq \dim(R_p)_s$$

Putting these together we get:

$$\Phi_{\mathrm{vs}}(s) \leq \dim_K(\mathfrak{m}_s/\mathfrak{m}_s^2) \geq \Phi_{\mathrm{Krull}}(s).$$

I need to research this more!

Thank you

These slides are available at
`mysite.verizon.net/jkovacic/lectures/leeds.dvi`