Morley sequences in Dependent Theories

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Basic notations

- **We assume that the theory** $T$ **is dependent and** $T = T^{eq}$. 
- **We write** $a \equiv_A b$ **for** $tp(a/A) = tp(b/A)$. 
- **We say that** $a$ **and** $b$ **are of Lascar distance 1 over a set** $A$ **if there exists an** $A$-indiscernible sequence containing both. This is not an equivalence relation, but its transitive closure $E^L_A(x, y)$ is. **We say that** $a$ **and** $b$ **have the same Lascar type if they are** $E^L_A$-equivalent. 
- **We write** $Lstp(a/A) = Lstp(b/A)$ **or** $a \equiv_{Lstp,A} b$. 

(Here, $tp$ denotes the type of a sequence, $E^L_A$ denotes the Lascar equivalence relation, and $Lstp$ denotes the Lascar type of a sequence.)
Let $I$ be an indiscernible sequence over a set $A$. Then $a \models \text{Av}(I, A \cup I)$ if and only if $I \prec \{a\}$ is indiscernible over $A$.

All indiscernible sequences we mention will be assumed to be *endless*. We write $I \equiv_A J$ meaning $\text{EM}(I/A) = \text{EM}(J/A)$. We say that $J$ continues $I$ over $A$ if $I \prec J$ is $A$-indiscernible.

For $I$ an indiscernible sequence over $A$, we often denote $\text{Av}(I, A \cup I)$ by $\text{Av}(I)$. So this is just the type of the “next element” of $I$ over $A$. 
Basic definitions - forking

- A formula $\varphi(x, a)$ divides over a set $A$ if there exists an $A$-indiscernible sequence $I = \langle a_i : i < \omega \rangle$ containing $a$ such that the set

$$\{\varphi(x, a_i) : i < \omega\}$$

is inconsistent.

- $\varphi(x, a)$ forks over $A$ if it implies a finite disjunction of formulas that divide over $A$.

- Equivalently, $\varphi(x, a)$ forks over $A$ if every global type $p$ which contains $\varphi(x, a)$ divides over $A$.

- A type $p$ divides/forks over a set $A$ if it contains a dividing/forking formula.
Basic definitions (splitting)

- A type $p \in S(B)$ does not split over a set $A$ if whenever $b, c \in B$ have the same type over $A$, we have $\varphi(x, b) \in p \iff \varphi(x, c) \in p$ for every formula $\varphi(x, y)$.

- A type $p \in S(B)$ does not split strongly over a set $A$ if whenever $b, c \in B$ are of Lascar distance 1 over $A$, we have $\varphi(x, b) \in p \iff \varphi(x, c) \in p$ for every formula $\varphi(x, y)$.

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- Note that a global type doesn’t split over a set $A$ if it is invariant under the action of the automorphism group of $\mathcal{C}$ over $A$. 
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Trivialities on splitting

- A type $p$ over $B$ does not split over $A$ if and only if whenever $b, c \in B$ have the same type over $A$ and $a \models p$, we have $ab \equiv_A ac$.

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- Let $M$ be a $(|A| + \aleph_0)^+$-saturated model containing $A$, $p \in S(M)$. Then $p$ does not Lascar-split over $A$ if and only if $p$ does not split strongly over $A$.

- Let $A$ be a set. Then there are at most $2^{2|A|+|T|}$ types over $C$ which do not split over $A$. Same is true for splitting replaced with Lascar splitting or strong splitting.
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Basic definitions (Morley sequences)

- Let $O$ a linear order, $A$ a set. We call a sequence $I = \langle a_i : i \in O \rangle$ a **Morley sequence over $A$** if it is an indiscernible sequence over $A$ of realizations of $p$ and $\text{tp}(a_i/Aa_{<i})$ does not fork over $A$ for all $i \in O$.

- If a sequence $I$ is indiscernible over $B$ and Morley over $A \subseteq B$, we sometimes say that $I$ is **based on $A$**.

- Let $p \in S(B)$ be a type. We call a sequence $I$ a **Morley sequence in $p$** if it is a Morley sequence over $B$ of realizations of $p$.

- (Existence of Morley sequences). Let $a, A \subseteq B$ be such that $\text{tp}(a/B)$ does not fork over $A$. Then there exists a Morley sequence in $\text{tp}(a/B)$ based on $A$. 
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Hence Lascar-splitting implies forking (follows since for global types strong splitting coincides with Lascar-splitting).
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Important consequences:

- There are boundedly many global types which do not fork over a given set \( A \).
- Let \( I = \langle a_i : i < \lambda \rangle \) be such that
  - \( \text{tp}(a_i/Aa_{<i}) \) does not fork over \( A \)
  - \( \text{Lstp}(a_i/Aa_{<i}) = \text{Lstp}(a_j/Aa_{<i}) \) for every \( j \geq i \).

Then \( I \) is a Morley sequence over \( A \) (that is, it is indiscernible over \( A \)).
Important consequences:

- There are boundedly many global types which do not fork over a given set $A$.
- Let $I = \langle a_i : i < \lambda \rangle$ be such that
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Forking - equivalences

(Shelah, Adler)
The following are equivalent for a global type $p$ and a set $A$:

- $p$ forks over $A$
- $p$ divides over $A$
- $p$ splits strongly over $A$
- $p$ Lascar splits over $A$
- $p$ is not Lascar-invariant over $A$
Morley sequence and average type

- Let $I = \langle b_i : i < \omega \rangle$ be an indiscernible sequence in $p \in S(A)$. The following are equivalent:
  - $I$ is a Morley sequence in $p$.
  - $\text{Av}(I)$ is a nonforking extension of $p$.
  - There exists a global extension of $\text{Av}(I)$ which does not fork over $A$.

- A natural question is: what can be said about global extensions of $\text{Av}(I) = \text{Av}(I, A \cup I)$ as above? How many such extensions are there? Can we describe them?
- The “obvious” candidate $\text{Av}(I, \mathcal{C})$ does not always work.
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(Poizat) We call an $A$-indiscernible type sequence *special* if for every two realizations $I_1$ and $I_2$ of $\text{tp}(I/A)$, there exists $J$ such that $I_1 \bar{\sim} J$ and $I_2 \bar{\sim} J$ are $A$-indiscernible.

We call an $A$-indiscernible sequence *weakly special* if two realizations $I_1$ and $I_2$ of $\text{Lstp}(I/A)$, there exists $J$ such that $I_1 \bar{\sim} J$ and $I_2 \bar{\sim} J$ are $A$-indiscernible.

A Morley sequence over $A$ is weakly special over $A$.

Let $\varphi(x, b)$ be a formula. We say that an indiscernible sequence $J$ *eventually determines* $\varphi(x, b)$ if $\lim_{J'} \varphi(x, b)$ is constant for all $J'$ continuing $J$. 


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Let $I$ be a weakly special sequence over $A$, $\varphi(x, b)$ a formula. The following is very similar to Poizat’s (and Adler’s) treatment of special sequences:

- There exists $J \equiv_{\text{Lstp}, A} I$ which eventually determines $\varphi(x, b)$. Moreover, every $J_0 \equiv_{\text{Lstp}, A} I$ can be extended to $J$ that eventually determines $\varphi(x, b)$.

- For every $J, J' \equiv_{\text{Lstp}, A} I$ which eventually determine $\varphi(x, b)$ we have $\lim_J \varphi(x, b) = \lim_{J'} \varphi(x, b)$, that is, the “eventual value” of $\varphi(x, b)$ depends only on Lascar type of $J$ over $A$, and not on the choice of $J$. 
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Let $I$ be a weakly special sequence over $A$. We define the *Eventual type* of $I$ over a set $C$, $\text{Ev}(I, C)$: the truth value of a formula $\varphi(x, b)$ equals the “eventual value” of $\varphi(x, b)$ as in the previous slide (depends only on $\text{Lstp}(I/A)$). We denote $\text{Ev}(I) = \text{Ev}(I, C)$.

- If $I$ is a weakly special sequence over $A$, then $\text{Ev}(I)$ is a global type extending $\text{Av}(I)$.
- If $I$ is a weakly special sequence over $A$ which is also an indiscernible set over $A$, then $\text{Ev}(I) = \text{Av}(I, C)$.
- Example: an increasing sequence of elements in the structure $(\mathbb{Q}, <)$ is weakly special and $\text{Ev}(I) \neq \text{Av}(I, C)$.
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Let $I$ be a weakly special sequence over $A$. Then $\text{Ev}(I)$ is a global type which does not Lascar-split over $A$. Hence it does not fork over $A$.

Recall that $\text{Ev}(I)$ extends $\text{Av}(I)$. It follows that $I$ is a Morley sequence over $A$.

We have established: $I$ is a Morley sequence over $A$ if and only if it is weakly special over $A$. 
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Stationarity of average types

- So we have a partial answer to our question: given a Morley sequence $I$, the global type $Ev(I)$ is an extension of $Av(I)$ which does not fork over $A$.

- Let $I$ be a Morley (nonforking) sequence over a set $A$. Then there exists a unique global types extending $Av(I)$ which does not fork over $A$. In other words, $Av(I)$ is stationary over $A$.
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- So if $I$ is a Morley sequence over $A$, then $Ev(I)$ is the unique global type extending $Av(I)$ which does not fork over $A$. 
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- So if $I$ is a Morley sequence over $A$, then $Ev(I)$ is the unique global type extending $Av(I)$ which does not fork over $A$. 
Generic stability

- Recall: if $I$ is a weakly special indiscernible set over $A$, then $\text{Ev}(I) = \text{Av}(I, C)$. Hence $\text{Av}(I, C)$ does not fork over $A$.

- We call a type $p \in S(A)$ generically stable if there exists a Morley sequence $\langle b_i : i < \omega \rangle$ in $p$ (over $A$) which is an indiscernible set.

- This notion is based on Shelah’s “stable” types (Shelah only studied finitely satisfiable types and co-heir sequences). The general notion was studied independently by Hrushovski and Pillay. In fact, I adopted the name they proposed.
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Definability and Stationarity

- Let $p \in S(A)$ be generically stable. Then $p$ is definable almost over $A$.

- Let $p \in S(A)$ be a generically stable type such that the definition schema $d_p$ as before is over $A$ (e.g. $A = acl(A)$). Then $p$ is stationary.
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Some consequences

- A type $p$ is generically stable if and only if it is extensible (does not fork over its domain) and every Morley sequence in it is an indiscernible set.
- Let $p \in S(A)$ be generically stable, $q \in S(B)$ extending $p$. Then $q$ does not fork over $A$ if and only if it is definable almost over $A$.
- A type which is parallel to a generically stable type is generically stable.
- A type dominated by generically stable type is generically stable.
Symmetry Lemma

From now on we write \( a \downarrow_A b \) for “\( \text{tp}(a/Ab) \) does not fork over \( A \)” (although in general this relation does not need to be symmetric).

Let \( p \in S(A) \) be generically stable, \( q \in S(A) \) does not fork over \( A \), \( a \models p \), \( b \models q \). Then

- \( a \downarrow_A b \iff b \downarrow_A a \). Moreover, if \( A = \text{acl}(A) \) and \( a \downarrow_A b \), then there exists a unique nonforking extension of \( q \) to \( S(Aa) \) which equals \( \text{tp}(b/Aa) \).
- \( b \downarrow_A a \iff a \downarrow_A b \).
Properties of Stable Independence

Let $p, q \in S(A)$ be generically stable, $a, b$ realize $p, q$ respectively, and let $c, \bar{d}$ be any tuples (maybe infinite). Then:

- **Irreflexivity** $a \downarrow_A a$ if and only if $p$ is algebraic.
- **Monotonicity** If $a \downarrow_A bc\bar{d}$, then $a \downarrow_A cb$.
- **Symmetry** $a \downarrow_A b$ if and only if $b \downarrow_A a$.
- **Transitivity** $a \downarrow_A c\bar{d}$ if and only if $a \downarrow_{Ac} \bar{d}$ and $a \downarrow_A c$.
- **Existence** Let $B \supseteq A$, then there exists $a' \equiv_A a$ such that $tp(a'/B)$ is generically stable and $a' \downarrow_A B$.
- **Uniqueness** If $a \downarrow_A c$, $a' \downarrow_A c$ and $a' \equiv_{acl(A)} a$, then $a \equiv_{Ac} a'$.
- **Local Character** If $a \downarrow_A c$, then for some subset $A_0$ of $A$ of cardinality $|T|$, $a \downarrow_{A_0} c$. 

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More characterizations

- A type $p$ is generically stable if and only if nonforking is symmetric on the set of its realizations.

- This is essentially true because an indiscernible sequence $I = \langle a_\alpha : \alpha < \lambda \rangle$ which is a forking independent set, that is, $a_\alpha \perp a_\neq \alpha$ for all $\alpha$ is an indiscernible set.

- A type $p$ is generically stable if and only if there is a Morley sequence $I$ in it such that $\text{Av}(I, \mathcal{C})$ does not fork over the domain of $p$.

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Stable types

- (Lascar and Poizat) Recall that a type $p$ is called *stable* if every extension of it is definable.

- Every stable type is generically stable.

- Let $p \in S(A)$. The Following Are Equivalent:
  - $p$ is stable.
  - Every extension of $p$ is stable.
  - Every extension of $p$ is generically stable.
  - Every indiscernible sequence in $p$ is an indiscernible set.
  - There is no “order property” on $p$ (with or without external parameters)
  - On the set of realizations of $p$ there is no definable (with or without external parameters) partial order with infinite chains.

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Interesting Examples

Generically stable types which are not stable or stably dominated:

- $(Q, P_0, <_0, +)$, $p$ the “infinity” type. Then it is generically stable.

- Let $RV$ be a two-sorted theory of a real closed (ordered) field $R$ and an infinite dimensional vector space $V$ over it. There is a definable partial order on $V$:

$$v_1 \leq v_2 \iff \exists r \in R, r \geq 1_R \text{ such that } v_2 = r \cdot v_1$$

Let $M$ be a model and $p \in S(M)$ be the type of a generic vector. Then $p$ is generically stable and every Morley sequence is an indiscernible linearly independent set.
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(with Alf Onshuus)

Assume $A$ is an extension base.

- Let $\varphi(x, a)$ be a formula which divides over a set $A$. Then there exists a Morley sequence $I$ in $\text{tp}(a/A)$ witnessing dividing; that is, the set $\varphi(x, I) = \{\varphi(x, a') : a' \in I\}$ is inconsistent.

- Moreover, there exists a global type $q$ extending $\text{tp}(a/A)$ which does not fork over $A$ such that any Morley sequence in $q$ over $A$ exemplifies dividing of $\varphi(x, a)$.

- If $A$ is a model, $q$ can be chosen to be finitely satisfiable in $A$.

- Adler and Pillay noticed that the proof only uses $\text{NTP}_2$ (and not $\text{NIP}$).
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Not the same as the original notion of weak dividing! So what...

We say that $P = \text{tp}(a/B)$ weakly forks over $A$ if every extension of it to a global type weakly divides over $A$.

We say that $p$ is a strongly nonforking/nondividing extension of $p|A$ if it does not weakly fork/divide over $A$. 
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Shelah’s strong nonforking

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Morley sequences in Dependent Theories

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Morley sequences in dependent theories
Generic stability
Strong nonforking
Related issues and weight
Generically stable measures

Strongly nonforking extensions

Let $N$ be saturated enough over $A$. Then

- A type $p \in S(N)$ does not weakly fork over $A$ if and only if for every $a \models p$
  - $a \upharpoonright_A N$
  - $tp(N/Aa)$ does not divide over $A$.

- If $p \in S(N)$ is a heir of $p\upharpoonright A$ and does not fork over $A$, it is strongly nonforking over $A$. In particular this is the case if $p$ is both a heir and a co-heir of $p\upharpoonright A$.

- Note: a global type which is both definable in $p$ and a co-heir of $p\upharpoonright A$, is generically stable. Not so if $p$ is just a heir and a co-heir.
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- Note: a global type which is both definable in $p$ and a co-heir of $p|A$, is generically stable. Not so if $p$ is just a heir and a co-heir.
Let $O$ a linear order, $A$ a set. We call a sequence $I = \langle a_i : i \in O \rangle$ a strong Morley sequence over $B$ based on $A$ if it is an indiscernible sequence over $B$ and $\text{tp}(a_i/Ba_{<i})$ is strongly free over $A$ for all $i \in O$.

In the previous definition, we omit “based on $A$” if $A = B$.

Let $p \in S(B)$ be a type. We call a sequence $I$ a strong Morley sequence in $p$ if it is a strongly Morley sequence over $B$ of realizations of $p$. 
“Kim’s Lemma” for dependent theories.
Assume $A$ is an extension base.

- Let $\varphi(x, a)$ be a formula which divides over a set $A$. Then every strongly Morley sequence $I$ in $\text{tp}(a/A)$ witnesses dividing; that is, the set $\varphi(x, I) = \{\varphi(x, a') : a' \in I\}$ is inconsistent.

- Observed independently by Itay Kaplan and Artem Chernikov.
Existence?

- That is all very nice, but do strong Morley sequences actually exist?
- Shelah claimed that any co-heir over $M$ does not weakly fork over $M$. Turns out to be false.
- (Chernikov and Kaplan) Let $M$ be a model, $p \in S(M)$. Assume that

$$p \vdash \bigvee_{i<k} \varphi_i(x, b_i) \lor \bigvee_{j<n} \psi_j(x, c_j)$$

where $\varphi_i(x, y_i), \psi_i(x, z_j)$ are over $M$, $\varphi_i(x, b_i)$ does not divide over $M$ for all $i$, and $\psi_j(x, c_j)$ divides over $M$ for all $j$. Then there are $m < \omega$ and automorphisms $\sigma_0, \ldots, \sigma_{m-1}$ over $M$ such that

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Every type over a model has an extension which is a global nonforking heir.

Moreover, if $T$ is extendible (that is, every type is extendible), then every type has a global extension which does not weakly fork over $A$.

So strong Morley sequences exist, at least over models.
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Properties of strong nonforking

(with Itay Kaplan)

Let $M$ be a model.

- Strong nonforking over $M$ is symmetric.
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“Local character” for dependent theories

Let $M$ be a model (can be replaced with any set in an extendible dependent theory).

- **($NTP_2$)** Let $\langle a_\alpha : \alpha < \lambda \rangle$ be a strongly nonforking sequence (that is, $a_\alpha \downarrow_M a_{<\alpha}$), $b$ an element. Then for almost all (except $|T|$-many) $\alpha$ we have $b \downarrow_M a_\alpha$.

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- Question: what about assuming that $\{a_\alpha : \alpha < \lambda \}$ is a forking independent set, that is, $a_\alpha \downarrow_M a_{\neq \alpha}$ for all $\alpha$?
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“Local character” for dependent theories

Let $M$ be a model (can be replaced with any set in an extendible dependent theory).

- $(NTP_2)$ Let $\langle a_\alpha : \alpha < \lambda \rangle$ be a strongly nonforking sequence (that is, $a_\alpha \downarrow_M a_{<\alpha}$), $b$ an element. Then for almost all (except $|T|$-many) $\alpha$ we have $b \downarrow_M a_\alpha$.

- This is not true when strong nonforking is replaced with nonforking.

- Question: what about assuming that $\{a_\alpha : \alpha < \lambda\}$ is a forking independent set, that is, $a_\alpha \downarrow_M a_{\neq \alpha}$ for all $\alpha$?
Question: let \( \{a_\alpha : \alpha < \lambda\} \) be given. What is the weakest condition one needs to impose on this set such that whenever there are indiscernible sequences \( J_\alpha \) starting with \( a_\alpha \) there are indiscernible sequences \( J'_\alpha \) starting with the same \( a_\alpha \) such that \( J'_\alpha \) is indiscernible over \( J'_{\neq \alpha} \) and \( J'_\alpha \equiv J_\alpha \)?

(Shelah) True if this is a strongly nonforking sequence.

What if you only demand \( J'_\alpha \) indiscernible over \( a_{<\alpha} \)\( J'_{\alpha} \)? Or just over \( a_{\neq \alpha} \)?

Shelah claimed that \( \{a_\alpha : \alpha < \lambda\} \) being a nonforking sequence is enough for getting indiscernibility over \( a_{<\alpha} \)\( J'_{\alpha} \). Turns out to be false: there is an example which prevents getting indiscernibility even over the first elements.
Related problems

- Question: let \( \{ a_\alpha : \alpha < \lambda \} \) be given. What is the weakest condition one needs to impose on this set such that whenever there are indiscernible sequences \( J_\alpha \) starting with \( a_\alpha \) there are indiscernible sequences \( J'_\alpha \) starting with *the same* \( a_\alpha \) such that \( J'_\alpha \) is indiscernible over \( J' \neq \alpha \) and \( J'_\alpha \equiv J_\alpha \)?
- (Shelah) True if this is a strongly nonforking sequence.
- What if you only demand \( J'_\alpha \) indiscernible over \( a_{< \alpha} J'_{\alpha} \)? Or just over \( a_{\neq \alpha} \)?
- Shelah claimed that \( \{ a_\alpha : \alpha < \lambda \} \) being a nonforking sequence is enough for getting indiscernibility over \( a_{< \alpha} J'_{\alpha} \). Turns out to be false: there is an example which prevents getting indiscernibility even over the first elements.
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What if you only demand \( J'_\alpha \) indiscernible over \( a_{<\alpha} J'_{>\alpha} \)? Or just over \( a_{\neq \alpha} \)?

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Some unsatisfactory answers

- If $\{a_\alpha : \alpha < \lambda\}$ is a nonforking sequence, then there are mutually indiscernible $J_\alpha$ starting with $a_\alpha$ (but one cannot control their type!).

- If $\{a_\alpha : \alpha < \lambda\}$ is a nonforking set, then whenever there are indiscernible sequences $J_\alpha$ starting with $a_\alpha$ there are indiscernible sequences $J'_\alpha$ starting with the same $a_\alpha$ such that $J'_\alpha$ is indiscernible over $a_\neq \alpha$ and $J'_\alpha \equiv J_\alpha$.

- Of course, if the type of $a_\alpha$ is generically stable, any assumption suffices for the strongest conclusion.

- All these (and other similar results) use boundedness of nonforking.

- Question: can one get better results?
Some unsatisfactory answers

- If \( \{ a_\alpha : \alpha < \lambda \} \) is a nonforking sequence, then there are mutually indiscernible \( J_\alpha \) starting with \( a_\alpha \) (but one can not control their type!).

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So the “local character” mentioned above can be stated in a much more familiar way: every type has bounded weight.

There are slightly different ways of defining weight, some of which might be equivalent (one still has to sort out the relationship between strong nonforking, strong nondividing and nonforking; work in progress). But in any reasonable definition the statement above is true.

Moreover, $T$ is strongly dependent if and only if every type has almost finite weight.

Question: does almost finite weight imply finite weight? True in stable, and every simple theories (essentially due to Hyttinen).

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No relevant data available.