On AE-Axiomatisability of Generic Structures

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This is a joint work with Koichiro Ikeda and Akito Tsuboi.
Dimension Function

Let $\mathcal{L}$ be a finite relational language, $\alpha = (\alpha_R : R \in \mathcal{L})$ with $0 < \alpha_R \leq 1$ for each $R \in \mathcal{L}$. For a finite $\mathcal{L}$-structure $A$, let

$$\delta(A) = \delta_\alpha(A) = |A| - \sum_{R \in \mathcal{L}} \alpha_R e_R(A)$$

where $e_R(A) = \#\{x \in A^m | A \models R(x)\}$ for an $m$-ary relation $R \in \mathcal{L}$. When we assume that each $R$ in $\mathcal{L}$ is symmetric and irreflexive, we also consider $e_R(A) = \#\{x \in [A]^m | A \models R(x)\}$.
Theorem (Ikeda-K.-Tsuboi; Laskowski)

Let

\[ K_\alpha = \{ A : \text{finite} \mid \delta_\alpha(X) \geq 0 \text{ for all } X \subseteq A \} \].

The theory of the generic model of \( K_\alpha \) is AE-axiomatisable for any \( \alpha = (\alpha_R : R \in \mathcal{L}) \) with \( 0 < \alpha_R \leq 1 \) for each \( R \in \mathcal{L} \).

Laskowski proved the theorem when the \( \alpha_R \) for \( R \in \mathcal{L} \) and 1 are linearly independent over \( \mathbb{Q} \) (including Shelah-Spencer Theory).

We proved it for any case.
**Strong Substructure**

**Definition (Strong Substructure)**

Suppose $A \subseteq_{\text{fin}} B$ ($\mathcal{L}$-structures). $A \leq B$ (A is a **strong substructure** of $B$, or $A$ is **closed** in $B$) if

$$A \subseteq X \subseteq_{\text{fin}} B \Rightarrow \delta(A) \leq \delta(X).$$

With this notation,

$$K_\alpha = \{ A : \text{finite} \mid A \geq \emptyset \}.$$
Generic Structure

Definition (Generic Structure)

Suppose $K \subseteq K_\alpha$.
A countable $L$-structure $M$ is called a generic model of $K$ if

- $A \subseteq_{\text{fin}} M \Rightarrow$ there is $B$ such that $A \subseteq B \subseteq_{\text{fin}} M$ and $B \leq M$;
- $A \subset_{\text{fin}} M \Rightarrow A \in K$;
- for any $A, B$ in $K$ with $A \leq B$,

\[
\begin{align*}
B & \uparrow \\
\lor & \\
A & \longrightarrow M \\
\leq & \quad \\
\end{align*}
\]
An Elementary Property of a Generic Structure

Fact (Baldwin, Shi)
There is a generic model $M$ of $K_\alpha$. For any $A, B$ in $K_\alpha$ with $A \leq B$,

\[ A \rightarrow M \rightarrow B \]

This property can be expressed by a set of $\forall \exists$-formulas in $L$. 
Fact (Baldwin, Shi)

There is a generic model $M$ of $K_\alpha$. For any $A, B$ in $K_\alpha$ with $A \leq B$,

\[ A \rightarrow M \xrightarrow{\leq} B \]

This property can be expressed by a set of $\forall\exists$-formulas in $\mathcal{L}$. 

\[ \forall \exists \]
Theorem (Ikeda-K.-Tsuboi; Laskowski)

An $\mathcal{L}$-structure $M$ is elementarily equivalent to the generic model of $K_\alpha$ if and only if

**Axiom 1** $A \subseteq_{\text{fin}} M \Rightarrow A \in K_\alpha$. i.e., $M \in \bar{K}_\alpha$.

**Axiom 2** For any $A, B$ in $K_\alpha$ with $A \leq B$,
Proposition

Suppose $\alpha$ consists of rational numbers. If $M$ is a countably saturated model of Axioms 1, 2 for $K_\alpha$ then for any $A, B$ in $K_\alpha$ with $A \leq B$,

![Diagram showing the relationship between $A$, $B$, and $M$.]
Proposition

Suppose $\alpha$ consists of rational numbers. If $M$ is a countably saturated model of Axioms 1, 2 for $K_\alpha$ then for any $A, B$ in $K_\alpha$ with $A \leq B$,

We explain the proof later.
Completeness in Rational Case

**Theorem**

If $\alpha$ consists of rational numbers then the set of Axioms 1, 2 for $K_\alpha$ is complete.
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Let $M, N$ be countably saturated models of Axioms 1 and 2 for $K_{\alpha}$. 
Completeness in Rational Case

Theorem

If $\alpha$ consists of rational numbers then the set of Axioms 1, 2 for $K_\alpha$ is complete.

Let $M, N$ be countably saturated models of Axioms 1 and 2 for $K_\alpha$.

Since $\delta$-values are discrete, any finite subset of $M$ has a finite closure in $M$. 
Completeness in Rational Case

**Theorem**

If $\alpha$ consists of rational numbers then the set of Axioms 1, 2 for $K_\alpha$ is complete.

Let $M, N$ be countably saturated models of Axioms 1 and 2 for $K_\alpha$. Since $\delta$-values are discrete, any finite subset of $M$ has a finite closure in $M$. A standard back-and-forth argument shows that there are $M_0 \prec M$ and $N_0 \prec N$ such that $M_0 \cong N_0$. 
We prove the proposition in the following case: $K_\alpha = K_1 \subset$ the class of binary graphs.
$\delta(X) = |X| - e(X)$.
$A = \emptyset \leq b = B$ (singleton).

**Proposition**
Suppose $\alpha$ consists of rational numbers. If $M$ is a countably saturated model of Axioms 1, 2 for $K_\alpha$ then for any $A, B$ in $K_\alpha$ with $A \leq B$, 

\[ B \quad \text{VI} \quad A \quad \leq \quad M \]
"Genericity" of a Saturated Model

\[ \delta(X) = |X| - e(X). \quad A = \emptyset \leq b = B. \quad \delta(A) = 0, \quad \delta(B) = 1. \]

Aim: \[ B \xrightarrow{\leq} M. \]
\[ \delta(X) = |X| - e(X). \ A = \emptyset \leq b = B. \ \delta(A) = 0, \ \delta(B) = 1. \]

**Aim:** \( B \xrightarrow{\leq} M. \)

Consider the following structure \( C: \)

\[ B \]

"Genericity" of a Saturated Model
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Consider the following structure \( C \):

\[ \delta(\bullet - \bullet) = 1 + 1 - 1 = 1 \]
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Completeness of Axioms
Proof of “Genericity” in Rational Case

“Genericity” of a Saturated Model

\[ \delta(X) = |X| - e(X) \]

\[ A = \emptyset \leq b = B. \quad \delta(A) = 0, \quad \delta(B) = 1. \]

Aim: \( B \xrightarrow{\leq} M \).

Consider the following structure \( C \):

\[ B \]

\[ \delta(\text{red}) = 1 + 1 - 1 = 1 \]

\[ \delta(\text{other}) = 1 \]
"Genericity" of a Saturated Model

\[ \delta(X) = |X| - e(X). \]

Let \( A = \emptyset \leq b = B. \delta(A) = 0, \delta(B) = 1. \]

Aim: \( B \xrightarrow{\leq} M. \)

Consider the following structure \( C: \)

\[ \begin{align*}
\delta(\bullet - \bullet) &= 1 + 1 - 1 = 1 \\
\delta(\bullet - \bullet - \bullet - \ldots - \bullet) &= 1 \\
\delta(\bullet - \bullet - \bullet - \ldots - \bullet) &= 1 + 2 - 3 = 0
\end{align*} \]
“Genericity” of a Saturated Model

\[ \delta(X) = |X| - e(X). \quad A = \emptyset \leq b = B. \quad \delta(A) = 0, \quad \delta(B) = 1. \]

Aim: \( B \xrightarrow{\leq} M \).

Consider the following structure \( C \):

Given \( n \), with sufficiently large \( C \), we have \( A \leq B \leq_n C, \quad A \leq C, \quad \delta(C) = \delta(A)(= 0). \)
\[ \delta(X) = |X| - e(X). \]  
\[ A = \emptyset \leq b = B. \]  
\[ \delta(A) = 0, \delta(B) = 1. \]

**Aim:** \[ B \xrightarrow{\leq} M. \]

Consider the following structure \( C \):

Given \( n \), with sufficiently large \( C \), we have \( A \leq B \leq_n C, A \leq C, \delta(C) = \delta(A)(= 0). \)

Here, \( B \leq_n C \) means \( X \subseteq C, |X| \leq n \Rightarrow B \leq BX \).
\[ \delta(X) = |X| - e(X). \quad A = \emptyset \leq b = B. \quad \delta(A) = 0, \quad \delta(B) = 1. \]

**Aim:** \( B \xrightarrow{\leq} M \).

Consider the following structure \( C \):

Given \( n \), with sufficiently large \( C \), we have
\[ A \leq B \leq_n C, \quad A \leq C, \quad \delta(C) = \delta(A) (= 0). \]

By Axiom 2, we can embed \( C \) in \( M \) over \( A \): \( A \leq B \leq_n C \rightarrow M \).
"Genericity" of a Saturated Model

\[ \delta(X) = |X| - e(X). \]
\[ A = \emptyset \leq b = B. \]
\[ \delta(A) = 0, \delta(B) = 1. \]

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Consider the following structure \( C \):

Given \( n \), with sufficiently large \( C \), we have
\[ A \leq B \leq_n C, \quad A \leq C, \quad \delta(C) = \delta(A)(= 0). \]

By Axiom 2, we can embed \( C \) in \( M \) over \( A \):
\[ A \leq B \leq_n C \rightarrow M. \]

Since \( \delta(C) = \delta(A) \), this is a strong embedding.
\[ B \leq_n C \leq M. \]

Therefore, \( B \leq^n M \).
\( \delta(X) = |X| - e(X) \). \( A = \emptyset \leq b = B \). \( \delta(A) = 0 \), \( \delta(B) = 1 \).

**Aim:** \( B \xrightarrow{\leq} M \).

Consider the following structure \( C \):

![Diagram](image)

Given \( n \), with sufficiently large \( C \), we have
\( A \leq B \leq_n C \), \( A \leq C \), \( \delta(C) = \delta(A)(= 0) \).

By Axiom 2, we can embed \( C \) in \( M \) over \( A \): \( A \leq B \leq_n C \rightarrow M \).

Since \( \delta(C) = \delta(A) \), this is a strong embedding. \( B \leq_n C \xrightarrow{\leq} M \).

Therefore, \( B \xrightarrow{\leq_n} M \).

\( B \xrightarrow{\leq} M \) by saturation of \( M \).
When $\alpha$ consists of rational numbers, the following lemma is essential for the “Genericity” of saturated models of Axioms 1, 2 for $K_\alpha$.

**Lemma (Extension Property)**

Suppose $A, B \in K_\alpha$ and $A \leq B$. Then for any natural number $n$, there is $C \in K_\alpha$ such that $B \leq_n C$, $A \leq C$ and $\delta(C/A) = 0$.

In the case $\alpha = 1$,  is used to make $B \leq_n C$ and  is used to reduce $\delta(C)$ so that $\delta(C/A) = 0$. 
Definition (*s-component*)

Suppose $0 \leq s \leq 2$. A triple $(E, a, b)$ with $a, b \in E$ is an *s-component* if

For any non-empty substructure $X$ of $E$,

1. $\delta(X) \geq 1$ if $a \notin X$ or $b \notin X$,
2. $\delta(X) \geq s$ if $a, b \in X$, and
3. $\delta(E) = s$.

It is called *proper* if $\delta(E|ab) = 2$. 
Argument for Lemma

**Definition (s-component)**

Suppose $0 \leq s \leq 2$. A triple $(E, a, b)$ with $a, b \in E$ is an $s$-component if

For any non-empty substructure $X$ of $E$,

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2. $\delta(X) \geq s$ if $a, b \in X$, and
3. $\delta(E) = s$.

It is called proper if $\delta(E|ab) = 2$.

When $\delta(X) = |X| - e(X)$, $\bullet \rightarrow \bullet$ is a 1-component,

$\bullet \rightarrow \bullet \rightarrow \bullet$ is a proper 1-component, and $\bullet \rightarrow \bullet \rightarrow \bullet$ is a 0-component.
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Argument for Lemma

Definition (s-component)

Suppose $0 \leq s \leq 2$. A triple $(E, a, b)$ with $a, b \in E$ is an $s$-component if
For any non-empty substructure $X$ of $E$,

1. $\delta(X) \geq 1$ if $a \not\in X$ or $b \not\in X$,
2. $\delta(X) \geq s$ if $a, b \in X$, and
3. $\delta(E) = s$.

It is called proper if $\delta(E|ab) = 2$.

$x \leq E \setminus \{a\}$ for each $x \in E \setminus \{a\}$.
$x \leq E \setminus \{b\}$ for each $x \in E \setminus \{b\}$. 
Suppose $0 \leq s \leq 2$. A triple $(E, a, b)$ with $a, b \in E$ is an $s$-component if:

1. $\delta(X) \geq 1$ if $a \notin X$ or $b \notin X$,
2. $\delta(X) \geq s$ if $a, b \in X$, and
3. $\delta(E) = s$.

It is called proper if $\delta(E|ab) = 2$.

$x \leq E \setminus \{a\}$ for each $x \in E \setminus \{a\}$.
$x \leq E \setminus \{b\}$ for each $x \in E \setminus \{b\}$.
If $s \geq 1$, $x \leq E$ for each $x \in E$. 
1. From \( \bullet \) and \( \bullet \) with \( s, t \geq 1 \),
   an \( s \text{-cmp.} \) and a \( t \text{-cmp.} \)
   we get a proper \( (s + t - 1) \text{-cmp.} \).
1. From \( \bullet \) and \( \bullet \) with \( s, t \geq 1 \),
   we get a proper \((s + t - 1)\)-cmp.

2. From \( \bullet \) and \( \bullet \) with \( s, t \geq 1 \),
   we get a proper \((s + t - 2)\)-cmp.
1. From \( \bullet \) and \( \bullet \) with \( s, t \geq 1 \), we get a proper \( (s + t - 1) \)-cmp.

2. From \( \bullet \) and \( \bullet \) with \( s, t \geq 1 \), we get a proper \( (s + t - 2) \)-cmp.

3. Suppose \( r_0 = r_1 + r_2 + \cdots + r_k - m \) with an integer \( m \) and \( 1 \leq r_i < 2 \) for each \( i \). If there is a proper \( r_i \)-component for each \( i = 1, 2, \ldots, k \) then there is a proper \( r_0 \)-component.
Consider the case of binary graphs with \( \delta(X) = |X| - \frac{3}{5} e(X) \) (\( \alpha = 3/5 \)).

1-components and 4/5-components (\( 4/5 = 1 - 1/5 \)) will play rolls for 1-components and 0-components in the case \( \alpha = 1 \).

We can reduce the \( \delta \)-rank by 1/5 using 4/5-components.
A Case of Binary Graphs ($\alpha > 1/2$)

Consider the case of binary graphs with $\delta(X) = |X| - \frac{3}{5}e(X)$ ($\alpha = 3/5$).

1-components and 4/5-components ($4/5 = 1 - 1/5$) will play rolls for 1-components and 0-components in the case $\alpha = 1$. We can reduce the $\delta$-rank by 1/5 using 4/5-components. 

\[\bullet \longrightarrow \bullet\] is a 7/5-component.
Consider the case of binary graphs with $\delta(X) = |X| - \frac{3}{5}e(X)$ ($\alpha = 3/5$). 1-components and $4/5$-components ($4/5 = 1 - 1/5$) will play rolls for 1-components and 0-components in the case $\alpha = 1$. We can reduce the $\delta$-rank by $1/5$ using $4/5$-components.

- is a $7/5$-component.
- is a proper $9/5$-component ($2 \cdot 7/5 - 1 = 9/5$).
Consider the case of binary graphs with $\delta(X) = |X| - \frac{3}{5} e(X)$ ($\alpha = 3/5$). 1-components and 4/5-components ($4/5 = 1 - 1/5$) will play rolls for 1-components and 0-components in the case $\alpha = 1$. We can reduce the $\delta$-rank by 1/5 using 4/5-components.

- $\bigcirc - \bigcirc$ is a 7/5-component.
- $\bigcirc - \bigcirc - \bigcirc$ is a proper 9/5-component ($2 \cdot 7/5 - 1 = 9/5$).

\begin{align*}
5 \cdot 9/5 - 8 &= 1 \\
1 + 9/5 - 2 &= 4/5
\end{align*}
A Case of Binary Graphs \((\alpha \leq 1/2)\)

Consider the case of binary graphs with \(\delta(X) = |X| - \frac{3}{7}e(X)\) \((\alpha = 3/7)\).

1-components and 6/7-components \((6/7 = 1 - 1/7)\) will play rolls for 1-components and 0-components in the case \(\alpha = 1\).

\(\delta(K_2), \delta(K_3), \delta(K_4) > 1, \delta(K_5) = 5 - (3/7) \cdot 10 = 5/7 < 1.\)

\[\delta(K_5) + \alpha = 5/7 + 3/7 = 8/7\]

There is a proper 1-component by \(7 \cdot 8/7 - 7 = 1.\)

There is a proper 6/7-component by \(6 \cdot 8/7 - 6 = 6/7.\)
Once you get a 1-component with a single relation $R$ with coefficient $\alpha_R$, you get a proper $(1 + \alpha_R)$-component by removing one “edge”.

**Example.** $\delta(X) = |X| - \frac{2}{5}e(X)$. $K_5$ is a 1-component. 
$\delta(K_5) = 5 - \frac{2}{5} \cdot 10 = 1$.

$\delta(K_5) + \alpha = 1 + \frac{2}{5}$
An $s$-component in binary graph can be transformed into an $s$-component in hypergraph with the same domain if the domain is big enough. Transform each edge in the binary graph to a $m$-hyperedge by a 1-1 map $f : [D]^2 \rightarrow [D]^m$ such that $x, y \in f(\{x, y\})$ for any $\{x, y\} \in [D]^2$. 

![binary graph to hypergraph transformation](image.png)
Suppose $\mathcal{L} = \{ R_1, R_2, \ldots, R_l \}$ and $\alpha_{R_i} = n_i/d_i$ with relatively prime natural numbers $n_i, d_i$ for each $i$. For each $i$, there is a proper $(1 + n_i/d_i)$-component with only $R_i$-edges, and thus there is a proper $(1 + 1/d_i)$-component. Let

$$d = \text{the least common multiple of } d_1, d_2, \ldots, d_l$$

Then we have a proper $(1 + 1/d)$-component and thus a $(1 - 1/d)$-component in $K_\alpha$. 
Lemma (Extension Property)
Assume $\alpha$ consists of rational numbers. Suppose $A, B \in K_\alpha$ and $A \leq B$. Then for any natural number $n$, there is $C \in K_\alpha$ such that $B \leq_n C$, $A \leq C$ and $\delta(C/A) = 0$.

Proposition
Suppose $\alpha$ consists of rational numbers. If $M$ is a countably saturated model of Axioms 1, 2 for $K_\alpha$ then for any $A, B$ in $K_\alpha$ with $A \leq B$, 
Proposition

Suppose $\alpha$ consists of rational numbers. If $M$ is a countably saturated model of Axioms 1, 2 for $K_\alpha$ then for any $A, B$ in $K_\alpha$ with $A \leq B$,

\[
\begin{array}{c}
A \\
\leq \\
\longrightarrow \\
M
\end{array}
\]

Theorem

Suppose $\alpha$ consists of rational numbers. The set of Axioms 1, 2 for $K_\alpha$ is complete.
When $\alpha$ may have some irrational element, we can prove the following:

**Lemma (Approximating Extension Property)**

Suppose $A, B \in K_{\alpha}$ and $A \leq B$. Then for any natural number $n$ and for any real number $\varepsilon > 0$, there is $C \in K_{\alpha}$ such that $B \leq_{n} C$, $A \leq C$ and $\delta(C/A) < \varepsilon$.

If $\alpha$ has irrational element, then there are proper $(1 + t)$-component and proper $(1 - t')$-component in $K_{\alpha}$ with $t, t' > 0$ arbitrarily close to 0.
General Case

**Definition**

For any $\mathcal{L}$-structures $A, B$ with $A \subseteq B$, $A \preceq B$ if $\delta(A \cap X) \leq \delta(X)$ for any $X \subseteq_{\text{fin}} B$.

**Proposition**

If $M$ is an $\aleph_1$-saturated model of Axioms 1, 2 for $K_\alpha$ then for any countable structures $A, B$ in $\overline{K}_\alpha$ with $A \preceq B$, $A \preceq M \overset{\frown}{\ni} B$. 
On AE-Axiomatisability of Generic Structures
Completeness of Axioms
General Case (Irrational Coefficients May Exist)

**General Case**

**Proposition**
If $M$ is an $\aleph_1$-saturated model of Axioms 1, 2 for $K_\alpha$ then for any countable structures $A, B$ in $\overline{K}_\alpha$ with $A \leq B$,

**Theorem**
The set of Axioms 1, 2 for $K_\alpha$ is complete.
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The Paper Available

The paper is available at

http://www2.kobe-u.ac.jp/~kikyo/

There will 10th Asian Logic Conference at Kobe, Japan from 1st – 6th September, 2008.
Please visit:

http://kurt.scitec.kobe-u.ac.jp/ALC10/