

DIFFERENTIAL TANNAKIAN FORMALISM

MOSHE KAMENSKY

The purpose of this note is to define differential tensor categories, and to give a model theoretic proof of the basic theorem, corresponding such categories, endowed with a suitably defined fibre functor, with linear differential algebraic groups. This result was proved by algebraic methods in [Ovc07a] and [Ovc07b].

1. DERIVATIVES OF CATEGORIES

All categories are abelian and all functors are additive unless stated otherwise. We also assume that in a tensor category (\mathcal{C}, \otimes) , the functor \otimes is exact; this is automatic if \mathcal{C} is rigid (see [DMOS82].)

1.1. Definition. Let \mathcal{C} be an abelian category. The *derivative* $\mathbb{D}(\mathcal{C})$ of \mathcal{C} is defined as follows: The objects are exact sequences $0 \rightarrow X_0 \xrightarrow{i_X} X_1 \xrightarrow{\pi_X} X_0 \rightarrow 0$ of \mathcal{C} , and the morphisms from such an object are morphisms of exact sequences whose two X parts coincide.

The category $\mathbb{D}(\mathcal{C})$ is again abelian. An exact functor $F : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ gives rise to an induced (exact) functor $\mathbb{D}(F) : \mathbb{D}(\mathcal{C}_1) \rightarrow \mathbb{D}(\mathcal{C}_2)$. We denote by Π_i ($i = 0, 1$) the functors from $\mathbb{D}(\mathcal{C})$ to \mathcal{C} assigning X_i to $0 \rightarrow X_0 \xrightarrow{i_X} X_1 \xrightarrow{\pi_X} X_0 \rightarrow 0$ (thus there is an exact sequence $0 \rightarrow \Pi_0 \xrightarrow{i_\Pi} \Pi_1 \xrightarrow{\pi_\Pi} \Pi_0 \rightarrow 0$.) $\Pi_i(X)$ is also abbreviated as X_i , and X is said to be over X_0 (and similarly for morphisms.)

1.2. Let A and B be two objects over X . Their *Yoneda sum* $A \star B$ is a new object over X , defined as follows: the combined map $X \times X \rightarrow A_1 \times B_1$ factors through $A_1 \times_X B_1$, and together with the map $X \xrightarrow{1, -1} X \times X$ gives rise to a map $f : X \rightarrow A_1 \times_X B_1$. Let W be the co-kernel of this map. The map f composed with the projection from $A_1 \times_X B_1$ to X is 0, so we obtain an induced map $p : W \rightarrow X$. The diagonal inclusion Δ of X in W together with p give rise to an exact sequence $0 \rightarrow X \xrightarrow{\Delta} W \xrightarrow{p} X \rightarrow 0$, which is the required object.

For any object A of $\mathbb{D}(\mathcal{C})$, we denote by $T(A)$ the object obtained by negating all arrows that appear in A .

1.3. Let $(\mathcal{C}, \otimes, \phi_0, \psi_0)$ be a tensor category. An object X_0 of \mathcal{C} gives rise to a functor from $\mathbb{D}(\mathcal{C})$ to itself, by tensoring the exact sequence pointwise. Since we assumed \otimes to be exact, this functor commutes with Yoneda sums: $(A \star B) \otimes X_0$ is canonically isomorphic with $(A \otimes X_0) \star (B \otimes X_0)$. Also, $T(A) \otimes X_0$ is isomorphic to $T(A \otimes X_0)$.

We endow $\mathbb{D}(\mathcal{C})$ with a tensor structure. The tensor product of the two $\mathbb{D}(\mathcal{C})$ objects $0 \rightarrow A_0 \xrightarrow{i_A} A_1 \xrightarrow{\pi_A} A_0 \rightarrow 0$ and $0 \rightarrow B_0 \xrightarrow{i_B} B_1 \xrightarrow{\pi_B} B_0 \rightarrow 0$ is defined as follows: After tensoring the first with B_0 and the second with A_0 , we obtain two objects over $A_0 \otimes B_0$. We now take their Yoneda sum.

We shall make use of the following exact sequence.

1.4. **Lemma.** for any two objects A and B of $\mathbb{D}(\mathcal{C})$, there is an exact sequence

$$\{\text{eq:tenexact}\} \quad 0 \rightarrow (A \otimes T(B))_1 \xrightarrow{i} A_1 \otimes B_1 \xrightarrow{\pi} (A \otimes B)_1 \rightarrow 0 \quad (1)$$

where π is the quotient of the map obtained from the maps $\pi_A \otimes 1$ and $1 \otimes \pi_B$, and i is the restriction of the map obtained from the maps $i_A \otimes 1$ and $-1 \otimes i_B$.

Proof. Exactness in the middle follows directly from the definitions. We prove that π is surjective, the injectivity of i being similar. We shall use the Mitchell embedding theorem (cf. [Fre64]), which reduces the question to the case of abelian groups.

We in fact prove that already the map

$$A_1 \otimes B_1 \xrightarrow{\pi} A_0 \otimes B_1 \times_{A_0 \otimes B_0} A_1 \otimes B_0 =: U$$

is surjective. Let y be an element of U , and let y_1 and y_2 be its two projections to the components of U . Since the map $A_1 \otimes B_1 \xrightarrow{\pi_A \otimes 1} A_0 \otimes B_1$ is surjective, y_1 can be lifted to an element \tilde{y}_1 of $A_1 \otimes B_1$. We have that

$$(\pi_A \otimes 1)((1 \otimes \pi_B)(\tilde{y}_1)) = (1 \otimes \pi_B)((\pi_A \otimes 1)(\tilde{y}_1)) = (1 \otimes \pi_B)(y_1) = (\pi_A \otimes 1)(y_2)$$

Let $z = (1 \otimes \pi_B)(\tilde{y}_1) - y_2$. Since z is killed by $\pi_A \otimes 1$, it comes from an element, also z , of $A_0 \otimes B_0$. Let \tilde{z} be a lifting of z to $A_0 \otimes B_1$, and denote by \tilde{z} also its image in $A_1 \otimes B_1$ under the inclusion $i_A \otimes 1$. Then $\tilde{y}_1 - \tilde{z}$ is a lifting of y . \square

1.5. Let A, B, C be three objects of $\mathbb{D}(\mathcal{C})$. The associativity constraint ϕ_0 of \mathcal{C} gives rise to an isomorphism of $(A \otimes B) \otimes C$ with the quotient of

$$A_1 \otimes B_0 \otimes C_0 \times_{A_0 \otimes B_0 \otimes C_0} A_0 \otimes B_1 \otimes C_0 \times_{A_0 \otimes B_0 \otimes C_0} A_0 \otimes B_0 \otimes C_1$$

that identifies the three natural inclusions of $A_0 \otimes B_0 \otimes C_0$, and similarly for $A \otimes (B \otimes C)$. We thus get an associativity constraint ϕ on $\mathbb{D}(\mathcal{C})$, over ϕ_0 .

Likewise, the commutativity constraint ψ_0 induces a commutativity constraint ψ on $\mathbb{D}(\mathcal{C})$ over ψ_0 .

1.6. **Proposition.** The data $(\mathbb{D}(\mathcal{C}), \otimes, \phi, \psi)$ as defined above forms a tensor category, and Π_0 is a tensor functor. It is rigid if \mathcal{C} is rigid.

Proof. We define the additional data. Verification of the axioms reduces, as in lemma 1.4, to the case of abelian groups, where it is easy.

Let $u : \mathbf{1}_0 \rightarrow \mathbf{1}_0 \otimes \mathbf{1}_0$ be an identity object of \mathcal{C} . We set $\mathbf{1} = 0 \rightarrow \mathbf{1}_0 \rightarrow \mathbf{1}_0 \oplus \mathbf{1}_0 \rightarrow \mathbf{1}_0 \rightarrow 0$. For any object A of $\mathbb{D}(\mathcal{C})$, $\mathbf{1} \otimes A$ is identified via u with $0 \rightarrow A_0 \rightarrow (A_1 \times_{A_0} (A_0 \oplus A_0))/A_0 \rightarrow A_0 \rightarrow 0$. This is canonically isomorphic (over \mathcal{C}) to A , and so $\mathbf{1}$ acquires a structure of an identity object.

Assume that \mathcal{C} is rigid. For an object A of $\mathbb{D}(\mathcal{C})$, we set \check{A} to be the dual exact sequence $0 \rightarrow \check{A}_0 \xrightarrow{i_{\check{A}}} \check{A}_1 \xrightarrow{\pi_{\check{A}}} \check{A}_0 \rightarrow 0$. We define an evaluation map $A \otimes \check{A} \rightarrow \mathbf{1}$ as follows: We need to define two maps from $A_0 \otimes \check{A}_1 \times A_1 \otimes \check{A}_0$ to $\mathbf{1}_0$, that agree on the two inclusions of $A_0 \otimes \check{A}_0$, and such that the resulting map restricts to the evaluation on $A_0 \otimes \check{A}_0$.

To construct the first map, we consider the exact sequence (1), for $B = \check{A}$. We claim that the evaluation map on $A_1 \otimes \check{A}_1$ restricts to 0 when composed with i . To prove this, it is enough to show that the pair of maps obtained from ev_{A_1} by composition with $i_A \otimes 1$ and $-1 \otimes i_{\check{A}}$ comes from a map $A_0 \otimes \check{A}_0 \rightarrow \mathbf{1}_0$. However, under the adjunction, this pair of maps corresponds to $(i_A, -\pi_A)$, and so comes from the identity map on A_0 . It follows that ev_{A_1} induces a map on $(A \otimes \check{A})_1$,

which is the required map. The second map is obtained by projecting to $A_0 \otimes \check{A}_0$, and using the evaluation map on A_0 . By definition, this second map commutes with the projections to $A_0 \otimes \check{A}_0$ and the second coordinate of $\mathbf{1}$, restricting to the evaluation on A_0 . To prove that the first map restricts to the evaluation as well, we note that there is a commutative diagram

$$\begin{array}{ccc} (A \otimes \check{A})_1 & \xrightarrow{i} & A_1 \otimes \check{A}_1 \\ \downarrow \pi_{A \otimes \check{A}} & & \downarrow \pi \\ A_0 \otimes \check{A}_0 & \xrightarrow{i_{A \otimes \check{A}}} & (A \otimes \check{A})_1 \end{array}$$

where i is the (restriction of the) map obtained from the two maps $i_A \otimes 1$ and $1 \otimes i_{\check{A}}$.

Since $\pi_{A \otimes \check{A}}$ is surjective, it is therefore enough to prove that the maps $ev_{A_1} \circ i$ and $ev_{A_0} \circ \pi_A$ coincide. This is indeed the case, since they both correspond to the inclusion of A_0 in A_1 . □

2. DIFFERENTIAL TENSOR CATEGORIES

2.1. Definition. A *differential structure* on a tensor category \mathcal{C} is a tensor functor from \mathcal{C} to $\mathbb{D}(\mathcal{C})$ which is a section of Π_0 . If D_1 and D_2 are two differential structures on \mathcal{C} , a morphism from D_1 to D_2 is a morphism of tensor functors that induces the identity morphism under Π_0 . A *differential tensor category* is a tensor category together with a differential structure.

Let D be a differential structure on \mathcal{C} . Since D is a section of Π_0 , it is determined by $\partial = \Pi_1 \circ D$. In other words, on the abelian level, it is given by a functor $\partial : \mathcal{C} \rightarrow \mathcal{C}$, together with an exact sequence $0 \rightarrow Id \rightarrow \partial \rightarrow Id \rightarrow 0$. However, this description does not include the tensor structure. We also note that D (and ∂) is necessarily exact.

2.2. Let (\mathcal{C}, D) be a differential tensor category, let $\partial = \Pi_1 \circ D$, and let $A = End(\mathbf{1})$. Recall that for any object X , $End(X)$ is an A -algebra. The functor ∂ defines another ring homomorphism $\partial_1 : A \rightarrow End(\partial(\mathbf{1}))$. Given $a \in A$, the morphism $\partial_1(a) - a$ in $End(\partial(\mathbf{1}))$ restricts to 0 on $\mathbf{1}$, and thus induces a morphism from $\mathbf{1}$ to $\partial(\mathbf{1})$. Similarly, its composition with the projection $\partial(\mathbf{1}) \rightarrow \mathbf{1}$ is 0, so it factors through $\mathbf{1}$. We thus get a new element a' of A .

Claim. *The map $a \mapsto a'$ is a derivation on A .*

Proof. We need to show that given elements $a, b \in A$, the maps $\partial(ab) - ab$ and $(\partial(a) - a)b + a(\partial(b) - b)$ coincide on $\mathbf{1}$. This follows from the formula $\partial(ab) - ab = \partial(a)(\partial(b) - b) + (\partial(a) - a)b$, together with the fact that $\partial(a)(\partial(b) - b)$ induces $a(\partial(b) - b)$ on $\mathbf{1}$. □

2.3. Example. Let \mathcal{C} be the tensor category $\mathcal{V}ec_{\mathbb{k}}$ of finite dimensional vector spaces over a field \mathbb{k} . Given a derivative $'$ on \mathbb{k} , we construct a differential structure on \mathcal{C} as follows: For a vector space X , define $d(X) = \mathfrak{D} \hat{\otimes} X$, where \mathfrak{D} is the vector space with basis $1, \partial$, and $\hat{\otimes}$ is the tensor product with respect to the *right* vector space structure on \mathfrak{D} , given by $1 \cdot a = a \cdot 1$ and $\partial \cdot a = a' \cdot 1 + a \cdot \partial$. The exact sequence $D(X)$ is defined by $x \mapsto 1 \hat{\otimes} x$, $1 \hat{\otimes} x \mapsto 0$ and $\partial \hat{\otimes} x \mapsto x$, for any $x \in X$. If $T : X \rightarrow Y$ is a linear map, $d(T) = 1 \hat{\otimes} T$. We shall write x for $1 \hat{\otimes} x$ and ∂x for

$\partial \hat{\otimes} x$. The structure of a tensor functor is obtained by sending $\partial(x \otimes y) \in d(X \otimes Y)$ to the image of $\partial(x) \otimes y \oplus x \otimes \partial(y)$ in $(D(X) \otimes D(Y))_1$.

2.4. Claim. The constructions in 2.3 and in 2.2 give a bijective correspondence between derivatives on \mathbb{k} and isomorphism classes of differential structures on $\mathcal{V}ec_{\mathbb{k}}$.

Proof. If D_1 and D_2 are two differential structures, then $D_1(\mathbf{1}_0)$ and $D_2(\mathbf{1}_0)$ are both identity objects, and are therefore canonically isomorphic to the same object $\mathbf{1}$. If D_1 and D_2 are isomorphic, then the maps $d_i : \mathit{End}(\mathbf{1}_0) \rightarrow \mathit{End}(\mathbf{1})$ are conjugate, and therefore equal, since $\mathit{End}(\mathbf{1})$ is commutative.

It is clear from the definition that the derivative on \mathbb{k} obtained from the differential structure associated with a derivative is the original one. Conversely, if D_1 and D_2 are two differential structures that give the same derivative on \mathbb{k} , then we may identify $D_1(\mathbf{1}_0)$ and $D_2(\mathbf{1}_0)$. Under this identification, we get that the maps d_i are the same. But the functors D_i are determined by d_i . \square

2.5. We now come to the definition of functors between differential tensor categories. For simplicity, we shall only define (and use) *exact* such functors.

Let $\omega : \mathcal{C} \rightarrow \mathcal{D}$ be an exact functor between abelian categories. There is an induced functor $\mathbb{D}(\omega) : \mathbb{D}(\mathcal{C}) \rightarrow \mathbb{D}(\mathcal{D})$, given by applying ω to each term. If \mathcal{C} and \mathcal{D} are tensor categories, the structure of a tensor functor on ω gives rise to a similar structure on $\mathbb{D}(\omega)$ (again, since ω is exact.) If $t : \omega_1 \rightarrow \omega_2$ is a (tensor) morphism of functors, we likewise get an induced morphism $\mathbb{D}(t) : \mathbb{D}(\omega_1) \rightarrow \mathbb{D}(\omega_2)$.

2.6. Definition. Let (\mathcal{C}_1, D_1) and (\mathcal{C}_2, D_2) be two differential tensor categories. A *differential tensor functor* from \mathcal{C}_1 to \mathcal{C}_2 is an exact tensor functor ω from \mathcal{C}_1 to \mathcal{C}_2 , together with an isomorphism of tensor functors $r : \mathbb{D}(\omega) \circ D_1 \rightarrow D_2 \circ \omega$.

A morphism between two such differential tensor functors (ω_1, r_1) and (ω_2, r_2) is a morphism t between them as tensor functors such that the following diagram (of tensor functors and tensor maps between them) commutes:

$$\{\text{eq:diffmorphism}\} \quad \begin{array}{ccc} \mathbb{D}(\omega_1) \circ D_1 & \xrightarrow{r_1} & D_2 \circ \omega_1 \\ \downarrow \mathbb{D}(t) \circ D_1 & & \downarrow D_2 \circ t \\ \mathbb{D}(\omega_2) \circ D_1 & \xrightarrow{r_2} & D_2 \circ \omega_2 \end{array} \quad (2)$$

where $D_2 \circ t$ is the map from $D_2 \circ \omega_1$ to $D_2 \circ \omega_2$ obtained by applying D_2 to t “pointwise”.

Given a differential tensor functor ω , we denote by $\underline{\mathit{Aut}}^\partial(\omega)$ the group of automorphisms of ω .

If \mathcal{C} is a differential tensor category, and $\mathbb{k} = \mathit{End}(\mathbf{1})$ is a field, a \mathbb{k} -linear differential tensor functor into $\mathcal{V}ec_{\mathbb{k}}$ (with the induced differential structure) is called a (*differential*) *fibre functor*. Given such a functor ω , we denote by G_ω the functor from differential \mathbb{k} -algebras to groups assigning to an algebra A the group $\underline{\mathit{Aut}}^\partial(A \otimes \omega)$.

2.7. Given a \mathbb{k} vector space V , the map $d : V \rightarrow \mathfrak{D} \hat{\otimes} V$ given by $v \mapsto \partial v$ is a derivation, in the sense that $d(av) = a'v + ad(v)$ (where V is identified with its image in $\mathfrak{D} \hat{\otimes} V$.) It is universal for this property: any pair $(i, d) : V \rightarrow W$, where i is linear, and d is a derivation with respect to i factors through it.

Therefore, a fibre functor on (\mathcal{C}, D) is a fibre functor ω in the sense of tensor categories, together with a functorial derivation $d_X : \omega(X) \rightarrow \omega(\partial X)$ (where

$\partial X = D(X)_1$), satisfying the Leibniz rule with respect to the tensor product (and additional conditions.)

Similarly, a differential automorphism of ω is an automorphism t of ω as a tensor functor, with the additional condition that for any object X , the diagram

$$\begin{array}{ccc} \omega(X) & \xrightarrow{d_X} & \omega(\partial X) \\ \downarrow t_X & & \downarrow t_{\partial X} \\ \omega(X) & \xrightarrow{d_X} & \omega(\partial X) \end{array} \quad (3) \quad \{\text{eq: fibauto}\}$$

commutes. Thus the condition (2) really is about preservation of the differentiation.

2.8. Example. Let G be a linear differential algebraic group over a differential field \mathbb{k} . A representation of G is given by a finite dimensional vector space V over \mathbb{k} , together with a morphism $G \rightarrow GL(V)$. A map of representations is a linear transformation that gives a map of group representations for each differential \mathbb{k} -algebra. The category of all such representations is denoted $\mathcal{R}ep_G$.

We endow $\mathcal{R}ep_G$ with a differential structure in the same way as for vector spaces. If V is a representation of G , assigning gv to (g, v) , then the action of G on $\mathcal{D} \hat{\otimes} V$ is given by $(g, x \hat{\otimes} v) \mapsto x \hat{\otimes} gv$. With this differential structure, the forgetful functor ω into $\mathcal{V}ec_{\mathbb{k}}$ has an obvious structure of a differential tensor functor.

A differential automorphism t of ω is given by a collection of vector space automorphisms t_V , for any representation V of G . The commutativity condition (3) above translates to the condition that $t_{\mathcal{D} \hat{\otimes} V} = 1 \hat{\otimes} t_V$.

In particular, given a differential \mathbb{k} -algebra A , and $g \in G(A)$, action by g gives an automorphism of $A \otimes \omega$ as a differential tensor functor, since the action of g on $\mathcal{D} \hat{\otimes} V$ is deduced from its action on V . Thus we get a map $G \rightarrow G_\omega$. We shall prove in 3.10 that the map is an isomorphism.

2.9. Remark. By a linear differential algebraic group, we mean a differential algebraic group which is represented by a differential Hopf algebra. A differential algebraic group which is affine as a differential algebraic variety need not be linear in this sense, since a morphism of affine differential varieties need not correspond to a map of differential algebras. Any linear differential algebraic group has a faithful representation. All these results appear in [Cas72], along with an example of an affine non-linear group. In [Cas75] it is shown that any representation of a *linear* group (and more generally, any morphism of linear groups) does correspond to a map of differential algebras.

2.10. Example. Let G_m be the (differential) multiplicative group, and let \bar{G}_m be the multiplicative group of the constants (thus, as differential varieties, G_m is given by the equation $xy = 1$, and \bar{G}_m is the subvariety given by $x' = 0$.) There is a differential algebraic group homomorphism $dlog$ from G_m to G_a , sending x to x'/x , and $x \mapsto x'$ is a differential algebraic group endomorphism of G_a . Let V be the standard 2-dimensional algebraic representation of G_a (identifying G_a with the maximal unipotent group of GL_2 .) Using $dlog$ and the derivative, we thus get for any $i \geq 0$ a 2-dimensional irreducible representation V_i of G_m , which are all unrelated in terms of the tensor structure (and unrelated with the non-trivial 1-dimensional algebraic representations of G_m .)

However, if X is the G_m representation corresponding to the identity map on G_m , an easy calculation shows that V_0 is isomorphic to $\partial X \otimes \check{X}$. Similarly, V_{i+1} is a quotient of ∂V_i .

The inclusion of \bar{G}_m in G_m gives a functor from $\mathcal{R}ep_{G_m}$ to $\mathcal{R}ep_{\bar{G}_m}$. But in $\mathcal{R}ep_{\bar{G}_m}$, V_0 is isomorphic to $\mathbf{1} \oplus \mathbf{1}$ (and ∂X to $X \oplus X$.)

3. MODEL THEORY OF DIFFERENTIAL FIBRE FUNCTORS

3.1. Throughout this section, we work with a fixed differential rigid tensor category $(\mathcal{C}, \otimes, \phi, \psi, D)$, with $\mathbb{k} = \mathit{End}(\mathbf{1})$ a field. We view \mathbb{k} as a differential field, with the differential structure induced from D , as in 2.2. We set $\partial = \Pi_1 \circ D$, and denote by i and p the maps in the exact sequence $0 \rightarrow \mathit{Id} \xrightarrow{i} \partial \xrightarrow{p} \mathit{Id} \rightarrow 0$.

3.2. **The theory associated with a fibre functor.** We consider the following theory $T_{\mathcal{C}}$:

- (1) For any object X of \mathcal{C} , T has a sort V_X . The sort $V_{\mathbf{1}}$ is denoted by also by L .
- (2) L is a differentially closed field, with constants for the elements of \mathbb{k} (with the prescribed differential field structure.) Every V_X is a vector space over L , of dimension $rk(X)$.
- (3) For every morphism of $f : X \rightarrow Y$ of \mathcal{C} , there is a corresponding function symbol $V_f : V_X \rightarrow V_Y$. The theory says that all these functions are linear, and reflects the abelian category structure.
- (4) For every two object X and Y there is a symbol for a bilinear map $b_{X,Y} : V_X \times V_Y \rightarrow V_{X \otimes Y}$, that induces an isomorphism (in any model) of $V_X \otimes V_Y$ with $V_{X \otimes Y}$. The associativity and commutativity constraints commute with their usual counterpart on sets: $V_{\psi_{X,Y}} \circ b_{X,Y} = b_{Y,X} \circ s$ (where s is the map $(x, y) \mapsto (y, x)$), and similarly for associativity.
- (5) For every object X , there is a function symbol $d_X : V_X \rightarrow V_{\partial(X)}$. This function is a derivation, in the sense that for any $a \in L$ and $v \in V_X$,

$$d_X(av) = a'V_{i_X}(v) + ad_X(v)$$

The theory furthermore says d_X identifies $V_{\partial(X)}$ with $\mathfrak{D} \hat{\otimes} V_X$ (in any model), in the sense of 2.7 (explicitly, it says that $V_{p_X} \circ d_X$ is the identity map.)

- (6) The maps d and b are compatible with the structure of tensor functor of D : given objects X and Y of \mathcal{C} , let $c_{X,Y} : \partial(X \otimes Y) \rightarrow (\partial(X) \otimes \partial(Y))_1$ be the isomorphism supplied with D . Then we require that $V_{c_{X,Y}} \circ d_{X \otimes Y} \circ b_{X,Y}$ coincides with $b_{\partial(X),Y} \circ d_X \times 1 + b_{X,\partial(Y)} \circ 1 \times d_Y$.

3.3. Let ω be a differential fibre functor on \mathcal{C} , and let $K = M_{\mathbf{1}}$ be a differentially closed field containing \mathbb{k} . We expand $M_{\mathbf{1}}$ to a model M of $T_{\mathcal{C}}$ as follows: For any object X , $M_X = V_X(M)$ is $\omega(X) \otimes_{\mathbb{k}} K$. For any morphism f , $f_M = V_{f_M}$ is the map $\omega(f) \otimes 1$ (this satisfies 3.2(3) since tensoring by K is an exact functor.)

The tensor functor structure on ω includes, for any objects X, Y , an isomorphism $c_{X,Y} : \omega(X) \otimes \omega(Y) \rightarrow \omega(X \otimes Y)$. The maps b_{X,Y_M} are obtained by composing $c_{X,Y} \otimes 1$ with the bilinear map from $M_X \times M_Y$ to $M_{X \otimes Y}$. The axioms of a tensor functor ensure that 3.2(4) is satisfied.

Finally, the differential structure of ω gives (as in 2.7) a universal derivation $\omega(X) \rightarrow \omega(\partial(X))$, which extends uniquely to a (universal) derivation $(d_X)_M$ on M_X . This concludes the construction of M .

3.4. The model M_ω just constructed contains, in a natural way, the subset of elements coming from the fibre functor ω . This set, which will also be denoted by ω is definably closed: If τ is any automorphism of K over \mathbb{k} , it extends to an automorphism of M by acting on the K part of each sort. Since \mathbb{k} is definably closed in K , the set of elements fixed by all such automorphisms is precisely ω . In particular, we get the following corollary:

Corollary. *Assume that \mathcal{C} has a differential fibre functor. Then $T_{\mathcal{C}}$ is consistent, and $dcl(0) \cap L = \mathbb{k}$.*

3.5. **Internality.** Each sort of $T_{\mathcal{C}}$, being a finite dimensional vector space over L , is internal to L . Furthermore, if B is a basis for some V_X , then $B \cup d_X(B)$ is a basis for $V_{\partial(X)}$. Therefore, if \mathcal{C} is generated as a differential tensor category by one object (in the sense that the objects $\partial^i X$ generate \mathcal{C} as a tensor category), then all of the sorts are internal using the same definable set: There is a definable set U (namely, the set of bases of V_X , where X is a generator), such that, for any sort V_Y there is a definable family $g : V_Y \times U \rightarrow L^m$ of injective maps from V_Y to some L^m , parametrised by U .

In these circumstances, the basic theory of internality gives rise to a definable automorphism group G of $T_{\mathcal{C}}$ over L , acting freely on U . If A is a definably closed set such that $U(A)$ is not empty, then $G(A)$ is canonically identified (together with the action on U) with the group of automorphisms of A over $L(A)$. If \mathcal{C} is not finitely generated, the automorphism group is the pro-definable group represented by the system of such group, for a presentation of the category as a union.

3.6. Let ω be a fibre functor, and recall (2.6) that G_ω denotes the group functor $A \mapsto \underline{Aut}^\partial(A \otimes \omega)$ on the category of differential \mathbb{k} -algebras. Let G be the (pro-) definable group associated with $T_{\mathcal{C}}$. If A is a differential field extension of \mathbb{k} , $A \otimes \omega$ can be considered as a subset of a model of $T_{\mathcal{C}}$, which is definably closed, by the same argument as in 3.4. Therefore, $G(A \otimes \omega)$ is a well defined group, identified with the automorphisms of $A \otimes \omega$ over A (as $T_{\mathcal{C}}$ structures.) Since any automorphism in this sense is an automorphism of $A \otimes \omega$ as a fibre functor, we get a map $G(- \otimes \omega) \rightarrow G_\omega$ (restricted to fields), which is obviously injective. On the other hand, given an element of $G_\omega(A)$ we may extend it to a model by tensoring with the identity (on a differentially closed field extending A .) Therefore, the map is an isomorphism.

Since all our information is given in terms of definably closed sets, namely differential fields, the following lemma is useful.

3.7. **Lemma.** Let G_1 and G_2 be two definable groups in a stable theory T , and let f be an homomorphism between the group functors they induce on definably closed sets. Then f comes from a definable homomorphism.

We note that in the event that the G_i are differential algebraic groups in DCF , f is automatically an algebraic morphism.

Proof. The graph of f is a sub-sheaf of $G_1 \times G_2$, and is therefore a union of types.¹ Since it is the graph of a function, the (preimage of) the generic types of G_1 must be included in this set of types. But since f is a group homomorphism, f is determined by its values on these generic types. \square

¹TODO: explain this a bit more

3.8. Let H be a differential algebraic group over \mathbb{k} , let $\mathcal{C} = \mathcal{R}ep_H$ (example 2.8), and let ω be the forgetful functor. We consider H as a functor on differential field extensions of \mathbb{k} . The action of H on its representations gives rise to a map $H \rightarrow G_\omega = G(- \otimes \omega)$, as in 2.8. By 3.7, this map is definable. We would like to prove the following theorem:

Theorem. *The natural map from H to G_ω is surjective.*

Let \bar{H} be the image. Galois theory implies, that to prove the theorem, it is enough to prove that every element of $dcl(\omega)$ fixed by \bar{H} is definable over 0. However, here dcl and “definable” include imaginaries. Therefore, to use this method, it is essential to describe the imaginaries in $T_{\mathcal{C}}$. The following is an analogue of [Hru06, Proposition 4.2].

3.9. **Proposition.** Let T be a theory with sorts L and $(V_X)_X$, where the restriction of T to L is DCF , L is stably embedded, each V_X is a finite-dimensional vector space over L , and the V_X are closed under tensor products, duals and derivations, in the sense of 3.2(4), 3.2(5) and 3.2(6). Then T eliminates imaginaries to the level of projective spaces.

Proof. We need to show that any definable set S over parameters can be defined with a canonical parameter. Since, by assumption, no new structure is induced on L , and any set is internal to L , every such set is Kolchin constructible. By Noetherian induction, it is enough to consider S Kolchin closed.

We note that if U is a definable subspace of some V , and V carries a differential data, then the restriction of the differential data to U gives a differential data for U , and similarly for quotients. We denote by $d_U : U \rightarrow \partial(U)$ the universal derivation.

A Kolchin closed set is given by a finite number of differential polynomial equations, i.e., by a finite set of elements of structure algebra of some $U = V_X$. This algebra is the symmetric algebra, $S(U^*)$, on the vector space $U^* = \mathbb{k}[\partial] \hat{\otimes} \check{U}$, where $\mathbb{k}[\partial]$ is the algebra of differential operators. The vector space U^* has a natural filtration, $U^{(i)}$, by the degree of the differential operator. We claim, by induction, that $U^{(i)}$ (with its evaluation map on U) is definable in T . It will follow that the algebra of functions is defined as a differential algebra, since multiplication is given by the $b_{X,Y}$, and the derivation by the d_X .

More precisely, we shall construct by induction $U^{(i)}$, the exact sequence $0 \rightarrow U^{(0)} \rightarrow U^{(i)} \rightarrow U^{(i-1)} \rightarrow 0$, and a map t_i from $U^{(i)}$ to $\partial(U^{(i-1)})$. For $i = 0$, $U^{(0)}$ is the dual \check{U} of U , so setting $U^{(-1)} = 0$ we are done.

Assuming $U^{(i)}$, with the auxiliary data, is definable, we consider two maps from $\partial(U^{(i)})$ to $\partial(U^{(i-1)})$. The first, p , is defined by composing the map p_i from $\partial(U^{(i)})$ to $U^{(i)}$, given by the differential structure, with t_i . The second, q , is obtained by applying the functor ∂ to the exact sequence for $U^{(i)}$ (in other words, the projection from $U^{(i)}$ to $U^{(i-1)}$, composed with the universal derivation on $U^{(i-1)}$, gives a derivation on $U^{(i)}$, and therefore a linear map on $\partial(U^{(i)})$.) We set $U^{(i+1)} = \text{Ker}(p-q)$, t_{i+1} the inclusion map, and the exact sequence is obtain from restriction of the differential structure on $\partial(U^{(i)})$. Also, since both p and q send $U^{(i)}$ to 0, we get an inclusion map $U^{(i)} \rightarrow U^{(i+1)}$. The evaluation map $e_{i+1} : U^{(i+1)} \times U \rightarrow L$ is defined inductively as follows: if $u \in U$, the map $d \mapsto e_i(d, u)'$ is a derivation on $U^{(i)}$, and so defines, together with the inclusion of $U^{(0)}$ in $U^{(i)}$, a linear map from

$\partial(U^{(i)})$ to L . $e_{i+1}(-, u)$ is the restriction of this map to $U^{(i+1)}$. Finally, e_0 is the usual evaluation map.

The rest of the proof is the same as in [Hru06], namely, the Kolchin closed set S is determined by the finite dimensional linear space spanned by the defining equations, and this space is an elements of some Grassmanian, which is, in turn, a closed subset of some projective space. \square

3.10. Proof of 3.8. As explained in 3.8, we should prove that any element of $dcl(\omega)$ fixed by H is 0-definable, where dcl is computed in the theory $T_{\mathcal{C}}$ expanded by sorts for the projective spaces. Thus we need to consider two kinds of elements, vector space elements, and projective space elements.

If v is a vector in ω fixed by H , the map from L sending 1 to v is a map of H representations, hence is 0-definable. Since over ω , everything is in definable bijection with the field, and since G_m has no non-trivial torsors, every ω -definable line has a ω -definable point. Therefore, a line fixed by H comes from a fixed subspace of some representation, hence the inclusion map is also 0-definable. \square

3.11. We note that conversely, given \mathcal{C} and ω , the automorphism group G is a Kolchin closed subgroup of $GL(V_X)$, where X generates \mathcal{C} as a differential tensor category. Furthermore, by [Hru06], ω gives an equivalence of categories between \mathcal{C} and $\mathcal{R}ep_G$.

We can thus summarise the above results as follows.

3.12. Theorem. Let \mathcal{C} be a differential rigid tensor category, ω a fibre functor. Then ω is a differential tensor equivalence between \mathcal{C} and $\mathcal{R}ep_G$, where G is the pro-linear differential algebraic group $\underline{Aut}^{\partial}(\omega)$.

REFERENCES

- [Cas72] Phyllis Joan Cassidy, *Differential algebraic groups*, Amer. J. Math. **94** (1972), 891–954. MR MR0360611 (50 #13058) 2.9
- [Cas75] ———, *The differential rational representation algebra on a linear differential algebraic group*, J. Algebra **37** (1975), no. 2, 223–238. MR MR0409426 (53 #13181) 2.9
- [DMOS82] Pierre Deligne, James S. Milne, Arthur Ogus, and Kuang-yen Shih, *Hodge cycles, motives, and Shimura varieties*, Lecture Notes in Mathematics, vol. 900, Springer-Verlag, Berlin, 1982. MR MR654325 (84m:14046) 1
- [Fre64] Peter Freyd, *Abelian categories. An introduction to the theory of functors*, Harper’s Series in Modern Mathematics, Harper & Row Publishers, New York, 1964. MR MR0166240 (29 #3517) 1.4
- [Hru06] Ehud Hrushovski, *Groupoids, imaginaries and internal covers*, arXiv:math.LO/0603413, Mar 2006. 3.8, 3.9, 3.11
- [Ovc07a] Alexey Ovchinnikov, *Tannakian approach to linear differential algebraic groups*, arxiv:math/0702846v1, Feb 2007. (document)
- [Ovc07b] ———, *Tannakian categories, linear differential algebraic groups, and parameterized linear differential equations*, arxiv:math/0703422v1, Mar 2007. (document)

DEPARTMENT OF MATHS, UNIVERSITY OF EAST-ANGLIA, NORWICH, NR4 7TJ, ENGLAND
E-mail address: <mailto:m.kamensky@uea.ac.uk>
URL: <http://mkamensky.notlong.com>