# Compact and generic compact domination

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# Definitions I

- Let $M$ be a saturated model, $A \subseteq M$ small, $X$ an $A$-definable set, and $\pi : X \to C$ a surjective map from $X$ to a compact space $C$. Call $f$ $A$-definable if for any closed $D \subseteq C$, $\pi^{-1}(D)$ is type-definable over $A$.
- Special case: $X = G$ is an $A$-definable group, $C = H$ is a compact group, and $\pi$ a (surjective) homomorphism $G \to H$.
- A definable group $G$ is said to be compactly dominated if there is a definable surjective homomorphism $\pi$ from $G$ to a compact group $H$ such that for any definable subset $Y$ of $G$, the set of $h \in H$ such that $\pi^{-1}(h)$ meets both $Y$ and its complement, has Haar measure $0$.

# Definitions II

- Here is a basic example, which needs a little work to prove.
- Let $H$ be a compact Lie group definable in the real field $\mathbb{R}$, let $M$ be a saturated elementary extension of $\mathbb{R}$ and $G = H(M)$. Then the standard part map $G \to H$ is $\mathbb{R}$-definable in $M$ and moreover witnesses compact domination of $G$ (by $H$).

# The theorem I

**Theorem 1.1**

*Let $G$ be a "definably compact" group definable in a saturated $o$-minimal expansion of a real closed field. Then $G$ is compactly dominated (by a compact Lie group $H$ of the same dimension as the $o$-minimal dimension of $G$).*

- We concentrate on the case where $G$ is definably connected and commutative (which suffices).
- We already know that $G^{00}$ exists and $G/G^{00}$ is a compact Lie group of the right dimension (and of course $\pi : G \to G/G^{00}$ is definable).
- The last item depends on work by a number of people, in particular Edmundo-Otero on torsion points in commutative $o$-minimal groups.

# Idea of proof I

- (With major simplifications.)
- Let $M^*$ be the Shelah (or Baisalov-Poizat) expansion of $M$ by adding relations for all externally definable sets. $M^*$ has QE and is weakly $o$-minimal. Moreover $G^{00}$ is definable in $M^*$.
- $G/G^{00}$ has a dual existence: first as a hyperdefinable compact group in $M$. Secondly, as a group interpretable in $M^*$. But note that $M^*$ is not saturated.
- Write $J$ for $G/G^{00}$ considered as a definable (or interpretable) group in $M^*$. Let $n$ be the $o$-minimal dimension of $G$.
- Show there is $o$-minimal $(I, <) \cong (\mathbb{R}, <)$, definable in $M^*$ with $J \subseteq I^n$.

# Idea of proof II

- So $J$ gets another structure of a Lie group, and by considering its torsion, must be a compact Lie group of (both Lie and $o$-minimal) dimension $n$. In fact its topology coincides with the logic topology on $G/G^{00}$.
- Show that $G$ is $o$-minimally dominated by $J$, that is for a definable (in $M$) subset $X$ of $G$, the set of $h \in J$ such that $\pi^{-1}(h)$ meets both $X$ and its complement has dimension $< n$ so Haar measure $0$. Finish.
- Anyway among the interesting aspects of the situation is that the dual existences (compact hyperdefinable in $M$ and semi-$o$-minimal in $M^*$) cohere, and this is is worth further study.

# More details I

- A recent result of Otero and Peterzil (using cohomological arguments) says that there are 1-dimensional subsets $I_1, .., I_n$ subsets of $G$ such that $G = I_1 + ... + I_n$.
- Hence $J(= G/G^{00}) = \pi(I_1) + ... + \pi(I_n)$.
- Fix attention on $I = I_1$ which we may identify with an interval in $M$.
- The map $G \to J$ induces a definable (in $M^*$) equivalence relation $E$ on $I$, so the classes are finite (bounded) unions of convex sets.
- To simplify, assume that each $E$-class is convex. So $\pi(I)$ has an induced ordering, which we also denote $<$.

# More details II

- Using the fact that $E$ is defined by a countable set of formulas, as well as compactness of the logic topology on $\pi(I)$, and weak $o$-minimality of $M^*$, deduce that $(\pi(I), <) \cong (\mathbb{R}, <)$ and moreover is (strongly) $o$-minimal as a definable set in $M^*$.
- As $\pi(I)$ is a definable subset of the definable group $J$ in $M^*$, $\pi(I)$ has definable Skolem functions.
- Hence $J$ is definably isomorphic to a definable subset of $\pi(I_1) \times ... \times \pi(I_n)$, so of $\mathbb{R} \times .. \times \mathbb{R}$ where each copy of $\mathbb{R}$ has an additional $o$-minimal structure.
- So the proof that a group definable in an $o$-minimal structure is a "group manifold" over the structure concerned, gives $J$ the structure of a connected Lie group, hence by looking at torsion, of an $n$-dimensional compact Lie group. Continue as in the sketch.

# The fsg property and generic compact domination I

- Definably compact groups in $o$-minimal structures have an interesting property *finitely satisfiable generics* (fsg) shared by many other definable groups (stable groups, stably dominated groups, definably compact groups in $ACVF$ ($DCVF$?)...).
- The $fsg$ property says that "half" the properties of stable groups are present.
- Specifically, for $G$ a definable group in a saturated structure: we call definable $X \subseteq G$ left generic if finitely many left translates of $X$ cover $G$.
- $G$ has $fsg$ if (i) left generic = right generic, (ii) the nongenerics form an ideal, (iii) every generic $X$ meets any model $M_0$ (i.e. generic types are almost finitely satisfiable in $\emptyset$).

# The fsg property and generic compact domination II

- Assuming only that $T$ is a theory with $NIP$ and $G$ a definable group with $fsg$, we can prove "generic compact domination" (a weaker statement than compact domination), which can be considered as a measure-theoretic version of the (equivariant) finite equivalence relation theorem.

### Theorem 1.2

*(Under above assumptions of $NIP$ and $fsg$.) Let $X$ be a definable subset of $G$. Then for almost all (in sense of Haar measure) $h \in G/G^{00}$, either all global generic types in $\pi^{-1}(h)$ contain $X$ or all global generic types in $\pi^{-1}(h)$ contain $\neg X$*

# The fsg property and generic compact domination III

- Why do we say this is an analogue of the finite equivalence relation theorem?
- In a stable theory the FERT says that any complete type over an algebraically closed set has a unique global nonforking extension.
- The stable group version is that each coset of $G^0 (= G^{00})$ in $G$ contains a unique generic type of $G$.
- In the $fsg$ $NIP$ context it is NOT true that each coset of $G^{00}$ in $G$ contains a unique generic type. BUT Theorem 1.2 says that it is true locally (formula-by-formula) outside a measure $0$ set.

# Ingredients/Proof I

- The current proof in the paper "NIP and invariant measures", needs some modification.
- Thanks to the Wroclaw group for pointing out problems. Ideas of Pierre Simon are involved in the corrections.
- One new ingredient:
  (*) in a $NIP$ environment, if $\mu_x$ is a global Keisler measure which is finitely satisfiable in a small model, and $\lambda_y$ is a global Keisler measure which is definable, then $\lambda \ltimes \mu = \mu \ltimes \lambda$.
- Here $(\lambda_y \ltimes \mu_x)(\phi(x,y)) = \int_x \lambda(\phi(x,y)d\mu_x$.
- There exists a $G$-invariant measure $\mu$ on $G$ which is generic.
- (*) $+NIP$ implies that $\mu$ is definable, and is moreover is the *unique* $G^{00}$-invariant measure lifting the Haar measure $h$ on $G/G^{00}$.

# Ingredients/Proof II

- Suppose $F \subseteq G/G^{00}$ has Haar measure $> 0$. Define the "localization" or "conditional measure", $h_F$ by $h_F(B) = h(F \cap B)/h_F$. Define $\mu_F$ likewise.
- Then the item above on the consequences of (*)+$NIP$, extends to showing that $\mu_F$ is the unique global Keisler measure on $G$ which extends $h_F$ and is $G^{00}$-invariant.
- This is enough to prove Theorem 1.2.