

Minimal first order structures

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Around Classification Theory, Leeds 2008

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Let $M \preceq M_1, M_2$ be first-order structures. M_1 and M_2 are isomorphic over M if there is an isomorphism fixing M pointwise.

- **Conjecture (Pillay)**

Every infinite, countable first-order structure has infinitely many pairwise non-isomorphic (over M) countable elementary extensions.

- **Equivalent formulation**

If M is infinite, countable then $I(\aleph_0, \text{Th}_M(M)) \geq \aleph_0$.

- $\text{Th}_M(M)$ is not \aleph_0 -categorical so $I(\aleph_0, \text{Th}_M(M)) \geq 2$.
- By Vaught's theorem $I(\aleph_0, \text{Th}_M(M)) \geq 3$.

Theorem (Pillay)

$$I(\aleph_0, \text{Th}_M(M)) \geq 4 .$$

Theorem (Pillay)

If M has no order then $I(\aleph_0, \text{Th}_M(M)) \geq \aleph_0$.

M **has an order** if for some $\phi(\bar{x}, \bar{y})$ and $\{\bar{a}_i \mid i \in \omega\} \subset M^{<\omega}$

$$\text{Th}_M(M) \models \phi(\bar{a}_i, \bar{a}_j) \quad \text{iff} \quad i \leq j .$$

Remark

It suffices to prove the conjecture for minimal structures M .

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Definition

A first order structure is **minimal** if its each definable (with parameters) subset is either finite or co-finite.

Examples of minimal structures are:

- Strongly minimal structures ;
- Ordered structures: $(\omega, <)$ and $(\omega + \omega^*, <)$.

Theorem (Baldwin, Lachlan)

For M strongly minimal $I(\aleph_0, \text{Th}_M(M)) = \aleph_0$.

Theorem (Shelah)

For $M = (\omega, <, \dots)$ or $M = (\omega + \omega^, <, \dots)$:*

$$I(\aleph_0, \text{Th}_M(M)) = 2^{\aleph_0} .$$

Assumptions

Fix a minimal structure M , absorb its elements into the language.
Let $\mathcal{M} \models \text{Th}_M(M)$ be \aleph_1 -saturated.

- There is a unique non-algebraic 1-type in $S_1(\emptyset)$:

$$p(x) = \{\phi(x) \mid \phi(x) \text{ is over } M \text{ and } \phi(M) \text{ is co-finite in } M\}$$

$$(\phi(A) = \{a \in A \mid \models \phi(a)\} \quad ; \quad p(A) = \{a \in A \mid a \models p\})$$
- $\mathcal{M} = M \cup p(\mathcal{M})$.

Baldwin, Lachlan

- $(\mathcal{M}, \text{acl})$ is a pregeometry:
 - $X \subseteq Y$ implies $\text{acl}(X) \subseteq \text{acl}(Y)$
 - $X \subseteq \text{acl}(X) = \text{acl}(\text{acl}(X))$,
 - $\text{acl}(X) = \bigcup \{ \text{acl}(X_0) \mid X_0 \subseteq X \text{ finite} \}$
 - $a \in \text{acl}(Xb) \setminus \text{acl}(X)$ implies $b \in \text{acl}(Xa)$
- For M strongly minimal: acl -closed subsets are $\preceq \mathcal{M}$.

Definition

$a \in p(\mathcal{M})$ is semi-isolated by $A \subset p(\mathcal{M})$ iff:

$\phi(x) \vdash p(x)$ for some $\phi(x) \in \text{tp}(a/A)$.

$\text{Sem}(A) = \{a \in p(\mathcal{M}) \mid a \text{ is semi-isolated by } A\}$.

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- $\text{Sem} = \text{cl}_q$ on the locus of a strongly regular type q in a stable theory; if M is strongly minimal then $\text{Sem} = \text{acl}$ on $p(\mathcal{M})$.
- In general $\text{Sem} \neq \text{acl}$ on $p(\mathcal{M})$.
- $(p(\mathcal{M}), \text{Sem})$ is not necessarily a pregeometry.
- Sem-closed subsets are $\preceq \mathcal{M}$.

Theorem

Let M be a minimal first-order structure, \mathcal{M} its saturated elementary extension and $p \in S_1(M)$ the unique non-algebraic 1-type. Then exactly one of the following two options holds:

- (I)** *$(p(\mathcal{M}), \text{Sem})$ is a pregeometry.*
- (II)** *There exists a type directed by constants from M .*

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- (II)** *There exists a type directed by constants from M .*

Pillay's conjecture follows from the following two:

Fact

In case (I) $I(\aleph_0, \text{Th}_M(M)) \geq \aleph_0$.

Theorem

In case (II) $I(\aleph_0, \text{Th}_M(M)) = 2^{\aleph_0}$.

Assumptions

- M is a minimal structure; $\mathcal{M} \models \text{Th}_M(M)$ is saturated;
- $p \in S_1(\emptyset)$ is the unique non-algebraic type;
- $\mathcal{C} = \text{dcl}(\emptyset) = M$.

Notation

Let $\mathcal{C} \subset \text{dcl}(E)$ be infinite.

- A **\mathcal{C} -type** (**\mathcal{C} -coheir**) is a non-algebraic (possibly incomplete) type finitely satisfiable in \mathcal{C} .
- $A = (a_1, a_2, \dots)$ is a **\mathcal{C} -sequence over B** if $\text{tp}(a_{k+1}/a_1 a_2 \dots a_k B)$ is a \mathcal{C} -type for all relevant k .
- $(a_1, a_2, \dots, a_k, \dots)$ is **symmetric** if for all permutations $\text{tp}(a_1 a_2 \dots) = \text{tp}(a_{\pi(1)} a_{\pi(2)} \dots)$.

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Proposition

The following two are equivalent:

- $(p(\mathcal{M}), \text{Sem})$ is a pregeometry;
- Every \mathcal{C} -sequence of elements of $p(\mathcal{M})$ is symmetric.

$(S)_n$ Every \mathcal{C} -sequence of length n is symmetric.

$(U)_n$ \mathcal{C} -sequences of length n have a unique type:
if (a_1, a_2, \dots, a_n) and (b_1, b_2, \dots, b_n) are \mathcal{C} -sequences then

$$\text{tp}(a_1 a_2 \dots a_n) = \text{tp}(b_1 b_2 \dots b_n).$$

Remark $(S)_{n+1} \Rightarrow (U)_n$.

Proposition

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Proof of the dichotomy theorem: Assuming that there are no types directed by constants around, Propositions 2 and 3 imply:

$$(S)_n \wedge (U)_n \Rightarrow (S)_{n+1} \wedge (U)_{n+1} ;$$

since $(S)_1$ and $(U)_1$ are trivially satisfied, every \mathcal{C} -sequence must be symmetric and, by Proposition 1, option (I) holds. \square

Dichotomy Theorem

There are two types of minimal structures:

Symmetric type *Every \mathcal{C} -sequence of elements of \mathcal{M} is symmetric.*

- $(p(\mathcal{M}), \text{Sem})$ is a pregeometry;
- There is a unique \mathcal{C} -type in $S_1(A)$ for all $A \subset p(\mathcal{M})$.

Asymmetric type *There is an asymmetric \mathcal{C} -sequence of elements of \mathcal{M} .*

- There are E and $\mathcal{C}_0 \subset \mathcal{C}$ directing the type over E .
- $E = (e_1, e_2, \dots, e_n)$ where n is maximal for which both $(S)_n$ and $(U)_n$ are satisfied.

Definition and examples

Fix \mathcal{M} which is \aleph_1 -saturated in a countable language. A, B, E, \dots denote countable subsets of \mathcal{M} .

Definition

An infinite set \mathcal{C} **directs a type over** $A \subset \mathcal{M}$ if $\mathcal{C} \subseteq \text{dcl}(A)$ and there is an A -definable irreflexive and transitive $<$ on \mathcal{M} such that $(\mathcal{C}, <)$ is downwards directed and satisfies:

- (D1) If (a, b) is a \mathcal{C} -sequence over A then $\models a < b$; and
- (D2) For all $c \in \mathcal{C}$: if $\models c < d$ then $d \in \mathcal{C}$.

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- (D2) For all $c \in \mathcal{C}$: if $\models c < d$ then $d \in \mathcal{C}$.

In this case we also say that the set of all accumulation points of \mathcal{C}

$$p(x) = \{\phi(x) \mid \phi \text{ is over } A \text{ and } \phi(\mathcal{C}) \text{ is co-finite in } \mathcal{C}\}$$

is **$(\mathcal{C}, <)$ -directed over A** or simply **\mathcal{C} -directed over A** . A type is **directed by constants** if it is \mathcal{C} -directed over A for some A and $\mathcal{C} \subseteq \text{dcl}(A)$.

Example

- $(\omega^*, <)$

Here $\omega^* \subseteq dcl(\emptyset)$ and there is a unique non-algebraic 1-type $p \in S_1(\emptyset)$. p is $(\omega^*, <)$ -directed.

- $(\omega + \omega^*, <)$

There is a unique non-algebraic 1-type $p \in S_1(\emptyset)$.

p is both $(\omega, >)$ -directed and $(\omega^*, <)$ -directed but there are no $\omega + \omega^*$ -directed types.

- $(2^{<\omega}, \subset, c_n)_{n \in \omega}$ where $c_n : n \rightarrow \{0\}$.

Here $c_0 \subset c_1 \subset \dots$ and our structure is a binary tree of height ω with named all the elements from the branch

$$\mathcal{C} = \{c_n \mid n \in \omega\}$$

$\{c_n < x \mid n \in \omega\}$ determines the unique \mathcal{C} -type $p \in S_1(\emptyset)$.
 p is $(\mathcal{C}, >)$ -directed.

Assumptions

- \mathcal{M} is \aleph_1 -saturated in a countable language.
- $T = \text{Th}(\mathcal{M})$ is small ($|S(\emptyset)| = \aleph_0$).
- A type directed by constants exists.

After absorbing a few parameters into the language we may find $(\mathcal{C}, <)$ -directed type $p \in S_1(\emptyset)$ satisfying $(U)_n$ for all n :

- There is a unique type of a \mathcal{C} -sequence of a fixed length.

$A \subset p(\mathcal{M})$ is called **\mathcal{C} -independent** if:
every finite subset of A can be arranged into a \mathcal{C} -sequence.

$(\mathcal{C}, <)$ can be chosen satisfying in addition:

- For all $N \prec \mathcal{M}$ any two maximal \mathcal{C} -independent subsets of $p(N)$ are $<$ -isomorphic.
- For any linear order I there is a model N_I such that any maximal \mathcal{C} -independent subset of $p(N_I)$ has $<$ -order type isomorphic to I .

Theorem

$$I(\aleph_0, T) = 2^{\aleph_0} .$$

Minimal fields

Podewski's Conjecture A minimal field is algebraically closed.

Theorem (Wagner)

Minimal fields of characteristic p are algebraically closed.

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Is there a minimal field of asymmetric type?

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Question

Is there a minimal structure $(F, +, \cdot, <)$ where $(F, +, \cdot)$ is a field and $<$ has arbitrarily large finite chains?

Kueker's conjecture

Assumptions

- T is a countable, complete theory with infinite models;
- All uncountable models of T are \aleph_0 -saturated;
- T is not \aleph_0 -categorical.

Kueker's conjecture

T is \aleph_1 -categorical.

Kueker's Conjecture

Kueker's Conjecture has been verified in the following cases:

- (Buechler) T superstable;
- (Hrushovski) T stable;
- (Hrushovski) Linear orders;
- (Hrushovski) T has Skolem functions.
- Ziv Shami reduced the case T - simple to T - weakly minimal.

Kueker's conjecture

The conjecture can be reduced to the case of small profinite structures:

Theorem

If $|\text{dcl}(\emptyset)| = \aleph_0$ then T is \aleph_1 -categorical.

Question

Is there a dichotomy for small profinite structures?

Strongly regular types

$p, q \in S(A)$ are **weakly orthogonal** ($p^w \perp q$) iff

$p(\bar{x}) \cup q(\bar{y})$ determines a complete type.

Definition

$p \in S(A)$ is **strongly regular via** $\phi(x) \in p$ if it is non-isolated and

$p^w \perp \text{tp}(\bar{b}/A)$ for all $\bar{b} \subset \phi(\mathcal{M}) \setminus p(\mathcal{M})$.

p is **strongly regular** if it is so via some $\phi(x) \in p$.

Theorem

If p is strongly regular type then either $(p(\mathcal{M}), \text{Sem})$ is a pregeometry or there is a **directed type**.

Theories with Skolem functions

T is countable, complete, with infinite models and with built-in Skolem functions.

Theorem (Lascar)

If T is stable but not ω -stable then

$$I(\aleph_0, T) = 2^{\aleph_0}.$$

Theorem

If T is unstable NIP or NSOP then

$$I(\aleph_0, T) = 2^{\aleph_0}.$$