On pairs and the geometry of thorn rank one structures

Yevgeniy Vasilyev

Sir Wilfred Grenfell College
Memorial University of Newfoundland

June 28, 2008

Elementary pairs

Given an L-theory T, an *elementary pair of models of* T is a structure (M, P) in the language $L_P = L \cup \{P\}$, where P is a new unary relation distinguishing an elementary substructure of M (i.e. $P(M) \preceq M$).
The class of all such pairs is axiomatizable in L_P.

Pairs in the strongly minimal case

Let T be strongly minimal and let T_P be the theory of all infinite-dimensional (meaning $dim(M/P(M))$ is infinite) pairs of models of T. It is known that T_P is complete, and coincides with Poizat's theory of "beautiful pairs".

Theorem (Buechler, 1991):
T_P is ω-stable and has
U-rank 1 iff T is trivial
U-rank 2 iff T is non-trivial and locally modular (linear)
U-rank ω otherwise.

Pairs in the SU-rank 1 case

Let T be supersimple of SU-rank 1.

A pair (M, P) of models of T is *lovely (generic)*
if any nonalgebraic L-type $q(x, A)$ over a small $A \subset M$
has realizations in $P(M)$ and in $M \setminus acl_L(A \cup P(M))$.

It turns out that any pair embeds in a lovely one, all lovely pairs are elementarily equivalent and are exactly the sufficiently saturated models of their (complete) theory T_P.

Theorem (V, 2001):
T_P is supersimple of SU-rank 1, 2 or ω.

Linearity

We have the following characterization of linearity in the SU-rank 1 case (note that in the SU-rank 1 case linearity is strictly weaker than local modularity):

Theorem (V, 2001):
For an SU-rank 1 theory T the following are equivalent:

(a) T is linear (Cb of any plane curve has rank ≤ 1)

(b) T is 1-based (A is independent from B over $acl^{eq}(A) \cap acl^{eq}(B)$)

(c) T_P has SU-rank ≤ 2 (=2 if non-trivial)

(d) $acl_L = acl_{L_P}$ in T_P

(e) for some (any) lovely pair (M, P) the quotient pregeometry $(M, acl(- \cup P(M)))$ is modular

(f) T_P is model complete

Geometry in the linear SU-rank 1 case

What can we say about the geometry $(M/P, cl)$ of $(M, acl(- \cup P(M))$?

The relation "$|cl(a/P, b/P)| \geq 3$ or $a/P = b/P$" turns out to be an equivalence on $(M/P, cl)$ with no interaction between the classes.

Theorem (V, 2001):

Let T be a linear SU-rank 1 theory. Then

• $(M/P, cl)$ is a disjoint union of trivial geometries and/or projective geometries over division rings.

•The original geometry of M is a disjoint union of "subgeometries" of projective geometries over division rings.

• In the ω-categorical case, the division rings are finite fields, and the corresponding vector spaces are definable in $(T_P)^{eq}$.

Alternative approach using canonical bases

De Piro and Kim have shown (working with a non-trivial linear Lascar strong type D of SU-rank 1) that one can extend the geometry of D to a projective geometry over division ring by adding canonical bases of surfaces in D^3. In the ω-categorical case, they deduce definability of vector spaces in T^{eq} (rather than $(T_P)^{eq}$).

Pairs in the thorn rank 1 case

(joint work with Alexander Berenstein)

Thorn rank 1 structures include both SU-rank 1 structures and o-minimal structures, and in the thorn rank 1 case *acl* still induces a pregeometry.

It has been shown by Berenstein that the definition and basic properties of lovely pairs still hold in this wider class of structures, assuming elimination of $\exists^\infty$.

Theorem (Boxall, 2008): If T is of thorn rank 1 and eliminates $\exists^\infty$, then T_P is superrosy of thorn rank $\leq \omega$.

Linearity in the thorn rank 1¡¡¡ case

What about linearity? What is the "right" version of linearity/1-basedness in the absence of (almost) canonical bases?

Theorem: Let T be a thorn rank 1 theory eliminating $\exists^{\infty}$.

Then the following are equivalent:

(a) $acl_L = acl_{L_P}$ in T_P

(b) for some (any) lovely pair (M, P) the quotient pregeometry $(M, acl(- \cup P(M)))$
is modular

(c) T_P has thorn rank ≤ 2 (=2 if non-trivial)

Model completeness

In the presence of almost canonical bases, the conditions (a-c) are equivalent to the usual notion of linearity/ 1-basedness, as well as to model completeness of T_P.

In general,

T_P model complete $\Rightarrow (a - c)$,

but the converse is still open.

Geometry in the "linear" thorn rank 1 case

As in the SU-rank 1 case, we get

Theorem: Let T be a thorn rank 1 theory eliminating $\exists^\infty$, satisfying the equivalent conditions (a-c). Then

• For any lovely pair (M, P) of T, the geometry of $(M, acl(- \cup P(M)))$ is a disjoint union of trivial geometries and/or projective geometries over division rings.

• For any $M \models T$, the geometry of M is a disjoint union of subgeometries of projective geometries over division rings.

• If T is ω-categorical then so is T_P, and if T is non-trivial, the division rings in (a) are finite fields, and the corresponding vector spaces are definable in $(T_P)^{eq}$.

O-minimal case

Note that o-minimal theories expanding DLO eliminate $\exists^\infty$.

How does our version of linearity relate to the Trichotomy for o-minimal structures?

We know that conditions (a-c) imply non-interpretability of an infinite field.

What about the converse?

Theorem: Let T be an o-minimal theory expanding DLO with global addition, which does not interpret an infinite field, and (M, P) a lovely pair of T. Then the geometry of $(M, acl(- \cup P(M)))$ is modular, and moreover, it is a single projective geometry over a division ring.

"Linear" o-minimal theory which is not one-based in the usual sense

Example (Loveys, Peterzil, 1993):

Consider $T = Th(\mathbb{R}, +, 0, 1, f|[-1, 1])$ where $f(x) = \pi x$.

Then T is o-minimal, does not interpret an infinite field, and does not have almost canonical bases (hence is not 1-based in the usual sense).

However, by the above theorem it satisfies the conditions (a-c).

Open question

If T is a thorn rank 1 eliminating $\exists^\infty$, is it the case that T_P can only have thorn-rank 1, 2 or ω?

True in the SU-rank 1 case and in the o-minimal case (with global addition).