

# Borel complexity of some $\omega$ -stable theories

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# $\omega$ -stable theories

This work is joint with Saharon Shelah.

Fix a (complete)  $\omega$ -stable theory.

So:

- If  $p \in S(A)$  is stationary, there is a finite  $A_0 \subseteq A$  such that  $p$  does not fork over  $A_0$  and  $p|_{A_0}$  is stationary.
- Thus,  $M$  a-saturated  $\Leftrightarrow M$   $\omega$ -saturated.
- Prime models exist over arbitrary sets. (Unique up to  $\cong$ , but not unique!)

## Definition

A type  $p \in S(A)$  is ENI (eventually non-isolated) if  $p$  is stationary, regular, and for some finite  $A_0 \subseteq A$  over which  $p$  is based and stationary, there is a countable  $M \supseteq A_0$  with  $\dim(p|_{A_0}, M) < \aleph_0$ .

We do **not** require  $p$  to be strongly regular!

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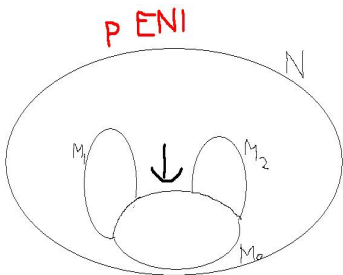
## Advantage

The class of ENI types is closed under non-orthogonality and automorphisms of  $\mathfrak{C}$ .

## Definition

A theory  $T$  has ENI-NDOP if, for all independent triples  $M_0 = M_1 \cap M_2$  of  $a$ -saturated models, for all  $a$ -prime models  $N \supseteq M_1 \cup M_2$ , and for all ENI  $p$ ,

$$p \not\perp N \Rightarrow p \not\perp M_i \quad \text{for some } i < 3.$$



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- *The class of  $\omega$ -saturated models is excellent*
- *$T$  has NOTOP **i.e.**, there is NO  $p(\bar{x}, \bar{y}, \bar{z})$  such that for any graph  $(G, E)$  we can find  $\{\bar{a}_g : g \in G\}$  and a model  $M_G$  such that  $p(\bar{x}, \bar{a}_g, \bar{a}_h)$  is omitted in  $M_G$  iff  $G \models E(g, h)$*

# ENI-supportive types

A stationary, regular type  $tp(b/A)$  is *ENI-supportive* if  $tp(b/A)$  is ENI OR there is an ENI  $q \in S(C)$ ,  $C \supseteq Ab$  and dominated by  $b$  over  $A$ , with  $q \perp A$ .

Think:  $tp(b/A)$  is ENI or lies below an ENI type in a decomposition tree.

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Think:  $tp(b/A)$  is ENI or lies below an ENI type in a decomposition tree.

Fact:  $T$  superstable,  $\mathbb{P}$  any class of stationary, regular types closed under non-orthogonality and automorphisms of  $\mathfrak{C}$ . Then:

- $T$  has  $\mathbb{P}$ -NDOP implies  $T$  has  $supp(\mathbb{P})$ -NDOP
- $q \in supp(\mathbb{P})$  of depth  $> 0$  implies  $q$  trivial.

# ENI-supportive decompositions

## Definition

A **prime decomposition** of  $M^*$  is a sequence  $\langle M_\eta, a_\eta : \eta \in I \rangle$  indexed by a tree  $(I, \trianglelefteq)$  satisfying:

- $\{M_\eta : \eta \in I\}$  is an independent tree of countable, elementary substructures of  $M^*$
- $M_{\langle \rangle}$  is prime
- For every  $\eta \in I$ ,  $\{a_\nu : \nu \in \text{Succ}_I(\eta)\}$  is a maximal independent (over  $M_\eta$ ) set of ENI-supportive types satisfying  $\text{tp}(a_\nu/M_\eta) \perp M_{\eta^-}$  (when  $\eta \neq \langle \rangle$ )
- For all  $\nu \neq \langle \rangle$ ,  $M_\nu$  is prime over  $M_{\nu^-} \cup \{a_\nu\}$
- $M^*$  is prime over  $\bigcup \{M_\eta : \eta \in I\}$ .

# Existence of decompositions

Theorem (proved independently by Koerwien)

*If  $T$  is  $\omega$ -stable with ENI-NDOP, then every model of  $T$  has a prime decomposition.*

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*If  $T$  is  $\omega$ -stable with ENI-NDOP, then every model of  $T$  has a prime decomposition.*

An  $\omega$ -stable  $T$  with ENI-NDOP is *ENI-deep* if some model of  $T$  has a prime decomposition with an infinite branch.

# Borel reducibility

Fix countable vocabularies  $\tau_1, \tau_2$ .

For  $i = 1, 2$

- $\mathcal{S}_i = \{\text{all } \tau_i\text{-structures with universe } \omega\}$  with the usual topology.
- $X_i$  be a Borel subset of  $\mathcal{S}_i$ , closed under  $\cong$ .
- $E_i$  be an equivalence relation on  $X_i$  extending  $\cong$ .

$(X_1, E_1) \leq_B (X_2, E_2)$  iff there is a Borel  $f : X_1 \rightarrow X_2$  (relative to topologies on  $\mathcal{S}_1, \mathcal{S}_2$ ) such that for all  $\mathfrak{A}, \mathfrak{B} \in X_1$

$$\mathfrak{A} E_1 \mathfrak{B} \iff f(\mathfrak{A}) E_2 f(\mathfrak{B})$$

# Borel completeness

Fact:  $(\text{Graphs}, \cong)$  is maximal w.r.t. Borel reducibility.

## Definition

$(X, E)$  is *Borel complete* if  $(\text{Graphs}, \cong) \leq_B (X, E)$ .

Routine: If  $T$  is  $\omega$ -stable with ENI-DOP, then  $(\text{Mod}_{\aleph_0}(T), \cong)$  is Borel complete.

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Routine: If  $T$  is  $\omega$ -stable with ENI-DOP, then  $(\text{Mod}_{\aleph_0}(T), \cong)$  is Borel complete. **Pf:** Use OTOP!

**A surprise:** (Friedman-Stanley) (subtrees of  $\langle{}^\omega\omega, \cong\rangle$ ) is Borel complete.

**This suggests:**  $T$  ENI-NDOP, ENI-deep implies  $(\text{Mod}_{\aleph_0}(T), \cong)$  Borel complete.

**Sketch:** Given  $I$ , form an independent tree  $\langle M_\eta, a_\eta : \eta \in I \rangle$  indexed by  $I$  in a canonical way, and let  $M_I$  be prime over  $\bigcup \{M_\eta : \eta \in I\}$ .

- If  $I \cong J$  as trees, then  $M_I \cong M_J$  (easy).
- If  $M_I \cong M_J$ , then  $I \cong J$  ???

The issue: Given a countable model of  $T$ , how unique is its decomposition?

## Definition

A prime decomposition  $\langle M_\eta, a_\eta : \eta \in I \rangle$  is **tidy** if for all  $\eta \in I$ ,

- $\{\text{tp}(a_\nu/M_\eta) : \nu \in \text{Succ}_I(\eta)\}$  is finite;
- For each  $\nu \in \text{Succ}_I(\eta)$  there are infinitely many  $\nu' \in \text{Succ}_I(\eta)$  such that  $\text{tp}(a_{\nu'}/M_\eta) = \text{tp}(a_\nu/M_\eta)$ ;
- If  $\nu, \gamma \in \text{Succ}_I(\eta)$  and  $\text{tp}(a_\nu/M_\eta) \neq \text{tp}(a_\gamma/M_\eta)$ , then  $\text{tp}(a_\nu/M_\eta) \perp \text{tp}(a_\gamma/M_\eta)$ .

### Definition

A nonempty subtree  $J \subseteq I$  of a tree is **large** if for all  $\eta \in J$ ,  $\text{Succ}_I(\eta) \setminus J$  is finite.

### Definition

Two trees  $I_1, I_2$  are **almost isomorphic** ( $I_1 \cong^* I_2$ ) if they have isomorphic large subtrees.

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## Theorem (Weak Uniqueness)

$T$   $\omega$ -stable, ENI-NDOP,  $M \models T$ . If  $\langle M_\eta^1, a_\eta : \eta \in I \rangle$  and  $\langle M_\eta^2, b_\eta : \eta \in J \rangle$  are both tidy decompositions of  $M$ , then  $(I \setminus \text{Leaves}(I))$  and  $(J \setminus \text{Leaves}(J))$  are almost isomorphic.

## Ideas:

- By removing the leaves, all relevant types are trivial.
- We handle each nonorthogonality class separately.
- If  $q$  is a trivial regular type,  $A \subseteq M$  finite, 'dependence over  $A$ ' is an equivalence relation on  $Q(M) = \{c \in M : tp(c/A) \text{ is regular, stationary, } \neq q\}$ .
- For any two maximal, independent/ $A$  subsets  $E, F \subseteq Q(M)$ , relation  $h : E \rightarrow F$  given by ' $\{e, h(e)\}$  fork over  $A$ ' is a bijection.

If

$$I_1^q = \{\nu \in I_1 : \text{tp}(a_\nu/M_{\nu^-}^1) \not\equiv q\}$$

and

$$I_2^q = \{\eta \in I_2 : \text{tp}(a_\eta/M_{\eta^-}^2) \not\equiv q\}$$

then there is a natural bijection  $h_q : J_1^q \rightarrow J_2^q$  between cofinite subsets  $J_1^q \subseteq I_1^q$  and  $J_2^q \subseteq I_2^q$ .

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**But...** In order to piece these 'almost bijections' together to obtain an almost isomorphism  $h : I_1 \rightarrow I_2$ , we need to show that the set of maps  $\{h_q\}_q$  preserves the tree order.

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For that we need to leave the realm of countable models!

Fix an  $\omega$ -stable theory  $T$  with ENI-NDOP and fix an uncountable,  $\omega$ -saturated  $M^* \models T$ .

$$E\text{supp}(M^*) = \{p \in S(M^*) : p \text{ is ENI-supportive}\}$$

We can use subsets of  $E\text{supp}(M^*)$  to 'measure' ENI-supportive types  $\text{tp}(c/A)$  with  $A \subseteq M^*$  finite and  $c \in M^*$

## Definition

A  $(c, A)$ -decomposition of  $M^*$  is a sequence  $\langle M_\eta, a_\eta : \eta \in I \rangle$  such that:

- $(I, \trianglelefteq)$  is a tree in which  $\langle 0 \rangle$  is the unique successor of  $\langle \rangle$
- $A \subseteq M_{\langle \rangle}$ ,  $c = a_{\langle 0 \rangle}$  and  $\text{tp}(c/M_{\langle \rangle})$  does not fork over  $A$
- $\{M_\eta : \eta \in I\}$  is an independent tree of  $\omega$ -saturated submodels of  $M^*$ ;
- For all  $\nu \neq \langle \rangle$  if  $\text{tp}(b/M_{\nu-})$  ENI-supportive and  $\text{tp}(b/M_\nu)$  forks over  $M_{\nu-}$ , then  $\text{tp}(b/M_{\nu-} \cup \{a_\nu\})$  forks over  $M_{\nu-}$ .
- For all  $\eta \neq \langle \rangle$   $\{a_\nu : \nu \in \text{Succ}_I(\eta)\} \subseteq M^*$  is a maximal independent over  $M_\eta$  set of realizations of ENI-supportive types.
- $(I, \trianglelefteq)$  is maximal with respect to these conditions.

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- $(I, \trianglelefteq)$  is maximal with respect to these conditions.

$M^*$  need not be a-prime over  $\bigcup \{M_\eta : \eta \in I\}$ !



Fix  $c, A$  from  $M^*$  with  $A$  finite and  $\text{tp}(c/A)$  ENI-supportive.

$\text{Trace}(c, A) = \{q \in \text{Esupp}(M^*) : q \not\leq M_\eta \text{ for some } (c, A)\text{-decomposition } \langle M_\eta, a_\eta : \eta \in I \rangle \text{ of } M^* \text{ and some } \eta \neq \langle \rangle\}$ .

### Theorem (Shelah)

- $\text{Trace}(c, A)$  DOES NOT DEPEND on our choice of  $(c, A)$ -decompositions (!)
- For any  $\omega$ -saturated decomposition  $\langle N_\eta, b_\eta : \eta \in I \rangle$  of  $M^*$ , for any  $\eta, \nu \in I$ ,
  - If  $\eta \trianglelefteq \nu$  then  $\text{Trace}(b_\nu/N_{\nu-}) \subseteq \text{Trace}(b_\eta/N_{\eta-})$
  - If  $\eta, \nu$  are incomparable then  $\text{Trace}(b_\eta, N_\eta)$  and  $\text{Trace}(b_\nu, N_{\nu-})$  are disjoint.

## Back to countable models

Given a countable  $M$  with two tidy decompositions, let  $M^* \succeq M$  be an uncountable, saturated elementary extension.

### Fact

*Within each nonorthogonality class, the almost bijection  $h_q : J_1^q \rightarrow J_2^q$  'preserves the trace' i.e.,*

$$\text{Trace}(a_\nu / M_{\nu^-}^1) = \text{Trace}(h_q(a_\nu) / M_{\eta^-}^2)$$

*on cofinite subsets  $J_1^q \subseteq I_1^q$ ,  $J_2^q \subseteq I_2^q$ . Thus, the composite map  $h : \bigcup_q J_1^q \rightarrow \bigcup_q J_2^q$  preserves the tree ordering.*

# Back to Borel reducibility

## Lemma

*(subtrees of  $\langle{}^\omega\omega, \cong^*\rangle$ ) are Borel complete.*

## Theorem (Weak Uniqueness)

*$T$   $\omega$ -stable, ENI-NDOP,  $M \models T$ . If  $\langle M_\eta^1, a_\eta : \eta \in I \rangle$  and  $\langle M_\eta^2, b_\eta : \eta \in J \rangle$  are both tidy decompositions of  $M$ , then  $(I \setminus \text{Leaves}(I))$  and  $(J \setminus \text{Leaves}(J))$  are almost isomorphic.*

### Corollary

*$T$   $\omega$ -stable, ENI-NDOP, ENI-deep  $\Rightarrow (\text{Mod}_{\aleph_0}(T), \cong)$  is Borel complete.*

**Question:** Are there  $\omega$ -stable, ENI-shallow theories that are Borel complete?

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### Example (Koerwien)

There is an  $\omega$ -stable, ENI-depth 2 theory for which  $\{SH(M) : M \in Mod_{\aleph_0}(T)\}$  is unbounded in  $\omega_1$ .

Koerwien's example is a 'pun on  $\omega$ '.

All of the complexity arises from the complicated structure of the automorphisms of  $ac/\emptyset$  !

# Borel reducibility revisited

**Recall:** Fix countable vocabularies  $\tau_1, \tau_2$ .

For  $i = 1, 2$

- $\mathcal{S}_i = \{\text{all } \tau_i\text{-structures with universe } \omega\}$  with the usual topology.
- $X_i$  be a Borel subset of  $\mathcal{S}_i$ , closed under  $\cong$ .
- $E_i$  be an equivalence relation on  $X_i$  extending  $\cong$ .

$(X_1, E_1) \leq_B (X_2, E_2)$  iff there is a Borel  $f : X_1 \rightarrow X_2$  (relative to topologies on  $\mathcal{S}_1, \mathcal{S}_2$ ) such that for all  $\mathfrak{A}, \mathfrak{B} \in X_1$

$$\mathfrak{A} E_1 \mathfrak{B} \iff f(\mathfrak{A}) E_2 f(\mathfrak{B})$$

$\mathcal{S}_2$ -Basic Open Sets:  $\mathcal{U}_{R, \bar{n}} = \{M_2 \in \mathcal{S}_2 : M_2 \models R(\bar{n})\}$   
for every  $R \in \tau_2$  and  $\bar{n} = (n_1, \dots, n_k) \in \omega^{\text{arity}(R)}$ .

$f : \mathcal{S}_1 \rightarrow \mathcal{S}_2$  Borel means  $f^{-1}$ (Basic open set in  $\mathcal{S}_2$ ) is a Borel subset of  $\mathcal{S}_1$ .

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$f : \mathcal{S}_1 \rightarrow \mathcal{S}_2$  Borel means  $f^{-1}$ (Basic open set in  $\mathcal{S}_2$ ) is a Borel subset of  $\mathcal{S}_1$ .

**So:** For every  $R, \bar{n}$  there is a quantifier-free  $\Phi_{R, \bar{n}} \in L_{\omega_1, \omega}$  in the vocabulary  $\tau_1(\omega)$  such that

$$M_1 \models \Phi_{R, \bar{n}} \iff f(M_1) \models R(\bar{n})$$

# $\lambda$ -Borel reducibility

Generalize to  $\lambda \geq \aleph_0$ :

$$\mathcal{S}_1^\lambda = \{\tau_1\text{-structures with universe } \lambda\}$$

$$\mathcal{S}_2^\lambda = \{\tau_2\text{-structures with universe } \lambda\}.$$

$f : \mathcal{S}_1^\lambda \rightarrow \mathcal{S}_2^\lambda$  is  $\lambda$ -Borel if for every  $R \in \tau_2$  and  $\bar{\alpha} = (\alpha_1, \dots, \alpha_k) \in \lambda^k$  there is a q.f.  $L_{\lambda^+, \omega}$ -sentence  $\Phi_{R, \bar{\alpha}}$  such that

$$M_1 \models \Phi_{R, \bar{\alpha}} \iff f(M_1) \models R(\bar{\alpha})$$

## Definition

If, for  $i = 1, 2$   $X_i$  is a  $\lambda$ -Borel subset of  $\mathcal{S}_i$  and  $E_i$  an equivalence relation on  $X_i$  extending  $\equiv_{\lambda^+, \omega}$ , then  $(X_1, E_1)$  is  $\lambda$ -Borel reducible to  $(X_2, E_2)$  if there is a  $\lambda$ -Borel  $f : X_1 \rightarrow X_2$  such that  $\mathfrak{A}E_1\mathfrak{B} \Leftrightarrow f(\mathfrak{A})E_2f(\mathfrak{B})$ .

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As before  $(\text{Graphs on } \lambda, \equiv_{\lambda^+, \omega})$  is maximal w.r.t.  $\lambda$ -Borel reducibility.

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As before  $(\text{Graphs on } \lambda, \equiv_{\lambda^+, \omega})$  is maximal w.r.t.  $\lambda$ -Borel reducibility.

**A surprise:**  $(\text{subtrees of } <^\omega \lambda, \equiv_{\lambda^+, \omega})$  is  $\lambda$ -Borel complete.

## Theorem

*TFAE for  $\omega$ -stable theories  $T$ :*

- *$T$  is ENI-NDOP, ENI-shallow*
- *For some  $\lambda \geq \aleph_0$   $(\text{Mod}_\lambda(T), \equiv_{\lambda^+, \omega})$  is not  $\lambda$ -Borel complete*
- *For some  $\lambda \geq \aleph_0$   $\{SH_{\lambda^+, \omega}(M) : M \in \text{Mod}_\lambda(T)\}$  is bounded below  $\lambda^+$*

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- *For some  $\kappa_0$ ,  $I_{\infty, \omega}(T, \kappa) < 2^\kappa$  for all  $\kappa \geq \kappa_0$ .*

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There are  $\omega$ -stable, ENI-NDOP, ENI-deep theories  $T$  with DOP !

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There are  $\omega$ -stable, ENI-NDOP, ENI-deep theories  $T$  with DOP !  
 Such theories  $T$  have  $2^\kappa$  nonisomorphic models, but fewer than  $2^\kappa$   
 $L_{\infty, \omega}$ -inequivalent models for most uncountable cardinals  $\kappa$ .

## Theorem

*TFAE for  $\omega$ -stable theories  $T$ :*

- *$T$  is ENI-NDOP, ENI-shallow*
- *For some  $\lambda \geq \aleph_0$   $(Mod_\lambda(T), \equiv_{\lambda^+, \omega})$  is not  $\lambda$ -Borel complete*
- *For some  $\lambda \geq \aleph_0$   $\{SH_{\lambda^+, \omega}(M) : M \in Mod_\lambda(T)\}$  is bounded below  $\lambda^+$*
- *For some  $\kappa_0$ ,  $I_{\infty, \omega}(T, \kappa) < 2^\kappa$  for all  $\kappa \geq \kappa_0$ .*

There are  $\omega$ -stable, ENI-NDOP, ENI-deep theories  $T$  with DOP !  
 Such theories  $T$  have  $2^\kappa$  nonisomorphic models, but fewer than  $2^\kappa$   
 $L_{\infty, \omega}$ -inequivalent models for most uncountable cardinals  $\kappa$ .

This partially explains why it is hard to prove "many-models" for theories with DOP.