Exponential Differential Equations of Semiabelian Varieties

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Differential Fields meeting
Leeds, 9th June 2007

Outline

1. Exponential differential equations
2. The theory
3. Some applications and motivation

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Exponential maps

Fact

Let G be a connected commutative complex algebraic group, of dimension n. Then there is a lattice $\Lambda \subseteq \mathbb{C}^n$ with $\Lambda \cong \mathbb{Z}^d$ for $d \leqslant 2n$ and an exact sequence of holomorphic group homomorphisms:

$$\begin{tikzcd} 0 \arrow[r] & \Lambda \arrow[r] & \mathbb{C}^n \arrow[r, "\exp_G"] & G \arrow[r] & 0 \end{tikzcd}$$

Examples

- $\mathbb{G}_{\mathrm{a}}$ $(n = 1, d = 0)$
- $\mathbb{G}_{\mathrm{m}}$ $(n = 1, d = 1)$
- Abelian varieties $(d = 2n)$

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- G is a quotient $G = \mathbb{C}^n/\Lambda$
- The quotient map is called the exponential map of G

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Tangent bundles

$$\begin{tikzcd} 0 \arrow[r] & \Lambda \arrow[r] & \mathbb{C}^n \arrow[r, "\exp_G"] & G \arrow[r] & 0 \end{tikzcd}$$

- We identify $\mathbb{C}^n$ with the Lie algebra LG, the tangent space at the identity of G.
- The Lie bracket is trivial.
- The tangent bundle is $TG \cong LG \times G$ (isomorphism of algebraic groups)
- The graph of $\exp_G$ is a subgroup: $\mathcal{G} \subseteq TG(\mathbb{C})$

Equation as a differential variety

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- Consider also the differential field $\mathbb{C}\langle\langle t \rangle\rangle$ of germs of meromorphic functions at 0, considered as Laurent series.
- The map $\exp_G$ is defined on $\mathbb{C}\langle\langle t \rangle\rangle$, so we get $\mathcal{G} \subseteq TG(\mathbb{C}\langle\langle t \rangle\rangle)$

Definition (Motivating definition)

Write Γ_G for the Kolchin closure of $\mathcal{G}$. This is a differential subvariety of TG, which we consider in any differential field F, of characteristic 0.

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Logarithmic derivatives

Fact

For any commutative algebraic group G, and differential field $\langle F, D\rangle$, there is a logarithmic derivative map, a group homomorphism $logD_G : G(F) \longrightarrow LG(F)$.

If $y = \exp_{\mathbb{G}_m}(x)$ then $Dx = \frac{Dy}{y}$, that is, $logD_{\mathbb{G}_a}(x) = logD_{\mathbb{G}_m}(y)$.

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Definition (Alternative definition)

In a differential field F, Γ_G is the solution set of the exponential differential equation

$$logD_{LG}(x) = logD_G(y)$$

under a chosen identification of LG and LLG.

Reducts of differential fields

Definition

Let $\langle F; +, \cdot, C, D\rangle$ *be a differential field (of characteristic 0). Let* C_0 *be a subfield of* C*, let* $\mathcal{S}$ *be a collection of semiabelian varieties, each defined over* C_0*, and assume* $\mathcal{S}$ *is closed under taking products, connected subgroups, quotients, and under isogeny.*
Expand F *by adding a symbol for* Γ_S *for each* $S \in \mathcal{S}$*, and constant symbols for each element of* C_0*.*
Then forget the deriviation – consider the reduct $\langle F; +, \cdot, C, (\Gamma_S)_{S\in\mathcal{S}}, (\hat{c})_{c\in C_0}\rangle$*.*

Goal

Find the complete first order theory $T_{\mathcal{S}}$ of this reduct, when F is differentially closed.

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Technical assumption

For technical reasons, we assume that $\mathcal{S}$ contains only products of simple semiabelian varieties, and groups isogenous to these.

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The algebraic theory

- U1 F is a field of char 0, C is a relatively algebraically closed subfield, and the parameters in C_0 have the appropriate type.
- U2 Γ_S is a subgroup of TS.
- U3 $TS(C) \subseteq \Gamma_S$
- U4 $(0, y) \in \Gamma_S \iff y \in S(C)$ and $(x, 1) \in \Gamma_S \iff x \in LS(C)$.
- U5 If $S_1 \xrightarrow{f} S_2$ is an algebraic group homomorphism then $(Tf)(\Gamma_{S_1}) \subseteq \Gamma_{S_2}$.
- U6 $S_1 \subseteq S_2 \implies \Gamma_{S_1} = \Gamma_{S_2} \cap TS_1$.
- U7 $\Gamma_{S_1 \times S_2} = \Gamma_{S_1} \times \Gamma_{S_2}$.

The Schanuel Property

Theorem

SP *If* $g \in \Gamma_S$ *and* $\operatorname{td}_C(g) < \dim S + 1$ *then there is a proper algebraic subgroup* H *of* S *and* $\gamma \in TS(C)$ *such that* g *lies in the coset* $\gamma \cdot TH$*.*

Special case, $S = \mathbb{G}_m{}^n$, a torus

Let $x_1, \ldots, x_n, y_1, \ldots, y_n \in F$ with $Dx_i = \frac{Dy_i}{y_i}$ for each i. Suppose $\operatorname{td}_C(\bar{x}, \bar{y}) < n + 1$. Then there are $m_1, \ldots, m_n \in \mathbb{Z}$, not all zero, such that $\sum_{i=1}^{n} m_i x_i \in C$ and $\prod_{i=1}^{n} y_i^{m_i} \in C$.

This case is due to Ax, 1971.

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The universal theory

Theorem

The algebraic axioms U1—U7, together with the Schanuel property SP, hold in any differential field. They are expressible by universal first-order sentences.

Schanuel property - geometric version

- The Schanuel property can be viewed as a necessary condition for a system of equations to have a solution.

Schanuel Property – version 2

Let V be an irreducible subvariety of TS, defined over C. Suppose $g \in V \cap \Gamma_S$, and $\dim V < \dim S + 1$. Then g lies in a C-coset of TH for some proper algebraic subgroup H of S.

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Freeness

Definition

An irreducible subvariety V of TS is free iff V is not contained in a coset of TH for any proper algebraic subgroup H of S.
It is absolutely free iff, for all such H, $\mathrm{pr}_S V$ is not contained in a coset of H and $\mathrm{pr}_{LS} V$ is not contained in a coset of LH.

It is enough to consider varieties which are absolutely free over C.

Schanuel Property – version 3

Let V be an irreducible subvariety of TS, defined over C, and suppose V is absolutely free. If $\dim V < \dim S + 1$ then $V \cap \Gamma_S$ does not contain a point of V which is generic over C.

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Rotundity

Definition

An irreducible subvariety V of TS is rotund *iff for every quotient map $S \xrightarrow{f} H$,*

$$\dim(Tf)(V) \geqslant \dim H$$

and strongly rotund *iff for every such f*

$$\dim(Tf)(W) \geqslant \dim H + 1.$$

Lemma

If V is defined over C and V is absolutely free, and $V \cap \Gamma$ contains a point of V which is generic over C, then V is strongly rotund.

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Existential closedness – 1

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If V is defined over C and V is absolutely free, and $V \cap \Gamma$ contains a point of V which is generic over C, then V is strongly rotund.

The existential closedness property is a converse to this.

Existential closedness – first approximation

For each strongly rotund irreducible subvariety V of TS, defined over C, the intersection $V \cap \Gamma_S$ contains a point of V which is generic over C.

The special case of this statement for powers of $\mathbb{G}_{\mathrm{m}}$ is a theorem due to Cecily Crampin, in her DPhil thesis.

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Existential closedness – 2

We must deal with varieties V which are not necessarily defined over C. This actually makes things easier!

Strong existential closedness

For each rotund irreducible subvariety V of TS, and each finitely generated field of definition A of V, the intersection $V \cap \Gamma_S$ contains a point of V which is generic over A.

Existential closedness

For each rotund irreducible subvariety V of TS, the intersection $V \cap \Gamma_S$ is nonempty.

Theorem

If F is differentially closed, then its reduct satisfies the above existential closedness property. Furthermore, this property can be axiomatized in a first order way.

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The complete theory

Theorem

The complete first order theory T_S of the reducts of differentially closed fields is given by

- *The algebraic axioms U1 — U7*
- *The Schanuel property*
- *The existential closedness property*
- *F is algebraically closed (implicit in EC)*
- *F contains a nonconstant point.*

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Categoricity

- There is a natural pregeometry defined on models of the theory $T_{\mathcal{S}}$, coming from the Schanuel property.
- We add two non-first order conditions to $T_{\mathcal{S}}$: CCP (the countable closure property for the pregeometry) and $\mathrm{td}(C/C_0) = d$ for some countable cardinal d.
- The resulting theory is quasiminimal excellent and hence uncountably categorical.
- The countable models of $T_{\mathcal{S}}$ are determined by their dimension and $\mathrm{td}(C/C_0)$, and there are only countably many.
- The reduct of a differentially closed field is always infinite dimensional, hence the reduct of any countable differentially closed field is the unique model of dimension $\aleph_0$, with given $\mathrm{td}(C/C_0)$.

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Blurred Complex Exponentiation

- Given $C_0 \subseteq \mathbb{C}$ and a collection $\mathcal{S}$ now of complex semiabelian varieties, consider the structure $\mathbb{C}_{\mathcal{S}} = \langle \mathbb{C}; +, \cdot, (\exp_S)_{S \in \mathcal{S}} \rangle$.
- Let C be a countable $\mathcal{S}$-exponentially algebraically closed subfield of $\mathbb{C}$, and define

$$\Gamma_S = \{(x, y) \in TS \mid y / \exp_S(x) \in S(C)\}$$

- Then "blurred complex $\mathcal{S}$-exponentiation", $\langle \mathbb{C}; +, \cdot, C, (\Gamma_S)_{S \in \mathcal{S}} \rangle$, is a model of $T_{\mathcal{S}}$ with CCP.

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- For the case $\mathcal{S} = \{\mathbb{G}_{\mathrm{m}}{}^{n} \mid n \in \mathbb{N}\}$, we can blur Zilber's [pseudo]-exponential field in the same way. This also produces a model of $T_{\mathcal{S}}$, with CCP, of cardinality $2^{\aleph_0}$.
- Blurred complex exponentiation and blurred pseudoexponentiation are isomorphic.
- The natural pregeometries on the exponential fields are the same as those controlling the blurred exponential fields, hence complex exponential algebraic closure is isomorphic to the natural (Schanuel) pregeometry on Zilber's exponential field.

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Differential Algebra Conclusions

Theorem

Let S be a semiabelian variety defined over C, and let V be a subvariety of TS. If V is defined over C and absolutely free then a necessary and sufficient condition for there to be a nonconstant point in $\Gamma_S \cap V$ in some differential field extension is for V to be strongly rotund.

If V is not defined over C but is free and $\operatorname{Loc}_C V$ is absolutely free then a sufficient condition for a point to exist is for V to be rotund. If in addition $\operatorname{Loc}_C V$ is strongly rotund then a nonconstant point exists.

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