Morley sequences in Dependent Theories

Alex Usvyatsov

Universidade de Lisboa - CMAF and UCLA

June 29, 2008

Basic notations

- *We assume that the theory T is dependent and $T = T^{eq}$.*
- We write $a \equiv_A b$ for $\mathrm{tp}(a/A) = \mathrm{tp}(b/A)$.
- We say that a and b are of Lascar distance 1 over a set A if there exists an A-indiscernible sequence containing both. This is not an equivalence relation, but its transitive closure $E^L_A(x, y)$ is. We say that a and b have the same Lascar type if they are E^L_A-equivalent.
- We write $\mathrm{Lstp}(a/A) = \mathrm{Lstp}(b/A)$ or $a \equiv_{\mathrm{Lstp},A} b$.

Basic notations - II

- Let I be an indiscernible sequence over a set A. Then $a \models \mathrm{Av}(I, A \cup I)$ if and only if $I^\frown\{a\}$ is indiscernible over A.
- All indiscernible sequences we mention will be assumed to be *endless*.
  We write $I \equiv_A J$ meaning $\mathrm{EM}(I/A) = \mathrm{EM}(J/A)$. We say that J *continues* I over A if $I^\frown J$ is A-indiscernible.
- For I an indiscernible sequence over A, we often denote $\mathrm{Av}(I, A \cup I)$ by $\mathrm{Av}(I)$.
  So this is just the type of the "next element" of I over A.

Basic definitions - forking

- A formula $\varphi(x, a)$ *divides* over a set A if there exists an A-indiscernible sequence $I = \langle a_i : i < \omega \rangle$ containing a such that the set

$$\{\varphi(x, a_i) : i < \omega\}$$

is inconsistent.

- $\varphi(x, a)$ *forks* over A if it implies a finite disjunction of formulas that divide over A.
- Equivalently, $\varphi(x, a)$ *forks* over A if every global type p which contains $\varphi(x, a)$ divides over A.
- A type p divides/forks over a set A if it contains a dividing/forking formula.

Basic definitions (splitting)

- A type $p \in S(B)$ *does not split* over a set A if whenever $b, c \in B$ have the same type over A, we have $\varphi(x, b) \in p \iff \varphi(x, c) \in p$ for every formula $\varphi(x, y)$.
- A type $p \in S(B)$ *does not split strongly* over a set A if whenever $b, c \in B$ are of Lascar distance 1 over A, we have $\varphi(x, b) \in p \iff \varphi(x, c) \in p$ for every formula $\varphi(x, y)$.
- A type $p \in S(B)$ *does not Lascar-split* over a set A if whenever $b, c \in B$ have the same Lascar type over A, we have $\varphi(x, b) \in p \iff \varphi(x, c) \in p$ for every formula $\varphi(x, y)$.
- Note that a global type doesn't split over a set A if it is invariant under the action of the automorphism group of $\mathfrak{C}$ over A.

Basic definitions (splitting)

- A type $p \in S(B)$ *does not split* over a set A if whenever $b, c \in B$ have the same type over A, we have $\varphi(x, b) \in p \iff \varphi(x, c) \in p$ for every formula $\varphi(x, y)$.
- A type $p \in S(B)$ *does not split strongly* over a set A if whenever $b, c \in B$ are of Lascar distance 1 over A, we have $\varphi(x, b) \in p \iff \varphi(x, c) \in p$ for every formula $\varphi(x, y)$.
- A type $p \in S(B)$ *does not Lascar-split* over a set A if whenever $b, c \in B$ have the same Lascar type over A, we have $\varphi(x, b) \in p \iff \varphi(x, c) \in p$ for every formula $\varphi(x, y)$.
- Note that a global type doesn't split over a set A if it is invariant under the action of the automorphism group of $\mathfrak{C}$ over A.

Basic definitions (splitting)

- A type $p \in S(B)$ *does not split* over a set A if whenever $b, c \in B$ have the same type over A, we have $\varphi(x, b) \in p \iff \varphi(x, c) \in p$ for every formula $\varphi(x, y)$.
- A type $p \in S(B)$ *does not split strongly* over a set A if whenever $b, c \in B$ are of Lascar distance 1 over A, we have $\varphi(x, b) \in p \iff \varphi(x, c) \in p$ for every formula $\varphi(x, y)$.
- A type $p \in S(B)$ *does not Lascar-split* over a set A if whenever $b, c \in B$ have the same Lascar type over A, we have $\varphi(x, b) \in p \iff \varphi(x, c) \in p$ for every formula $\varphi(x, y)$.
- Note that a global type doesn't split over a set A if it is invariant under the action of the automorphism group of $\mathfrak{C}$ over A.

Basic definitions (splitting)

- A type $p \in S(B)$ *does not split* over a set A if whenever $b, c \in B$ have the same type over A, we have $\varphi(x, b) \in p \iff \varphi(x, c) \in p$ for every formula $\varphi(x, y)$.
- A type $p \in S(B)$ *does not split strongly* over a set A if whenever $b, c \in B$ are of Lascar distance 1 over A, we have $\varphi(x, b) \in p \iff \varphi(x, c) \in p$ for every formula $\varphi(x, y)$.
- A type $p \in S(B)$ *does not Lascar-split* over a set A if whenever $b, c \in B$ have the same Lascar type over A, we have $\varphi(x, b) \in p \iff \varphi(x, c) \in p$ for every formula $\varphi(x, y)$.
- Note that a global type doesn't split over a set A if it is invariant under the action of the automorphism group of $\mathfrak{C}$ over A.

Trivialities on splitting

- A type p over B does not split over A if and only if whenever $b, c \in B$ have the same type over A and $a \models p$, we have $ab \equiv_A ac$.
- A type p over B does not Lascar-split over A if and only if whenever $b, c \in B$ have the same *Lascar* type over A and $a \models p$, we have $ab \equiv_A ac$.
- Let M be a $(|A| + \aleph_0)^+$-saturated model containing A, $p \in S(M)$. Then p does not Lascar-split over A if and only if p does not split strongly over A.
- Let A be a set. Then there are at most $2^{2^{|A|+|T|}}$ types over $\mathfrak{C}$ which do not split over A. Same is true for splitting replaced with Lascar splitting or strong splitting.

Trivialities on splitting

- A type p over B does not split over A if and only if whenever $b, c \in B$ have the same type over A and $a \models p$, we have $ab \equiv_A ac$.
- A type p over B does not Lascar-split over A if and only if whenever $b, c \in B$ have the same *Lascar* type over A and $a \models p$, we have $ab \equiv_A ac$.
- Let M be a $(|A| + \aleph_0)^+$-saturated model containing A, $p \in S(M)$. Then p does not Lascar-split over A if and only if p does not split strongly over A.
- Let A be a set. Then there are at most $2^{2^{|A|+|T|}}$ types over $\mathfrak{C}$ which do not split over A. Same is true for splitting replaced with Lascar splitting or strong splitting.

Trivialities on splitting

- A type p over B does not split over A if and only if whenever $b, c \in B$ have the same type over A and $a \models p$, we have $ab \equiv_A ac$.
- A type p over B does not Lascar-split over A if and only if whenever $b, c \in B$ have the same *Lascar* type over A and $a \models p$, we have $ab \equiv_A ac$.
- Let M be a $(|A| + \aleph_0)^+$-saturated model containing A, $p \in S(M)$. Then p does not Lascar-split over A if and only if p does not split strongly over A.
- Let A be a set. Then there are at most $2^{2^{|A|+|T|}}$ types over $\mathfrak{C}$ which do not split over A. Same is true for splitting replaced with Lascar splitting or strong splitting.

Basic definitions (Morley sequences)

- Let O a linear order, A a set. We call a sequence $I = \langle a_i : i \in O \rangle$ a *Morley sequence over A* if it is an indiscernible sequence over A of realizations of p and $\mathrm{tp}(a_i/Aa_{<i})$ does not fork over A for all $i \in O$.
- If a sequence I is indiscernible over B and Morley over $A \subseteq B$, we sometimes say that I is *based on A*.
- Let $p \in S(B)$ be a type. We call a sequence I a *Morley sequence in p* if it is a Morley sequence over B of realizations of p.
- (*Existence of Morley sequences*). Let $a, A \subseteq B$ be such that $\mathrm{tp}(a/B)$ does not fork over A. Then there exists a Morley sequence in $\mathrm{tp}(a/B)$ based on A.

Basic definitions (Morley sequences)

- Let O a linear order, A a set. We call a sequence $I = \langle a_i : i \in O \rangle$ a *Morley sequence over A* if it is an indiscernible sequence over A of realizations of p and $\text{tp}(a_i/Aa_{<i})$ does not fork over A for all $i \in O$.
- If a sequence I is indiscernible over B and Morley over $A \subseteq B$, we sometimes say that I is *based on A*.
- Let $p \in S(B)$ be a type. We call a sequence I a *Morley sequence in p* if it is a Morley sequence over B of realizations of p.
- (*Existence of Morley sequences*). Let $a, A \subseteq B$ be such that $\text{tp}(a/B)$ does not fork over A. Then there exists a Morley sequence in $\text{tp}(a/B)$ based on A.

Basic definitions (Morley sequences)

- Let O a linear order, A a set. We call a sequence $I = \langle a_i : i \in O \rangle$ a *Morley sequence over* A if it is an indiscernible sequence over A of realizations of p and $\text{tp}(a_i/Aa_{<i})$ does not fork over A for all $i \in O$.
- If a sequence I is indiscernible over B and Morley over $A \subseteq B$, we sometimes say that I is *based on* A.
- Let $p \in S(B)$ be a type. We call a sequence I a *Morley sequence in* p if it is a Morley sequence over B of realizations of p.
- (*Existence of Morley sequences*). Let $a, A \subseteq B$ be such that $\text{tp}(a/B)$ does not fork over A. Then there exists a Morley sequence in $\text{tp}(a/B)$ based on A.

Basic definitions (Morley sequences)

- Let O a linear order, A a set. We call a sequence $I = \langle a_i : i \in O \rangle$ a *Morley sequence over A* if it is an indiscernible sequence over A of realizations of p and $\operatorname{tp}(a_i/Aa_{<i})$ does not fork over A for all $i \in O$.
- If a sequence I is indiscernible over B and Morley over $A \subseteq B$, we sometimes say that I is *based on A*.
- Let $p \in S(B)$ be a type. We call a sequence I a *Morley sequence in p* if it is a Morley sequence over B of realizations of p.
- (*Existence of Morley sequences*). Let $a, A \subseteq B$ be such that $\operatorname{tp}(a/B)$ does not fork over A. Then there exists a Morley sequence in $\operatorname{tp}(a/B)$ based on A.

Strong splitting and dividing

- (Shelah) Strong splitting implies dividing, hence forking
- Hence Lascar-splitting implies forking (follows since for global types strong splitting coincides with Lascar-splitting).

Strong splitting and dividing

- (Shelah) Strong splitting implies dividing, hence forking
- Hence Lascar-splitting implies forking (follows since for global types strong splitting coincides with Lascar-splitting).

Important consequences:

- There are boundedly many global types which do not fork over a given set A.
- Let $I = \langle a_i : i < \lambda \rangle$ be such that
  - $\text{tp}(a_i/Aa_{<i})$ does not fork over A
  - $\text{Lstp}(a_i/Aa_{<i}) = \text{Lstp}(a_j/Aa_{<i})$ for every $j \geq i$.

  Then I is a Morley sequence over A (that is, it is indiscernible over A).

Important consequences:

- There are boundedly many global types which do not fork over a given set A.
- Let $I = \langle a_i : i < \lambda \rangle$ be such that
  - $\mathrm{tp}(a_i/Aa_{<i})$ does not fork over A
  - $\mathrm{Lstp}(a_i/Aa_{<i}) = \mathrm{Lstp}(a_j/Aa_{<i})$ for every $j \geq i$.

  Then I is a Morley sequence over A (that is, it is indiscernible over A).

Forking - equivalences

(Shelah, Adler)

The following are equivalent for a global type p and a set A:

- p forks over A
- p divides over A
- p splits strongly over A
- p Lascar splits over A
- p is not Lascar-invariant over A

Morley sequence and average type

- Let $I = \langle b_i : i < \omega \rangle$ be an indiscernible sequence in $p \in S(A)$. The following are equivalent:
  - I is a Morley sequence in p.
  - $\mathrm{Av}(I)$ is a nonforking extension of p.
  - There exists a global extension of $\mathrm{Av}(I)$ which does not fork over A.
- A natural question is: what can be said about global extensions of $\mathrm{Av}(I) = \mathrm{Av}(I, A \cup I)$ as above? How many such extensions are there? Can we describe them?
- The "obvious" candidate $\mathrm{Av}(I, \mathfrak{C})$ does not always work.

Morley sequence and average type

- Let $I = \langle b_i : i < \omega \rangle$ be an indiscernible sequence in $p \in S(A)$. The following are equivalent:
  - I is a Morley sequence in p.
  - $\mathrm{Av}(I)$ is a nonforking extension of p.
  - There exists a global extension of $\mathrm{Av}(I)$ which does not fork over A.
- A natural question is: what can be said about global extensions of $\mathrm{Av}(I) = \mathrm{Av}(I, A \cup I)$ as above? How many such extensions are there? Can we describe them?
- The "obvious" candidate $\mathrm{Av}(I, \mathfrak{C})$ does not always work.

Morley sequence and average type

- Let $I = \langle b_i : i < \omega \rangle$ be an indiscernible sequence in $p \in S(A)$. The following are equivalent:
  - I is a Morley sequence in p.
  - $\mathrm{Av}(I)$ is a nonforking extension of p.
  - There exists a global extension of $\mathrm{Av}(I)$ which does not fork over A.
- A natural question is: what can be said about global extensions of $\mathrm{Av}(I) = \mathrm{Av}(I, A \cup I)$ as above? How many such extensions are there? Can we describe them?
- The "obvious" candidate $\mathrm{Av}(I, \mathfrak{C})$ does not always work.

Weakly special sequences

- (Poizat) We call an A-indiscernible type sequence I *special* if for every two realizations I_1 and I_2 of $\mathrm{tp}(I/A)$, there exists J such that $I_1 ^\frown J$ and $I_2 ^\frown J$ are A-indiscernible.
- We call an A-indiscernible sequence *weakly special* if two realizations I_1 and I_2 of $\mathrm{Lstp}(I/A)$, there exists J such that $I_1 ^\frown J$ and $I_2 ^\frown J$ are A-indiscernible.
- A Morley sequence over A is weakly special over A.
- Let $\varphi(x, b)$ be a formula. We say that an indiscernible sequence J *eventually determines* $\varphi(x, b)$ if $\lim_{J'} \varphi(x, b)$ is constant for all J' continuing J.

Weakly special sequences

- (Poizat) We call an A-indiscernible type sequence I *special* if for every two realizations I_1 and I_2 of $\mathrm{tp}(I/A)$, there exists J such that $I_1^\frown J$ and $I_2^\frown J$ are A-indiscernible.
- We call an A-indiscernible sequence *weakly special* if two realizations I_1 and I_2 of $\mathrm{Lstp}(I/A)$, there exists J such that $I_1^\frown J$ and $I_2^\frown J$ are A-indiscernible.
- A Morley sequence over A is weakly special over A.
- Let $\varphi(x, b)$ be a formula. We say that an indiscernible sequence J *eventually determines* $\varphi(x, b)$ if $\lim_{J'} \varphi(x, b)$ is constant for all J' continuing J.

Weakly special sequences

- (Poizat) We call an A-indiscernible type sequence I *special* if for every two realizations I_1 and I_2 of $tp(I/A)$, there exists J such that $I_1 ^\frown J$ and $I_2 ^\frown J$ are A-indiscernible.
- We call an A-indiscernible sequence *weakly special* if two realizations I_1 and I_2 of $Lstp(I/A)$, there exists J such that $I_1 ^\frown J$ and $I_2 ^\frown J$ are A-indiscernible.
- A Morley sequence over A is weakly special over A.
- Let $\varphi(x, b)$ be a formula. We say that an indiscernible sequence J *eventually determines* $\varphi(x, b)$ if $\lim_{J'} \varphi(x, b)$ is constant for all J' continuing J.

Weakly special sequences

- (Poizat) We call an A-indiscernible type sequence I *special* if for every two realizations I_1 and I_2 of $\text{tp}(I/A)$, there exists J such that $I_1^\frown J$ and $I_2^\frown J$ are A-indiscernible.
- We call an A-indiscernible sequence *weakly special* if two realizations I_1 and I_2 of $\text{Lstp}(I/A)$, there exists J such that $I_1^\frown J$ and $I_2^\frown J$ are A-indiscernible.
- A Morley sequence over A is weakly special over A.
- Let $\varphi(x, b)$ be a formula. We say that an indiscernible sequence J *eventually determines* $\varphi(x, b)$ if $\lim_{J'} \varphi(x, b)$ is constant for all J' continuing J.

Eventual type - I

Let I be a weakly special sequence over A, $\varphi(x,b)$ a formula. The following is very similar to Poizat's (and Adler's) treatment of special sequences:

- There exists $J \equiv_{\text{Lstp},A} I$ which eventually determines $\varphi(x,b)$. Moreover, every $J_0 \equiv_{\text{Lstp},A} I$ can be extended to J that eventually determines $\varphi(x,b)$.
- For every $J, J' \equiv_{\text{Lstp},A} I$ which eventually determine $\varphi(x,b)$ we have $\lim_J \varphi(x,b) = \lim_{J'} \varphi(x,b)$, that is, the "eventual value" of $\varphi(x,b)$ depends only on Lascar type of J over A, and not on the choice of J.

Eventual type - I

Let I be a weakly special sequence over A, $\varphi(x, b)$ a formula. The following is very similar to Poizat's (and Adler's) treatment of special sequences:

- There exists $J \equiv_{\mathrm{Lstp},A} I$ which eventually determines $\varphi(x, b)$. Moreover, every $J_0 \equiv_{\mathrm{Lstp},A} I$ can be extended to J that eventually determines $\varphi(x, b)$.
- For every $J, J' \equiv_{\mathrm{Lstp},A} I$ which eventually determine $\varphi(x, b)$ we have $\lim_J \varphi(x, b) = \lim_{J'} \varphi(x, b)$, that is, the "eventual value" of $\varphi(x, b)$ depends only on Lascar type of J over A, and not on the choice of J.

Eventual type - I

Let I be a weakly special sequence over A, $\varphi(x,b)$ a formula. The following is very similar to Poizat's (and Adler's) treatment of special sequences:

- There exists $J \equiv_{\mathrm{Lstp},A} I$ which eventually determines $\varphi(x,b)$. Moreover, every $J_0 \equiv_{\mathrm{Lstp},A} I$ can be extended to J that eventually determines $\varphi(x,b)$.
- For every $J, J' \equiv_{\mathrm{Lstp},A} I$ which eventually determine $\varphi(x,b)$ we have $\lim_J \varphi(x,b) = \lim_{J'} \varphi(x,b)$, that is, the "eventual value" of $\varphi(x,b)$ depends only on Lascar type of J over A, and not on the choice of J.

Eventual type - II

- Let I be a weakly special sequence over A. We define the *Eventual type* of I over a set C, $\mathrm{Ev}(I, C)$: the truth value of a formula $\varphi(x, b)$ equals the "eventual value" of $\varphi(x, b)$ as in the previous slide (depends only on $\mathrm{Lstp}(I/A)$). We denote $\mathrm{Ev}(I) = \mathrm{Ev}(I, \mathfrak{C})$.
- If I is a weakly special sequence over A, then $\mathrm{Ev}(I)$ is a global type extending $\mathrm{Av}(I)$.
- If I is a weakly special sequence over A which is also an indiscernible *set* over A, then $\mathrm{Ev}(I) = \mathrm{Av}(I, \mathfrak{C})$.
- Example: an increasing sequence of elements in the structure $(\mathbb{Q}, <)$ is weakly special and $\mathrm{Ev}(I) \neq \mathrm{Av}(I, \mathfrak{C})$.

Eventual type - II

- Let I be a weakly special sequence over A. We define the *Eventual type* of I over a set C, $\mathrm{Ev}(I, C)$: the truth value of a formula $\varphi(x, b)$ equals the "eventual value" of $\varphi(x, b)$ as in the previous slide (depends only on $\mathrm{Lstp}(I/A)$). We denote $\mathrm{Ev}(I) = \mathrm{Ev}(I, \mathfrak{C})$.
- If I is a weakly special sequence over A, then $\mathrm{Ev}(I)$ is a global type extending $\mathrm{Av}(I)$.
- If I is a weakly special sequence over A which is also an indiscernible *set* over A, then $\mathrm{Ev}(I) = \mathrm{Av}(I, \mathfrak{C})$.
- Example: an increasing sequence of elements in the structure $(\mathbb{Q}, <)$ is weakly special and $\mathrm{Ev}(I) \neq \mathrm{Av}(I, \mathfrak{C})$.

Eventual type - II

- Let I be a weakly special sequence over A. We define the *Eventual type* of I over a set C, $\mathrm{Ev}(I, C)$: the truth value of a formula $\varphi(x, b)$ equals the "eventual value" of $\varphi(x, b)$ as in the previous slide (depends only on $\mathrm{Lstp}(I/A)$). We denote $\mathrm{Ev}(I) = \mathrm{Ev}(I, \mathfrak{C})$.
- If I is a weakly special sequence over A, then $\mathrm{Ev}(I)$ is a global type extending $\mathrm{Av}(I)$.
- If I is a weakly special sequence over A which is also an indiscernible *set* over A, then $\mathrm{Ev}(I) = \mathrm{Av}(I, \mathfrak{C})$.
- Example: an increasing sequence of elements in the structure $(\mathbb{Q}, <)$ is weakly special and $\mathrm{Ev}(I) \neq \mathrm{Av}(I, \mathfrak{C})$.

Eventual type - II

- Let I be a weakly special sequence over A. We define the *Eventual type* of I over a set C, $\mathrm{Ev}(I, C)$: the truth value of a formula $\varphi(x, b)$ equals the "eventual value" of $\varphi(x, b)$ as in the previous slide (depends only on $\mathrm{Lstp}(I/A)$). We denote $\mathrm{Ev}(I) = \mathrm{Ev}(I, \mathfrak{C})$.
- If I is a weakly special sequence over A, then $\mathrm{Ev}(I)$ is a global type extending $\mathrm{Av}(I)$.
- If I is a weakly special sequence over A which is also an indiscernible *set* over A, then $\mathrm{Ev}(I) = \mathrm{Av}(I, \mathfrak{C})$.
- Example: an increasing sequence of elements in the structure $(\mathbb{Q}, <)$ is weakly special and $\mathrm{Ev}(I) \neq \mathrm{Av}(I, \mathfrak{C})$.

Eventual types - III

- Let I be a weakly special sequence over A. Then $\text{Ev}(I)$ is a global type which does not Lascar-split over A. Hence it does not *fork* over A.
- Recall that $\text{Ev}(I)$ extends $\text{Av}(I)$. It follows that I is a *Morley sequence* over A.
- We have established: I is a Morley sequence over A if and only if it is weakly special over A.

Eventual types - III

- Let I be a weakly special sequence over A. Then $\mathrm{Ev}(I)$ is a global type which does not Lascar-split over A. Hence it does not *fork* over A.
- Recall that $\mathrm{Ev}(I)$ extends $\mathrm{Av}(I)$. It follows that I is a *Morley sequence* over A.
- We have established: I is a Morley sequence over A if and only if it is weakly special over A.

Eventual types - III

- Let I be a weakly special sequence over A. Then $\mathrm{Ev}(I)$ is a global type which does not Lascar-split over A. Hence it does not *fork* over A.
- Recall that $\mathrm{Ev}(I)$ extends $\mathrm{Av}(I)$. It follows that I is a *Morley sequence* over A.
- We have established: I is a Morley sequence over A if and only if it is weakly special over A.

Stationarity of average types

- So we have a partial answer to our question: given a Morley sequence I, the global type $\mathrm{Ev}(I)$ is an extension of $\mathrm{Av}(I)$ which does not fork over A.
- Let I be a Morley (nonforking) sequence over a set A. Then there exists a unique global types extending $\mathrm{Av}(I)$ which does not fork over A. In other words, $\mathrm{Av}(I)$ is stationary over A.
  (Requires a bit of work)
- So if I is a Morley sequence over A, then $\mathrm{Ev}(I)$ is *the unique global type extending* $\mathrm{Av}(I)$ *which does not fork over* A.

Stationarity of average types

- So we have a partial answer to our question: given a Morley sequence I, the global type $\mathrm{Ev}(I)$ is an extension of $\mathrm{Av}(I)$ which does not fork over A.
- Let I be a Morley (nonforking) sequence over a set A. Then there exists a unique global types extending $\mathrm{Av}(I)$ which does not fork over A. In other words, $\mathrm{Av}(I)$ is stationary over A.
  (Requires a bit of work)
- So if I is a Morley sequence over A, then $\mathrm{Ev}(I)$ is *the unique global type extending* $\mathrm{Av}(I)$ *which does not fork over* A.

Stationarity of average types

- So we have a partial answer to our question: given a Morley sequence I, the global type $\mathrm{Ev}(I)$ is an extension of $\mathrm{Av}(I)$ which does not fork over A.
- Let I be a Morley (nonforking) sequence over a set A. Then there exists a unique global types extending $\mathrm{Av}(I)$ which does not fork over A. In other words, $\mathrm{Av}(I)$ is stationary over A.
  (Requires a bit of work)
- So if I is a Morley sequence over A, then $\mathrm{Ev}(I)$ is *the unique global type extending* $\mathrm{Av}(I)$ *which does not fork over* A.

Generic stability

- Recall: if I is a weakly special indiscernible set over A, then $\mathrm{Ev}(I) = \mathrm{Av}(I, \mathfrak{C})$. Hence $\mathrm{Av}(I, \mathfrak{C})$ *does not fork over* A.
- We call a type $p \in S(A)$ *generically stable* if there exists a Morley sequence $\langle b_i : i < \omega \rangle$ in p (over A) which is an indiscernible set.
- This notion is based on Shelah's "stable" types (Shelah only studied finitely satisfiable types and co-heir sequences). The general notion was studied independently by Hrushovski and Pillay. In fact, I adopted the name they proposed.

Generic stability

Morley sequences in Dependent Theories

Alex Usvyatsov

- Recall: if I is a weakly special indiscernible set over A, then $\mathrm{Ev}(I) = \mathrm{Av}(I, \mathfrak{C})$. Hence $\mathrm{Av}(I, \mathfrak{C})$ *does not fork over* A.
- We call a type $p \in S(A)$ *generically stable* if there exists a Morley sequence $\langle b_i : i < \omega \rangle$ in p (over A) which is an indiscernible set.
- This notion is based on Shelah's "stable" types (Shelah only studied finitely satisfiable types and co-heir sequences). The general notion was studied independently by Hrushovski and Pillay. In fact, I adopted the name they proposed.

Generic stability

- Recall: if I is a weakly special indiscernible set over A, then $\mathrm{Ev}(I) = \mathrm{Av}(I, \mathfrak{C})$. Hence $\mathrm{Av}(I, \mathfrak{C})$ *does not fork over* A.
- We call a type $p \in S(A)$ *generically stable* if there exists a Morley sequence $\langle b_i : i < \omega \rangle$ in p (over A) which is an indiscernible set.
- This notion is based on Shelah's "stable" types (Shelah only studied finitely satisfiable types and co-heir sequences). The general notion was studied independently by Hrushovski and Pillay. In fact, I adopted the name they proposed.

Definability and Stationarity

- Let $p \in S(A)$ be generically stable. Then p is definable almost over A.
- Let $p \in S(A)$ be a generically stable type such that the definition schema d_p as before is *over* A (e.g. $A = \mathrm{acl}(A)$). Then p is stationary.

Definability and Stationarity

- Let $p \in S(A)$ be generically stable. Then p is definable almost over A.
- Let $p \in S(A)$ be a generically stable type such that the definition schema d_p as before is *over* A (e.g. $A = \text{acl}(A)$). Then p is stationary.

Some consequences

- A type p is generically stable if and only if it is extensible (does not fork over its domain) and *every* Morley sequence in it is an indiscernible set.
- Let $p \in S(A)$ be generically stable, $q \in S(B)$ extending p. Then q does not fork over A if and only if it is definable almost over A.
- A type which is parallel to a generically stable type is generically stable.
- A type dominated by generically stable type is generically stable.

Symmetry Lemma

From now on we write $a \perp_A b$ for "tp(a/Ab) does not fork over A" (although in general this relation does not need to be symmetric).

Let $p \in S(A)$ be generically stable, $q \in S(A)$ does not fork over A, $a \models p$, $b \models q$. Then

- $a \perp_A b \Longrightarrow b \perp_A a$. Moreover, if $A = \text{acl}(A)$ and $a \perp_A b$, then there exists a unique nonforking extension of q to $S(Aa)$ which equals tp(b/Aa).
- $b \perp_A a \Longrightarrow a \perp_A b$.

Properties of Stable Independence

Let $p, q \in S(A)$ be generically stable, a, b realize p, q respectively, and let $c, \bar{d}$ be any tuples (maybe infinite). Then:

- *Irreflexivity* $a \perp_A a$ if and only if p is algebraic
- *Monotonicity* If $a \perp_A bc\bar{d}$, then $a \perp_A cb$.
- *Symmetry* $a \perp_A b$ if and only if $b \perp_A a$
- *Transitivity* $a \perp_A c\bar{d}$ if and only if $a \perp_{Ac} \bar{d}$ and $a \perp_A c$
- *Existence* Let $B \supseteq A$, then there exists $a' \equiv_A a$ such that $\mathrm{tp}(a'/B)$ is generically stable and $a' \perp_A B$.
- *Uniqueness* If $a \perp_A c$, $a' \perp_A c$ and $a' \equiv_{\mathrm{acl}(A)} a$, then $a \equiv_{Ac} a'$
- *Local Character* If $a \perp_A c$, then for some subset A_0 of A of cardinality $|T|$, $a \perp_{A_0} c$.

More characterizations

- A type p is generically stable if and only if nonforking is symmetric on the set of its realizations.
- This is essentially true because an indiscernible sequence $I = \langle a_\alpha : \alpha < \lambda \rangle$ which is a forking independent *set*, that is, $a_\alpha \perp a_{\neq \alpha}$ for all α is an indiscernible *set*.
- A type p is generically stable if and only if there is a Morley sequence I in it such that $\mathrm{Av}(I, \mathfrak{C})$ does not fork over the domain of p.
- A type p is generically stable if and only if it has a global nonforking extension which is both definable over and finitely satisfiable in a small model M.
- A type p is generically stable if and only if it has a global nonforking extension which is both definable over and finitely satisfiable in a countable Morley sequence I.

More characterizations

- A type p is generically stable if and only if nonforking is symmetric on the set of its realizations.
- This is essentially true because an indiscernible sequence $I = \langle a_\alpha : \alpha < \lambda \rangle$ which is a forking independent *set*, that is, $a_\alpha \downarrow a_{\neq\alpha}$ for all α is an indiscernible *set*.
- A type p is generically stable if and only if there is a Morley sequence I in it such that $\mathrm{Av}(I, \mathfrak{C})$ does not fork over the domain of p.
- A type p is generically stable if and only if it has a global nonforking extension which is both definable over and finitely satisfiable in a small model M.
- A type p is generically stable if and only if it has a global nonforking extension which is both definable over and finitely satisfiable in a countable Morley sequence I.

More characterizations

- A type p is generically stable if and only if nonforking is symmetric on the set of its realizations.
- This is essentially true because an indiscernible sequence $I = \langle a_\alpha : \alpha < \lambda \rangle$ which is a forking independent *set*, that is, $a_\alpha \perp a_{\neq \alpha}$ for all α is an indiscernible *set*.
- A type p is generically stable if and only if there is a Morley sequence I in it such that $\mathrm{Av}(I, \mathfrak{C})$ does not fork over the domain of p.
- A type p is generically stable if and only if it has a global nonforking extension which is both definable over and finitely satisfiable in a small model M.
- A type p is generically stable if and only if it has a global nonforking extension which is both definable over and finitely satisfiable in a countable Morley sequence I.

More characterizations

- A type p is generically stable if and only if nonforking is symmetric on the set of its realizations.
- This is essentially true because an indiscernible sequence $I = \langle a_\alpha : \alpha < \lambda \rangle$ which is a forking independent *set*, that is, $a_\alpha \downarrow a_{\neq\alpha}$ for all α is an indiscernible *set*.
- A type p is generically stable if and only if there is a Morley sequence I in it such that $\text{Av}(I, \mathfrak{C})$ does not fork over the domain of p.
- A type p is generically stable if and only if it has a global nonforking extension which is both definable over and finitely satisfiable in a small model M.
- A type p is generically stable if and only if it has a global nonforking extension which is both definable over and finitely satisfiable in a countable Morley sequence I.

More characterizations

- A type p is generically stable if and only if nonforking is symmetric on the set of its realizations.
- This is essentially true because an indiscernible sequence $I = \langle a_\alpha : \alpha < \lambda \rangle$ which is a forking independent *set*, that is, $a_\alpha \downarrow a_{\neq\alpha}$ for all α is an indiscernible *set*.
- A type p is generically stable if and only if there is a Morley sequence I in it such that $\mathrm{Av}(I, \mathfrak{C})$ does not fork over the domain of p.
- A type p is generically stable if and only if it has a global nonforking extension which is both definable over and finitely satisfiable in a small model M.
- A type p is generically stable if and only if it has a global nonforking extension which is both definable over and finitely satisfiable in a countable Morley sequence I.

Stable types

- (Lascar and Poizat) Recall that a type p is called *stable* if every extension of it is definable.
- Every stable type is generically stable.
- Let $p \in S(A)$. The Following Are Equivalent:
  - p is stable.
  - Every extension of p is stable.
  - Every extension of p is generically stable.
  - Every indiscernible sequence in p is an indiscernible set.
  - There is no "order property" on p (with or without external parameters)
  - On the set of realizations of p there is no definable (with or without external parameters) partial order with infinite chains.
- Every stably dominated type (as in Hrushovski, Haskell and Macpherson) is generically stable.

Stable types

- (Lascar and Poizat) Recall that a type p is called *stable* if every extension of it is definable.
- Every stable type is generically stable.
- Let $p \in S(A)$. The Following Are Equivalent:
  - p is stable.
  - Every extension of p is stable.
  - Every extension of p is generically stable.
  - Every indiscernible sequence in p is an indiscernible set.
  - There is no "order property" on p (with or without external parameters)
  - On the set of realizations of p there is no definable (with or without external parameters) partial order with infinite chains.
- Every stably dominated type (as in Hrushovski, Haskell and Macpherson) is generically stable.

Stable types

- (Lascar and Poizat) Recall that a type p is called *stable* if every extension of it is definable.
- Every stable type is generically stable.
- Let $p \in S(A)$. The Following Are Equivalent:
  - p is stable.
  - Every extension of p is stable.
  - Every extension of p is generically stable.
  - Every indiscernible sequence in p is an indiscernible set.
  - There is no "order property" on p (with or without external parameters)
  - On the set of realizations of p there is no definable (with or without external parameters) partial order with infinite chains.
- Every stably dominated type (as in Hrushovski, Haskell and Macpherson) is generically stable.

Stable types

- (Lascar and Poizat) Recall that a type p is called *stable* if every extension of it is definable.
- Every stable type is generically stable.
- Let $p \in S(A)$. The Following Are Equivalent:
  - p is stable.
  - Every extension of p is stable.
  - Every extension of p is generically stable.
  - Every indiscernible sequence in p is an indiscernible set.
  - There is no "order property" on p (with or without external parameters)
  - On the set of realizations of p there is no definable (with or without external parameters) partial order with infinite chains.
- Every stably dominated type (as in Hrushovski, Haskell and Macpherson) is generically stable.

Interesting Examples

Generically stable types which are not stable or stably dominated:

- $(Q, P_0, <_0, +)$, p the "infinity" type. Then it is generically stable.
- Let RV be a two-sorted theory of a real closed (ordered) field R and an infinite dimensional vector space V over it. There is a definable partial order on V:

$$v_1 \leq v_2 \iff \exists r \in R, r \geq 1_R \text{ such that } v_2 = r \cdot v_1$$

  Let M be a model and $p \in S(M)$ be the type of a generic vector. Then p is generically stable and every Morley sequence is an indiscernible linearly independent set.

Interesting Examples

Generically stable types which are not stable or stably dominated:

- $(Q, P_0, <_0, +)$, p the "infinity" type. Then it is generically stable.
- Let RV be a two-sorted theory of a real closed (ordered) field R and an infinite dimensional vector space V over it. There is a definable partial order on V:

$$v_1 \leq v_2 \iff \exists r \in R, r \geq 1_R \text{ such that } v_2 = r \cdot v_1$$

Let M be a model and $p \in S(M)$ be the type of a generic vector. Then p is generically stable and every Morley sequence is an indiscernible linearly independent set.

Morley sequences and dividing

(with Alf Onshuus)
Assume A is an extension base.

- Let $\varphi(x, a)$ be a formula which divides over a set A. Then there exists a Morley sequence I in $\text{tp}(a/A)$ witnessing dividing; that is, the set $\varphi(x, I) = \{\varphi(x, a') \colon a' \in I\}$ is inconsistent.
- Moreover, there exists a global type q extending $\text{tp}(a/A)$ which does not fork over A such that any Morley sequence in q over A exemplifies dividing of $\varphi(x, a)$.
- If A is a model, q can be chosen to be finitely satisfiable in A.
- Adler and Pillay noticed that the proof only uses NTP_2 (and not NIP).

Morley sequences and dividing

(with Alf Onshuus)
Assume A is an extension base.

- Let $\varphi(x, a)$ be a formula which divides over a set A. Then there exists a Morley sequence I in $\text{tp}(a/A)$ witnessing dividing; that is, the set $\varphi(x, I) = \{\varphi(x, a') : a' \in I\}$ is inconsistent.
- Moreover, there exists a global type q extending $\text{tp}(a/A)$ which does not fork over A such that any Morley sequence in q over A exemplifies dividing of $\varphi(x, a)$.
- If A is a model, q can be chosen to be finitely satisfiable in A.
- Adler and Pillay noticed that the proof only uses NTP_2 (and not NIP).

Morley sequences and dividing

(with Alf Onshuus)
Assume A is an extension base.

- Let $\varphi(x, a)$ be a formula which divides over a set A. Then there exists a Morley sequence I in $\text{tp}(a/A)$ witnessing dividing; that is, the set $\varphi(x, I) = \{\varphi(x, a') \colon a' \in I\}$ is inconsistent.
- Moreover, there exists a global type q extending $\text{tp}(a/A)$ which does not fork over A such that any Morley sequence in q over A exemplifies dividing of $\varphi(x, a)$.
- If A is a model, q can be chosen to be finitely satisfiable in A.
- Adler and Pillay noticed that the proof only uses NTP_2 (and not NIP).

Morley sequences and dividing

(with Alf Onshuus)

Assume A is an extension base.

- Let $\varphi(x, a)$ be a formula which divides over a set A. Then there exists a Morley sequence I in $\text{tp}(a/A)$ witnessing dividing; that is, the set $\varphi(x, I) = \{\varphi(x, a') : a' \in I\}$ is inconsistent.
- Moreover, there exists a global type q extending $\text{tp}(a/A)$ which does not fork over A such that any Morley sequence in q over A exemplifies dividing of $\varphi(x, a)$.
- If A is a model, q can be chosen to be finitely satisfiable in A.
- Adler and Pillay noticed that the proof only uses NTP_2 (and not NIP).

Morley sequences and dividing

(with Alf Onshuus)
Assume A is an extension base.

- Let $\varphi(x, a)$ be a formula which divides over a set A. Then there exists a Morley sequence I in $\text{tp}(a/A)$ witnessing dividing; that is, the set $\varphi(x, I) = \{\varphi(x, a') \colon a' \in I\}$ is inconsistent.
- Moreover, there exists a global type q extending $\text{tp}(a/A)$ which does not fork over A such that any Morley sequence in q over A exemplifies dividing of $\varphi(x, a)$.
- If A is a model, q can be chosen to be finitely satisfiable in A.
- Adler and Pillay noticed that the proof only uses NTP_2 (and not NIP).

Shelah's strong nonforking

- Let $A \subseteq B$. We say that a type $p = \text{tp}(a/B)$ *weakly divides* over A if $\text{tp}(a/B)$ divides over A or $\text{tp}(B/Aa)$ divides over A.
- Not the same as the original notion of weak dividing! So what...
- We say that $P = \text{tp}(a/B)$ *weakly forks* over A if every extension of it to a global type weakly divides over A.
- We say that p is a strongly nonforking/nondividing extension of $p \restriction A$ if it does not weakly fork/divide over A.

Shelah's strong nonforking

- Let $A \subseteq B$. We say that a type $p = \mathrm{tp}(a/B)$ *weakly divides* over A if $\mathrm{tp}(a/B)$ divides over A or $\mathrm{tp}(B/Aa)$ divides over A.
- Not the same as the original notion of weak dividing! So what...
- We say that $P = \mathrm{tp}(a/B)$ *weakly forks* over A if every extension of it to a global type weakly divides over A.
- We say that p is a strongly nonforking/nondividing extension of $p{\restriction}A$ if it does not weakly fork/divide over A.

Shelah's strong nonforking

- Let $A \subseteq B$. We say that a type $p = \text{tp}(a/B)$ *weakly divides* over A if $\text{tp}(a/B)$ divides over A or $\text{tp}(B/Aa)$ divides over A.
- Not the same as the original notion of weak dividing! So what...
- We say that $P = \text{tp}(a/B)$ *weakly forks* over A if every extension of it to a global type weakly divides over A.
- We say that p is a strongly nonforking/nondividing extension of $p \restriction A$ if it does not weakly fork/divide over A.

Shelah's strong nonforking

- Let $A \subseteq B$. We say that a type $p = \text{tp}(a/B)$ *weakly divides* over A if $\text{tp}(a/B)$ divides over A or $\text{tp}(B/Aa)$ divides over A.
- Not the same as the original notion of weak dividing! So what...
- We say that $P = \text{tp}(a/B)$ *weakly forks* over A if every extension of it to a global type weakly divides over A.
- We say that p is a strongly nonforking/nondividing extension of $p \restriction A$ if it does not weakly fork/divide over A.

Strongly nonforking extensions

Let N be saturated enough over A. Then

- A type $p \in S(N)$ does not weakly fork over A if and only if for every $a \models p$
  - $a \perp_A N$
  - $tp(N/Aa)$ does not divide over A.
- If $p \in S(N)$ is a heir of $p \restriction A$ and does not fork over A, it is strongly nonforking over A. In particular this is the case if p is both a heir and a co-heir of $p \restriction A$.
- Note: a global type which is both definable in p and a co-heir of $p \restriction A$, is generically stable. Not so if p is just a heir and a co-heir.

Strongly nonforking extensions

Let N be saturated enough over A. Then

- A type $p \in S(N)$ does not weakly fork over A if and only if for every $a \models p$
  - $a \mathop{\smile\!\!\!\!\!\!\downarrow}_A N$
  - $\text{tp}(N/Aa)$ does not divide over A.
- If $p \in S(N)$ is a heir of $p \restriction A$ and does not fork over A, it is strongly nonforking over A. In particular this is the case if p is both a heir and a co-heir of $p \restriction A$.
- Note: a global type which is both definable in p and a co-heir of $p \restriction A$, is generically stable. Not so if p is just a heir and a co-heir.

Strongly nonforking extensions

Let N be saturated enough over A. Then

- A type $p \in S(N)$ does not weakly fork over A if and only if for every $a \models p$
  - $a \perp_A N$
  - $\text{tp}(N/Aa)$ does not divide over A.
- If $p \in S(N)$ is a heir of $p \restriction A$ and does not fork over A, it is strongly nonforking over A. In particular this is the case if p is both a heir and a co-heir of $p \restriction A$.
- Note: a global type which is both definable in p and a co-heir of $p \restriction A$, is generically stable. Not so if p is just a heir and a co-heir.

Strong Morley sequences

- Let O a linear order, A a set. We call a sequence $I = \langle a_i : i \in O \rangle$ a *strong Morley sequence over B based on A* if it is an indiscernible sequence over B and $\mathrm{tp}(a_i/Ba_{<i})$ is strongly free over A for all $i \in O$.
- In the previous definition, we omit "based on A" if $A = B$.
- Let $p \in S(B)$ be a type. We call a sequence I a *strong Morley sequence in p* if it is a strongly Morley sequence over B of realizations of p.

Strong Morley sequences and dividing

"Kim's Lemma" for dependent theories.
Assume A is an extension base.

- Let $\varphi(x, a)$ be a formula which divides over a set A. Then every strongly Morley sequence I in $\text{tp}(a/A)$ witnesses dividing; that is, the set $\varphi(x, I) = \{\varphi(x, a') : a' \in I\}$ is inconsistent.
- Observed independently by Itay Kaplan and Artem Chernikov.

Existence?

- That is all very nice, but do strong Morley sequences actually exist?
- Shelah claimed that any co-heir over M does not weakly fork over M. Turns out to be false.
- (Chernikov and Kaplan) Let M be a model, $p \in S(M)$. Assume that

$$p \vdash \bigvee_{i<k} \varphi_i(x, b_i) \vee \bigvee_{j<n} \psi_j(x, c_j)$$

where $\varphi_i(x, y_i), \psi_i(x, z_j)$ are over M, $\varphi_i(x, b_i)$ does not divide over M for all i, and $\psi_j(x, c_j)$ divides over M for all j. Then there are $m < \omega$ and automorphisms $\sigma_0, \ldots, \sigma_{m-1}$ over M such that

$$p \vdash \bigvee_{i<k} \bigvee_{\ell<m} \varphi_i(x, \sigma_\ell(b_i))$$

Existence?

- That is all very nice, but do strong Morley sequences actually exist?
- Shelah claimed that any co-heir over M does not weakly fork over M. Turns out to be false.
- (Chernikov and Kaplan) Let M be a model, $p \in S(M)$. Assume that

$$p \vdash \bigvee_{i<k} \varphi_i(x, b_i) \vee \bigvee_{j<n} \psi_j(x, c_j)$$

where $\varphi_i(x, y_i), \psi_i(x, z_j)$ are over M, $\varphi_i(x, b_i)$ does not divide over M for all i, and $\psi_j(x, c_j)$ divides over M for all j. Then there are $m < \omega$ and automorphisms $\sigma_0, \ldots, \sigma_{m-1}$ over M such that

$$p \vdash \bigvee_{i<k} \bigvee_{\ell<m} \varphi_i(x, \sigma_\ell(b_i))$$

Existence?

- That is all very nice, but do strong Morley sequences actually exist?
- Shelah claimed that any co-heir over M does not weakly fork over M. Turns out to be false.
- (Chernikov and Kaplan) Let M be a model, $p \in S(M)$. Assume that

$$p \vdash \bigvee_{i<k} \varphi_i(x, b_i) \vee \bigvee_{j<n} \psi_j(x, c_j)$$

where $\varphi_i(x, y_i), \psi_i(x, z_j)$ are over M, $\varphi_i(x, b_i)$ does not divide over M for all i, and $\psi_j(x, c_j)$ divides over M for all j. Then there are $m < \omega$ and automorphisms $\sigma_0, \ldots, \sigma_{m-1}$ over M such that

$$p \vdash \bigvee_{i<k} \bigvee_{\ell<m} \varphi_i(x, \sigma_\ell(b_i))$$

Existence!

- Every type over a model has an extension which is a global nonforking heir.
- Moreover, if T is extendible (that is, every type is extendible), then every type has a global extension which does not weakly fork over A.
- So strong Morley sequences exist, at least over models.

Existence!

- Every type over a model has an extension which is a global nonforking heir.
- Moreover, if T is extendible (that is, every type is extendible), then every type has a global extension which does not weakly fork over A.
- So strong Morley sequences exist, at least over models.

Existence!

- Every type over a model has an extension which is a global nonforking heir.
- Moreover, if T is extendible (that is, every type is extendible), then every type has a global extension which does not weakly fork over A.
- So strong Morley sequences exist, at least over models.

Properties of strong nonforking

(with Itay Kaplan)

Let M be a model.

- Strong nonforking over M is symmetric.
- The following are equivalent for $p \in S(M)$
  - Strong nonforking satisfies transitivity on the set of realizations of p.
  - Strong nonforking coincides with nonforking for realizations of p.
  - p is generically stable.

Properties of strong nonforking

(with Itay Kaplan)
Let M be a model.

- Strong nonforking over M is symmetric.
- The following are equivalent for $p \in S(M)$
  - Strong nonforking satisfies transitivity on the set of realizations of p.
  - Strong nonforking coincides with nonforking for realizations of p.
  - p is generically stable.

Properties of strong nonforking

(with Itay Kaplan)

Let M be a model.

- Strong nonforking over M is symmetric.
- The following are equivalent for $p \in S(M)$
  - Strong nonforking satisfies transitivity on the set of realizations of p.
  - Strong nonforking coincides with nonforking for realizations of p.
  - p is generically stable.

"Local character" for dependent theories

Let M be a model (can be replaced with any set in an extendible dependent theory).

- (NTP_2) Let $\langle a_\alpha : \alpha < \lambda \rangle$ be a strongly nonforking sequence (that is, $a_\alpha \downarrow_M a_{<\alpha}$), b an element. Then for almost all (except $|T|$-many) α we have $b \downarrow_M a_\alpha$.
- This is not true when strong nonforking is replaced with nonforking.
- Question: what about assuming that $\{a_\alpha : \alpha < \lambda\}$ is a forking independent *set*, that is, $a_\alpha \downarrow_M a_{\neq\alpha}$ for all α?

"Local character" for dependent theories

Let M be a model (can be replaced with any set in an extendible dependent theory).

- (NTP_2) Let $\langle a_\alpha : \alpha < \lambda \rangle$ be a strongly nonforking sequence (that is, $a_\alpha \downarrow_M a_{<\alpha}$), b an element. Then for almost all (except $|T|$-many) α we have $b \downarrow_M a_\alpha$.
- This is not true when strong nonforking is replaced with nonforking.
- Question: what about assuming that $\{a_\alpha : \alpha < \lambda\}$ is a forking independent *set*, that is, $a_\alpha \downarrow_M a_{\neq\alpha}$ for all α?

"Local character" for dependent theories

Let M be a model (can be replaced with any set in an extendible dependent theory).

- (NTP_2) Let $\langle a_\alpha : \alpha < \lambda \rangle$ be a strongly nonforking sequence (that is, $a_\alpha \mathop{\smile\!\!\!\!\!\!|}_M a_{<\alpha}$), b an element. Then for almost all (except $|T|$-many) α we have $b \mathop{\smile\!\!\!\!\!\!|}_M a_\alpha$.
- This is not true when strong nonforking is replaced with nonforking.
- Question: what about assuming that $\{a_\alpha : \alpha < \lambda\}$ is a forking independent *set*, that is, $a_\alpha \mathop{\smile\!\!\!\!\!\!|}_M a_{\neq\alpha}$ for all α?

Related problems

- Question: let $\{a_\alpha : \alpha < \lambda\}$ be given. What is the weakest condition one needs to impose on this set such that whenever there are indiscernible sequences J_α starting with a_α there are indiscernible sequences J'_α starting with *the same* a_α such that J'_α is indiscernible over $J'_{\neq\alpha}$ and $J'_\alpha \equiv J_\alpha$?
- (Shelah) True if this is a strongly nonforking sequence.
- What if you only demand J'_α indiscernible over $a_{<\alpha}J'_{>\alpha}$? Or just over $a_{\neq\alpha}$?
- Shelah claimed that $\{a_\alpha : \alpha < \lambda\}$ being a nonforking sequence is enough for getting indiscernibility over $a_{<\alpha}J'_{>\alpha}$. Turns out to be false: there is an example which prevents getting indiscernibility even over the first elements.

Related problems

- Question: let $\{a_\alpha : \alpha < \lambda\}$ be given. What is the weakest condition one needs to impose on this set such that whenever there are indiscernible sequences J_α starting with a_α there are indiscernible sequences J'_α starting with *the same* a_α such that J'_α is indiscernible over $J'_{\neq\alpha}$ and $J'_\alpha \equiv J_\alpha$?
- (Shelah) True if this is a strongly nonforking sequence.
- What if you only demand J'_α indiscernible over $a_{<\alpha}J'_{>\alpha}$? Or just over $a_{\neq\alpha}$?
- Shelah claimed that $\{a_\alpha : \alpha < \lambda\}$ being a nonforking sequence is enough for getting indiscernibility over $a_{<\alpha}J'_{>\alpha}$. Turns out to be false: there is an example which prevents getting indiscernibility even over the first elements.

Related problems

- Question: let $\{a_\alpha : \alpha < \lambda\}$ be given. What is the weakest condition one needs to impose on this set such that whenever there are indiscernible sequences J_α starting with a_α there are indiscernible sequences J'_α starting with *the same* a_α such that J'_α is indiscernible over $J'_{\neq\alpha}$ and $J'_\alpha \equiv J_\alpha$?
- (Shelah) True if this is a strongly nonforking sequence.
- What if you only demand J'_α indiscernible over $a_{<\alpha} J'_{>\alpha}$? Or just over $a_{\neq\alpha}$?
- Shelah claimed that $\{a_\alpha : \alpha < \lambda\}$ being a nonforking sequence is enough for getting indiscernibility over $a_{<\alpha} J'_{>\alpha}$. Turns out to be false: there is an example which prevents getting indiscernibility even over the first elements.

Related problems

- Question: let $\{a_\alpha : \alpha < \lambda\}$ be given. What is the weakest condition one needs to impose on this set such that whenever there are indiscernible sequences J_α starting with a_α there are indiscernible sequences J'_α starting with *the same* a_α such that J'_α is indiscernible over $J'_{\neq\alpha}$ and $J'_\alpha \equiv J_\alpha$?
- (Shelah) True if this is a strongly nonforking sequence.
- What if you only demand J'_α indiscernible over $a_{<\alpha}J'_{>\alpha}$? Or just over $a_{\neq\alpha}$?
- Shelah claimed that $\{a_\alpha : \alpha < \lambda\}$ being a nonforking sequence is enough for getting indiscernibility over $a_{<\alpha}J'_{>\alpha}$. Turns out to be false: there is an example which prevents getting indiscernibility even over the first elements.

Some unsatisfactory answers

- If $\{a_\alpha : \alpha < \lambda\}$ is a nonforking sequence, then there are mutually indiscernible J_α starting with a_α (but one can not control their type!).
- If $\{a_\alpha : \alpha < \lambda\}$ is a nonforking set, then whenever there are indiscernible sequences J_α starting with a_α there are indiscernible sequences J'_α starting with *the same* a_α such that J'_α is indiscernible over $a_{\neq\alpha}$ and $J'_\alpha \equiv J_\alpha$.
- Of course, if the type of a_α is generically stable, any assumption suffices for the strongest conclusion.
- All these (and other similar results) use boundedness of nonforking.
- Question: can one get better results?

Some unsatisfactory answers

- If $\{a_\alpha : \alpha < \lambda\}$ is a nonforking sequence, then there are mutually indiscernible J_α starting with a_α (but one can not control their type!).
- If $\{a_\alpha : \alpha < \lambda\}$ is a nonforking set, then whenever there are indiscernible sequences J_α starting with a_α there are indiscernible sequences J'_α starting with *the same* a_α such that J'_α is indiscernible over $a_{\neq\alpha}$ and $J'_\alpha \equiv J_\alpha$.
- Of course, if the type of a_α is generically stable, any assumption suffices for the strongest conclusion.
- All these (and other similar results) use boundedness of nonforking.
- Question: can one get better results?

Some unsatisfactory answers

- If $\{a_\alpha \colon \alpha < \lambda\}$ is a nonforking sequence, then there are mutually indiscernible J_α starting with a_α (but one can not control their type!).
- If $\{a_\alpha \colon \alpha < \lambda\}$ is a nonforking set, then whenever there are indiscernible sequences J_α starting with a_α there are indiscernible sequences J'_α starting with *the same* a_α such that J'_α is indiscernible over $a_{\neq\alpha}$ and $J'_\alpha \equiv J_\alpha$.
- Of course, if the type of a_α is generically stable, any assumption suffices for the strongest conclusion.
- All these (and other similar results) use boundedness of nonforking.
- Question: can one get better results?

Some unsatisfactory answers

- If $\{a_\alpha : \alpha < \lambda\}$ is a nonforking sequence, then there are mutually indiscernible J_α starting with a_α (but one can not control their type!).
- If $\{a_\alpha : \alpha < \lambda\}$ is a nonforking set, then whenever there are indiscernible sequences J_α starting with a_α there are indiscernible sequences J'_α starting with *the same* a_α such that J'_α is indiscernible over $a_{\neq\alpha}$ and $J'_\alpha \equiv J_\alpha$.
- Of course, if the type of a_α is generically stable, any assumption suffices for the strongest conclusion.
- All these (and other similar results) use boundedness of nonforking.
- Question: can one get better results?

Some unsatisfactory answers

- If $\{a_\alpha : \alpha < \lambda\}$ is a nonforking sequence, then there are mutually indiscernible J_α starting with a_α (but one can not control their type!).
- If $\{a_\alpha : \alpha < \lambda\}$ is a nonforking set, then whenever there are indiscernible sequences J_α starting with a_α there are indiscernible sequences J'_α starting with *the same* a_α such that J'_α is indiscernible over $a_{\neq\alpha}$ and $J'_\alpha \equiv J_\alpha$.
- Of course, if the type of a_α is generically stable, any assumption suffices for the strongest conclusion.
- All these (and other similar results) use boundedness of nonforking.
- Question: can one get better results?

Weight

- So the "local character" mentioned above can be stated in a much more familiar way: every type has bounded weight.
- There are slightly different ways of defining weight, some of which might be equivalent (one still has to sort out the relationship between strong nonforking, strong nondividing and nonforking; work in progress). But in any reasonable definition the statement above is true.
- Moreover, T is strongly dependent if and only if every type has almost finite weight.
- Question: does almost finite weight imply finite weight? True in stable, and every simple theories (essentially due to Hyttinnen).
- All this is very much related to Hans Adler's work on "burden"; see his talk.

Weight

- So the "local character" mentioned above can be stated in a much more familiar way: every type has bounded weight.
- There are slightly different ways of defining weight, some of which might be equivalent (one still has to sort out the relationship between strong nonforking, strong nondividing and nonforking; work in progress). But in any reasonable definition the statement above is true.
- Moreover, T is strongly dependent if and only if every type has almost finite weight.
- Question: does almost finite weight imply finite weight? True in stable, and every simple theories (essentially due to Hyttinnen).
- All this is very much related to Hans Adler's work on "burden"; see his talk.

Weight

- So the "local character" mentioned above can be stated in a much more familiar way: every type has bounded weight.
- There are slightly different ways of defining weight, some of which might be equivalent (one still has to sort out the relationship between strong nonforking, strong nondividing and nonforking; work in progress). But in any reasonable definition the statement above is true.
- Moreover, T is strongly dependent if and only if every type has almost finite weight.
- Question: does almost finite weight imply finite weight? True in stable, and every simple theories (essentially due to Hyttinen).
- All this is very much related to Hans Adler's work on "burden"; see his talk.

Weight

- So the "local character" mentioned above can be stated in a much more familiar way: every type has bounded weight.
- There are slightly different ways of defining weight, some of which might be equivalent (one still has to sort out the relationship between strong nonforking, strong nondividing and nonforking; work in progress). But in any reasonable definition the statement above is true.
- Moreover, T is strongly dependent if and only if every type has almost finite weight.
- Question: does almost finite weight imply finite weight? True in stable, and every simple theories (essentially due to Hyttinnen).
- All this is very much related to Hans Adler's work on "burden"; see his talk.

Weight

- So the "local character" mentioned above can be stated in a much more familiar way: every type has bounded weight.
- There are slightly different ways of defining weight, some of which might be equivalent (one still has to sort out the relationship between strong nonforking, strong nondividing and nonforking; work in progress). But in any reasonable definition the statement above is true.
- Moreover, T is strongly dependent if and only if every type has almost finite weight.
- Question: does almost finite weight imply finite weight? True in stable, and every simple theories (essentially due to Hyttinnen).
- All this is very much related to Hans Adler's work on "burden"; see his talk.

Generically stable measures

No relevant data available.