

SUPERROSY GROUPS AND FIELDS

1. Rosiness, NIP, fsg

T - a first order theory

\mathfrak{C} - a monster model of T

Definition T is *rosy* if there is a ternary relation \perp^* on subsets of \mathfrak{C}^{eq} satisfying all the basic properties of forking independence in simple theories except for the independence theorem. \perp^* is called an *independence relation*.

Fact There exists the strongest independence relation, so-called \perp^b -forking, denoted by \perp^b .

Definition *Thorn U -rank* is a function

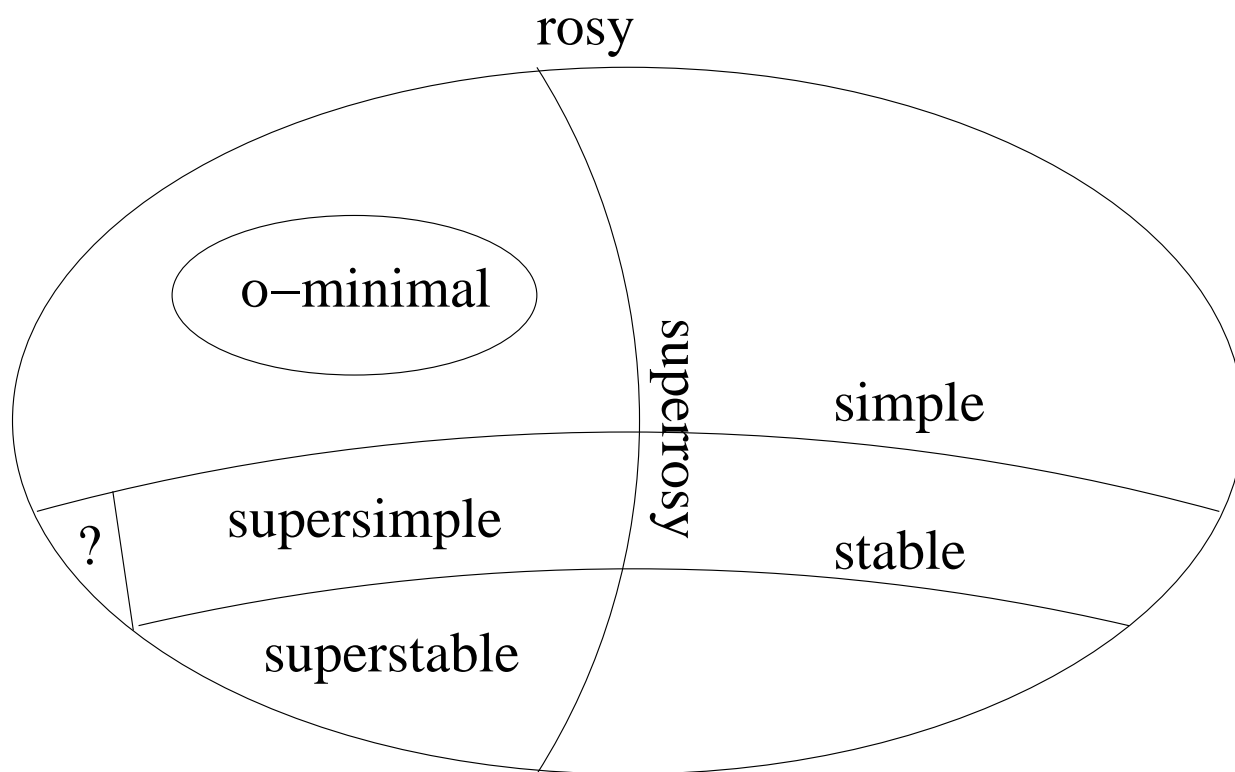
U^b : complete types $\rightarrow Ord \cup \{\infty\}$ defined by:

$U^b(a/A) \geq \alpha + 1$ iff there is $B \supseteq A$ such that

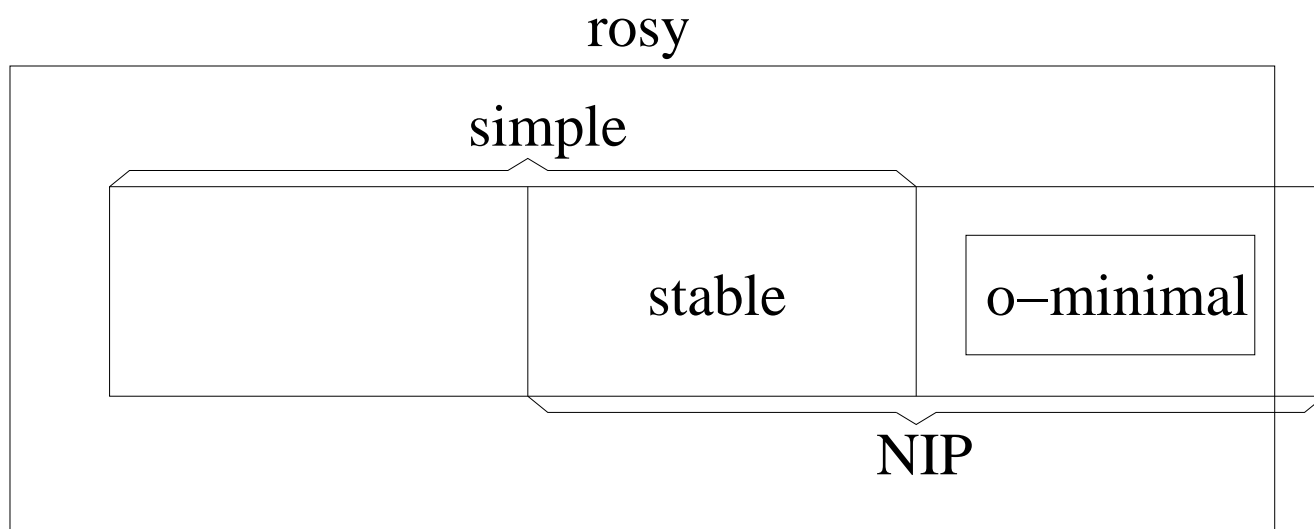
$U^b(a/B) \geq \alpha$ and $a \not\perp_A^b B$.

Definition $U^b(D) := \sup\{U^b(d/A) : d \in D\}$
for any A -definable set D .

Definition T is *superrosy* if $U^b(a) \in \text{Ord}$ for every $a \in \mathfrak{C}$.



Definition T has NIP if there is no formula $\varphi(x, y)$ for which there are sequences $\langle a_i \rangle_{i \in \omega}$ and $\langle b_w \rangle_{w \subseteq \omega}$ such that $\varphi(a_i, b_w) \iff i \in w$.



G - a group interpretable in \mathfrak{C}

Definition G has *fsg* (*finitely satisfiable generics*) if there is $p \in S(\mathfrak{C})$ implying ' $x \in G$ ' and a small model $M \prec \mathfrak{C}$ such that for all $g \in G$ the translate gp is finitely satisfiable in M .

Examples: stable groups (in particular algebraic groups), compact Lie groups

Definition G^{00} is the smallest type-definable subgroup of bounded index in G , if such exists.

Fact If T has NIP or G has fsg, then G^{00} exists.

Definition A formula $\varphi(x)$, or the set $\varphi(G)$, is *left generic* if finitely many left translates of $\varphi(G)$ cover G . A type is *left generic* if every formula in it is left generic.

Fact If G has fsg, then a global generic type exists, and every generic type is finitely satisfiable in a small submodel.

Remark Assume G has fsg. If G^{00} is definable, there is only one generic type in G^{00} .

2. Rosy groups (EKP)

Proposition A rosy group has ucc (the uniform chain condition).

Proposition A superrosy group has ω dcc.

Fact If G is a group with NIP, then for each formula φ there is $n \in \omega$ such that any finite family of φ -definable subgroups is an intersection of n members of the family.

Corollary A rosy group with NIP has icc (the uniform chain condition on intersections of uniformly definable groups).

Definition Let G be a rosy group. We say that $p \in S(A)$ is *left \mathfrak{p} -generic* if for all $a, b \in G$ with $a \models p$ and $a \perp_A^{\mathfrak{p}} b$ one has $b \cdot a \perp^{\mathfrak{p}} A, b$

Proposition In any rosy group, a global \mathfrak{p} -generic exists.

The proof uses suitably defined stratified local \mathfrak{p} -ranks that witness \mathfrak{p} -forking.

Question Suppose G is a rosy group. Does there exist a $*$ -generic type with respect to an abstract independence relation \perp^* ?

Many basic results concerning simple groups are true for rosy groups. However, we do not know a good notion of a stabilizer of a type and we do not have any version of Zilber's Indecomposables Theorem.

3. Groups of U^b -rank 1 (EKP)

G - a group interpretable in a monster model of a rosy theory T

Proposition If G contains a b -generic involution, then it contains a b -generic element g such that $[G : C(g)] < \omega$.

Corollary If T satisfies NIP and G has a b -generic involution, then G is abelian-by-finite.

Theorem If T has NIP, G has fsg, and $U^b(G) = 1$, then G is abelian-by-finite.

Sketch of the proof. Assume for a contradiction it is false. Taking the centralizer connected component and quotienting by its finite center, we can assume that every non-trivial element of G has a non-trivial, finite centralizer. We show that there are only finitely many conjugacy classes, and conclude that G^{00} is definable. By fsg, there is only one generic in G^{00} . Then we produce a contradiction using the first proposition. ■

More generally:

Theorem If T satisfies NIP, G has hereditarily fsg and $0 < U^{\mathfrak{p}}(G) < \infty$, then G contains an infinite definable abelian subgroup.

Theorem If T satisfies NIP, G has fsg, and $U^{\mathfrak{p}}(G) = \omega^{\alpha}$, then G is abelian-by-finite.

Theorem If T satisfies NIP, G has fsg, and at least one \mathfrak{p} -regular \mathfrak{p} -generic type, then G is abelian-by-finite.

Conjecture In each result, the hypothesis ' G has fsg' may be replaced with ' G is definably amenable', i.e. there is a left invariant, finitely additive probability measure on definable subsets of G .

Then we have a notion of generic with respect to the measure. The main obstacle is that we do not have uniqueness of such generics even if $G = G^{00}$.

4. Groups of U^b -rank 2 (EKP)

G - a group interpretable in a monster model of an arbitrary theory T .

Theorem If T satisfies NIP, G has hereditarily fsg, and $U^b(G) = 2$, then G is solvable-by-finite.

Some ideas are taken from the proof that each connected group of Morley rank 2 is solvable. The main obstacle in comparison with the Morley rank 2 case is that a b -generic may not be generic and there may be many b -generics even in the connected component.

The structure of the proof is as follows. We suppose for a contradiction that G is not solvable-by-finite. First we define Borel subgroups (albeit in a slightly different manner from that which is used in the Morley rank 2 case because we do not have definable connected components), and we study their properties. Then we use them to find involutions. In the last part of the proof we use Borels, involutions and some particular function

that comes from the theory of black box groups to get a final contradiction.

Definition We say that a subgroup B of G is a *Borel* if it is a minimal infinite intersection of centralizers, i.e. B is the intersection of centralizers, it is infinite and for every centralizer C either $C \cap B = B$ or $C \cap B$ is finite.

Conjecture Suppose T has NIP and $U^b(G) = 2$. If every definable subgroup of G is definably amenable, then G is solvable-by-finite.

5. Getting fields in groups

General question: Given an infinite group $\langle G, \cdot \rangle$, when does there exist an infinite field interpretable in $\langle G, \cdot \rangle$?

We will assume that G is finite dimensional, i.e. of finite U^b -rank.

Necessary condition: G is not abelian-by-finite.

Classical and new results:

	finite MR	o-minimal	$U^b < \omega$ and NIP
non solvable-by-finite	partial results	Yes	partial results
solvable-by-finite and non nilpotent-by-finite	Yes	Yes	Yes
nilpotent-by-finite	???	Yes	???

6. Getting fields in classical situations

We work in a monster model \mathfrak{C} of a theory T .

Definition If G is a definable group acting definably and by automorphisms on a definable group H , then we say that H is G -minimal if H is infinite and does not have infinite, proper, definable subgroups invariant under the action of G .

Fact 1 Assume T is of finite MR. Let A and M be infinite, definable, abelian groups, A act faithfully and definably on M by automorphisms, and M be A -minimal. Then there is an infinite field K interpretable in \mathfrak{C} . Moreover, $\langle K, + \rangle = \langle M, + \rangle$ and $\langle A, \cdot \rangle \hookrightarrow K^*$.

Main tools: Icc and Zlifer's Indecomposables Theorem

In o-minimal structures there is no ZIT.

New tool: \forall -definable rings

Definition We say that a ring $\langle R, \cdot, + \rangle$ is a \forall -*definable* (or rather \forall -*interpretable*) ring if $R = \bigcup_{i \in I} X_i$ where X_i 's are A -definable subsets of some sort of \mathfrak{C}^{eq} for some set A (and for every $i, j \in I$ there is $k \in I$ such that $X_i \cup X_j \subseteq X_k$), and the restrictions of addition and multiplication to $X_i \times X_j$ are definable functions.

Fact 2 (T o-minimal) Let R be a \forall -definable integral domain of positive dimension. Then the field of fractions of R is interpretable in \mathfrak{C} .

Main tool: certain topology on \forall -definable rings defined by means of the o-minimal topology

Fact 3 Fact 1 holds if T is o-minimal.

The proof uses Fact 2 and the topology on \forall -definable rings.

Let G be an infinite group interpretable in \mathfrak{C} .

Some variants of Fact 1 give:

Fact 4 (T of finite MR) Assume G is solvable but not nilpotent-by-finite. Then an infinite field is interpretable in $\langle G, \cdot \rangle$.

Fact 5 (T o-minimal) If G is not abelian-by-finite, then an infinite field is interpretable in $\langle G, \cdot \rangle$.

7. Getting fields in rosy theories (K)

We work in a monster model \mathfrak{C} of a rosy theory T .

Definition If $X = \bigcup_{i \in I} X_i$ where all X_i 's are definable over A , then $U^b(X) := \sup\{U^b(X_i) : i \in I\}$.

Theorem 1 Let R be a \bigvee -definable integral domain of positive finite U^b -rank. Then its field of fractions is interpretable in \mathfrak{C} .

Sketch of the proof. $R = \bigcup_{i \in I} X_i$ where X_i 's are definable over A . Let $f_r : R \rightarrow R$ be defined by $f_r(x) = rx$. Then $f_r \upharpoonright X_i, i \in I$, are definable.

Let $D := X_i$ be such that $U^b(D) = U^b(R)$.

Claim 1 For all $a, b \in R \setminus \{0\}$ we have $(Da - Da) \cap (Db - Db) \neq \{0\}$.

Proof of Claim 1. The function $f : D \times D \rightarrow R$ defined by $f(r_1, r_2) = r_1a + r_2b$ is definable.

Since $U^b(D \times D) = 2U^b(D) > U^b(D) = U^b(R)$,

f is not injective.

Thus there are distinct $(r_1, r_2), (r'_1, r'_2) \in D \times D$ such that $r_1a + r_2b = r'_1a + r'_2b$. Therefore, $(r_1 - r'_1)a = r_1a - r'_1a = r'_2b - r_2b = (r'_2 - r_2)b$.

Put $c = r_1a - r'_1a = r'_2b - r_2b \in (Da - Da) \cap (Db - Db)$.

As $a, b \neq 0$ and $(r_1 - r'_1 \neq 0$ or $r'_2 - r_2 \neq 0)$, and R is an integral domain, we get $c \neq 0$. \square

Choose any $a \in R \setminus \{0\}$ and put $X = Da - Da$.

Claim 2 For all $r_1, r_2 \in R \setminus \{0\}$ we have $r_1X \cap r_2X \neq \{0\}$.

By Claim 2, the field of fractions F can be identified with $(X \times (X \setminus \{0\})) / \sim$, and so it is interpretable. \blacksquare

Theorem 2 Let A and M be infinite, definable, abelian groups such that A acts faithfully and definably on M as a group of automorphisms, M is A -minimal and $U^b(M)$ is finite. Then there is an infinite field interpretable in \mathfrak{C} .

The proof is much more general and simpler than in the finite MR case, it uses Theorem 1 and the compactness theorem.

Proof.

For $a \in A$:

$$\text{Fix}(a) := \{m \in M : am = m\} <_{\text{def}} M.$$

For $m \in M$:

$$\text{Stab}(m) := \{a \in A : am = m\} <_{\text{def}} A.$$

Claim 1 There are $m_1, \dots, m_n \in M$ such that $\text{Stab}(m_1) \cap \dots \cap \text{Stab}(m_n) = \{e\}$.

Proof of Claim 1. For any $a \in A$, $\text{Fix}(a)$ is a proper, definable subgroup of M invariant under A . So by A -minimality of M , $\text{Fix}(a)$ is finite. Take any infinite countable set S . Then $\bigcap_{m \in S} \text{Stab}(m) = \{e\}$. So $\text{Stab}(m_1) \cap \dots \cap \text{Stab}(m_n) = \{e\}$ for some $m_1, \dots, m_n \in S$. \square

Let R be the ring of endomorphisms of M generated by A . Then R is an integral domain.

Claim 2 Every element $r \in R$ is determined by $(r(m_1), \dots, r(m_n))$.

Proof of Claim 2. If not, there is $r \in R \setminus \{0\}$ such that $r(m_1) = \dots = r(m_n) = 0$. Since R is commutative, we get that $\ker(r)$ is a proper, definable and invariant under A subgroup of M containing $\{m_1, \dots, m_n\}$. So

$$Am_1 + \dots + Am_n \subseteq \ker(r).$$

By Claim 1, the function $a \mapsto (am_1, \dots, am_n)$ is an injection from A to M^n . So there is i such that Am_i is infinite, and hence $\ker(r)$ is infinite. This contradicts the assumption that M is A -minimal. \square

Claim 3 The ring R is \forall -definable, contained in M^n with the addition inherited from M^n , and $0 < \text{Up}(R) < \omega$.

Proof of Claim 3. Let $H = \langle A(m_1, \dots, m_n) \rangle$, a \forall -definable subgroup of M^n . By Claim 2, the function $f : R \rightarrow H \subseteq M^n$ defined by $f(r) = (r(m_1), \dots, r(m_n))$ is a bijection. We define a ring multiplication on R so that f is a ring isomorphism. We check that R is \forall -definable and of positive finite Up -rank. \square

By Claim 3 and Theorem 1, we get a field. \blacksquare

Moreover, K can be chosen so that $\langle K, + \rangle = M/M_0$ for some finite subgroup M_0 of M invariant under A and A/A_0 is definably embeddable in K^* for some finite subgroup A_0 of A . In fact, the action of A on M induces a faithful and definable action of A/A_0 on M/M_0 by automorphisms, and after the embedding this action becomes the scalar multiplication.

Theorem 3 Let G be a group of finite U^b -rank definable in \mathfrak{C} and suppose that T has NIP. Assume that G is solvable-by-finite but not nilpotent-by-finite. Then there is an infinite field interpretable in $\langle G, \cdot \rangle$.

The proof uses Theorem 2, but it is more complicated than in the finite MR case. We need NIP to have icc and the existence of G^{00} .

Non-solvable case

G - a group interpretable in \mathfrak{C}

Fact If T is stable and G is a transitive group of permutations of a strongly minimal set, then $MR(G) \in \{1, 2, 3\}$. If $MR(G) \in \{2, 3\}$, then there is an infinite field interpretable in \mathfrak{C} .

Fact If G is a simple group of Morley rank $n > 0$ containing a subgroup of Morley rank $n - 1$, then there is an infinite field interpretable in \mathfrak{C} .

From now on suppose T has NIP, and G has hereditarily fsg.

Theorem 4 Assume $1 < U^b(G) < \infty$, and G acts definably on a definable set S of U^b -rank 1 so that there is $s \in S$ such that no infinite subgroup H of G_s (with $U^b(G) - U^b(H) \leq 2$) of finite index in an intersection of stabilizers of points in S has normalizer of finite index in G . Then there is an infinite field interpretable in \mathfrak{C} .

Corollary 5 Assume $U^b(G) < \infty$, and S is a definable set of U^b -rank 1 on which G acts definably so that G_S is finite. If G is not solvable-by-finite, then an infinite field is interpretable in \mathfrak{C} .

Corollary 6 Assume $U^b(G) = n + 1$, and there is a definable subgroup H with $U^b(H) = n$. Let $Z = \bigcap_{g \in G} H^g$. If G/Z is not solvable-by-finite, then an infinite field is interpretable in $\langle G, \cdot \rangle$.

Proof. Apply Corollary 5 for G/Z acting on G/H . ■

Corollary 7 Suppose $U^b(G) = 3$, there is a definable subgroup H of U^b -rank 2, and G is not solvable-by-finite. Then there is an infinite field interpretable in $\langle G, \cdot \rangle$.

Conjecture The assumptions of Theorem 4 [Corollary 5, Corollary 6] imply $U^b(G) \in \{1, 2, 3\}$.

8. Superrosy fields with NIP (EKP)

Main conjecture (EKP) Each superrosy field with NIP is either real or algebraically closed.

Theorem (KP) Each superrosy field with NIP whose additive group satisfies fsg is algebraically closed.

K - an infinite superrosy field with NIP

Proposition (K) $K = K^n - K^n$ for every natural number $n > 0$.

Main tool: Generics with respect to measures

Conjecture Assume $\sqrt{-1}$ exists in K . Then for every natural number n and $a \in K^*$ we have $K = K^n - aK^n$.

This conjecture implies that if $\sqrt{-1} \in K$, then $Br(K)$ is trivial.

Some very recent results (to be checked)

K - an infinite superrosy field

Proposition (K) Every valuation v on K has a divisible value group.

Conjecture (K) Assume K has NIP. Then the residue field of every non-trivial valuation on K is either real or algebraically closed.

Theorem (K) The conjecture is true in the case of $\text{char}(K) \neq 0$.

9. Groups of automorphisms (KP)

Fact Any definable group of definable automorphisms of an infinite superstable field is trivial.

K - a superrosy field with NIP

G - a definable group of definable automorphisms of K

Conjecture (KP) G is finite.

Proposition (KP) If $\text{char}(K) \neq 0$, then for any $\sigma \in G$ we have $[K : \text{Fix}(\sigma)] < \omega$.

Proposition (K) Assume that $\text{U}^{\text{b}}(K) < \omega$. If $\text{char}(K) = 0$ or $\overline{F}_p \subseteq K$, then G is finite.