On AE-Axiomatisability of Generic Structures

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Outline

Acknowledgement

This is a joint work with Koichiro Ikeda and Akito Tsuboi.

Dimension Function

Let $\mathcal{L}$ be a finite relational language,
$\alpha = (\alpha_R : R \in \mathcal{L})$ with $0 < \alpha_R \leq 1$ for each $R \in \mathcal{L}$.
For a finite $\mathcal{L}$-structure A, let

$$\delta(A) = \delta_\alpha(A) = |A| - \sum_{R \in \mathcal{L}} \alpha_R e_R(A)$$

where $e_R(A) = \#\{x \in A^m \mid A \models R(x)\}$ for an m-ary relation $R \in \mathcal{L}$.
When we assume that each R in $\mathcal{L}$ is symmetric and irreflexive, we also consider $e_R(A) = \#\{x \in [A]^m \mid A \models R(x)\}$.

Result

Theorem (Ikeda-K.-Tsuboi; Laskowski)

Let

$$\mathbf{K}_\alpha = \{A : \text{finite} \mid \delta_\alpha(X) \geq 0 \text{ for all } X \subseteq A\}.$$

The theory of the generic model of $\mathbf{K}_\alpha$ is AE-axiomatisable for any $\alpha = (\alpha_R : R \in \mathcal{L})$ with $0 < \alpha_R \leq 1$ for each $R \in \mathcal{L}$.

Laskowski proved the theorem when the α_R for $R \in \mathcal{L}$ and 1 are linearly independent over $\mathbb{Q}$ (including Shelah-Spencer Theory).
We proved it for any case.

Strong Substructure

Definition (Strong Substructure)

Suppose $A \subseteq_{\text{fin}} B$ ($\mathcal{L}$-structures).
$A \leq B$ (A is a strong substructure of B, or A is closed in B) if

$$A \subseteq X \subseteq_{\text{fin}} B \Rightarrow \delta(A) \leq \delta(X).$$

With this notation,

$$\mathbf{K}_\alpha = \{A : \text{finite} \mid A \geq \emptyset\}.$$

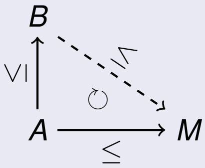
Generic Structure

Definition (Generic Structure)

Suppose $\mathbf{K} \subseteq \mathbf{K}_\alpha$.

A countable $\mathcal{L}$-structure M is called a generic model of $\mathbf{K}$ if

- $A \subseteq_{\text{fin}} M \Rightarrow$ there is B such that $A \subseteq B \subseteq_{\text{fin}} M$ and $B \leq M$;
- $A \subset_{\text{fin}} M \Rightarrow A \in \mathbf{K}$;
- for any A, B in $\mathbf{K}$ with $A \leq B$,



An Elementary Property of a Generic Structure

Fact (Baldwin, Shi)

There is a generic model M of $\mathbf{K}_\alpha$.
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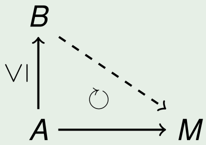

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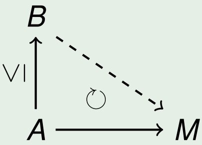


This property can be expressed by a set of $\forall\exists$-formulas in $\mathcal{L}$.

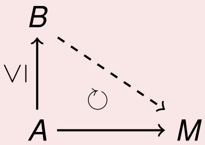
Axioms

Theorem (Ikeda-K.-Tsuboi; Laskowski)

An $\mathcal{L}$-structure M is elementarily equivalent to the generic model of $\mathbf{K}_\alpha$ if and only if

Axiom 1 $A \subseteq_{\text{fin}} M \Rightarrow A \in \mathbf{K}_\alpha$. i.e., $M \in \overline{\mathbf{K}}_\alpha$.

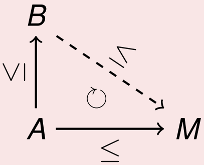
Axiom 2 For any A, B in $\mathbf{K}_\alpha$ with $A \leq B$,



“Genericity” of the Saturated Models

Proposition

Suppose α consists of rational numbers. If M is a countably saturated model of Axioms 1, 2 for $\mathbf{K}_\alpha$ then for any A, B in $\mathbf{K}_\alpha$ with $A \leq B$,



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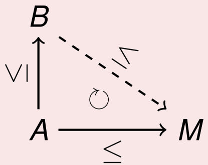

We explain the proof later.

Completeness in Rational Case

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If α consists of rational numbers then the set of Axioms 1, 2 for $\mathbf{K}_\alpha$ is complete.

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Since δ-values are discrete, any finite subset of M has a finite closure in M.
A standard back-and-forth argument shows that there are $M_0 \prec M$ and $N_0 \prec N$ such that $M_0 \cong N_0$.

“Genericity” of a Saturated Model

We prove the proposition in the following case:
$\mathbf{K}_\alpha = \mathbf{K}_1 \subset$ the class of binary graphs.
$\delta(X) = |X| - e(X)$.
$A = \emptyset \leq b = B$ (singleton).

Proposition

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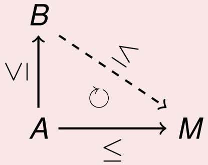

“Genericity” of a Saturated Model

$\delta(X) = |X| - e(X)$. $A = \emptyset \leq b = B$. $\delta(A) = 0$, $\delta(B) = 1$.

Aim: $B \xrightarrow{\leq} M$.

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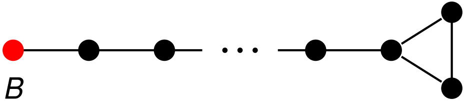

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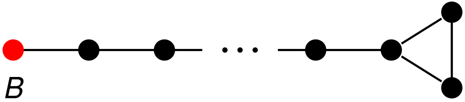
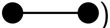


$\delta($  $) = 1 + 1 - 1 = 1$

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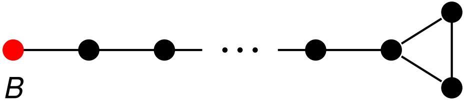
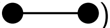
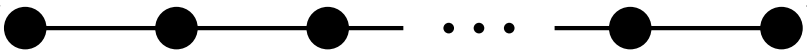


$\delta($$) = 1 + 1 - 1 = 1$

$\delta($$) = 1$

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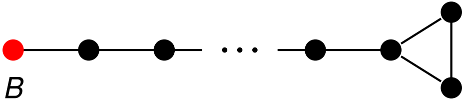
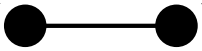
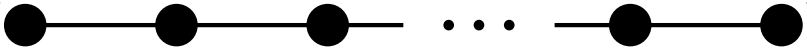
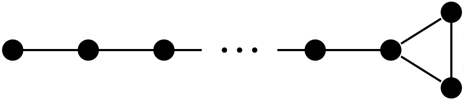


$\delta($  $) = 1 + 1 - 1 = 1$

$\delta($  $) = 1$

$\delta($  $) = 1 + 2 - 3 = 0$

“Genericity” of a Saturated Model

$\delta(X) = |X| - e(X)$. $A = \emptyset \leq b = B$. $\delta(A) = 0$, $\delta(B) = 1$.

Aim: $B \stackrel{\leq}{\longrightarrow} M$.

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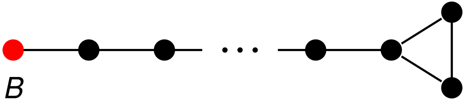

Given n, with sufficiently large C, we have
$A \leq B \leq_n C$, $A \leq C$, $\delta(C) = \delta(A)(= 0)$.

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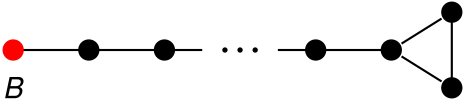

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Here, $B \leq_n C$ means $X \subseteq C$, $|X| \leq n \Rightarrow B \leq BX$.

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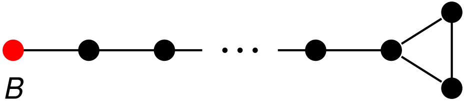

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By Axiom 2, we can embed C in M over A: $A \leq B \leq_n C \rightarrow M$.

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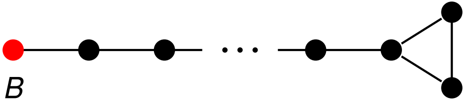

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Since $\delta(C) = \delta(A)$, this is a strong embedding. $B \leq_n C \xrightarrow{\leq} M$.
Therefore, $B \xrightarrow{\leq_n} M$.

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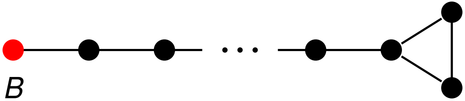

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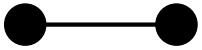
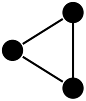
Since $\delta(C) = \delta(A)$, this is a strong embedding. $B \leq_n C \xrightarrow{\leq} M$.
Therefore, $B \xrightarrow{\leq_n} M$.
$B \xrightarrow{\leq} M$ by saturation of M.

A Property of Class $\mathbf{K}_\alpha$

When α consists of rational numbers, the following lemma is essental for the “Genericity” of saturated moodels of Axioms 1, 2 for $\mathbf{K}_\alpha$.

Lemma (Extension Property)

Suppose $A, B \in \mathbf{K}_\alpha$ and $A \leq B$. Then for any natural number n, there is $C \in \mathbf{K}_\alpha$ such that $B \leq_n C$, $A \leq C$ and $\delta(C/A) = 0$.

In the case $\alpha = 1$,  is used to make $B \leq_n C$ and 

is used to reduce $\delta(C)$ so that $\delta(C/A) = 0$.

Argument for Lemma

Definition (s-component)

Suppose $0 \leq s \leq 2$. A triple (E, a, b) with $a, b \in E$ is an s-component if
For any non-empty substructure X of E,

1. $\delta(X) \geq 1$ if $a \notin X$ or $b \notin X$,
2. $\delta(X) \geq s$ if $a, b \in X$, and
3. $\delta(E) = s$.

It is called proper if $\delta(E|ab) = 2$.

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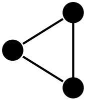
When $\delta(X) = |X| - e(X)$,  is a 1-component,

 is a proper 1-component, and  is a 0-component.

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$x \leq E \setminus \{a\}$ for each $x \in E \setminus \{a\}$.
$x \leq E \setminus \{b\}$ for each $x \in E \setminus \{b\}$.

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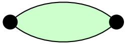
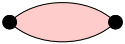
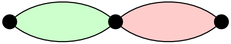
$x \leq E \setminus \{a\}$ for each $x \in E \setminus \{a\}$.
$x \leq E \setminus \{b\}$ for each $x \in E \setminus \{b\}$.
If $s \geq 1$, $x \leq E$ for each $x \in E$.

Arithmetic of Components

1. From  an s-cmp. and  a t-cmp. with $s, t \geq 1$,

we get a proper $(s + t - 1)$-cmp. 

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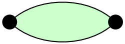
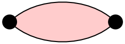
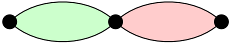
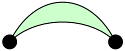
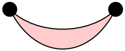
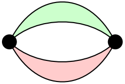
1. From  an s-cmp. and  a t-cmp. with $s, t \geq 1$,
   we get a proper $(s + t - 1)$-cmp. 
2. From  a proper s-cmp. and  a proper t-cmp. with
   $s, t \geq 1$, we get a proper $(s + t - 2)$-cmp. 

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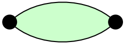
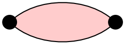
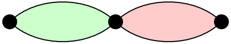
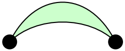
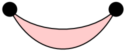
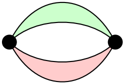
1. From  an s-cmp. and  a t-cmp. with $s, t \geq 1$,

   we get a proper $(s + t - 1)$-cmp. 

2. From  a proper s-cmp. and  a proper t-cmp. with

   $s, t \geq 1$, we get a proper $(s + t - 2)$-cmp. 

3. Suppose $r_0 = r_1 + r_2 + \cdots + r_k - m$ with an integer m and $1 \leq r_i < 2$ for each i. If there is a proper r_i-component for each $i = 1, 2, \ldots, k$ then there is a proper r_0-component.

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Consider the case of binary graphs with $\delta(X) = |X| - \frac{3}{5}e(X)$ ($\alpha = 3/5$).

1-components and $4/5$-components ($4/5 = 1 - 1/5$) will play rolls for 1-components and 0-components in the case $\alpha = 1$.

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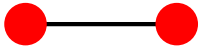 is a $7/5$-component.
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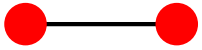 is a $7/5$-component.

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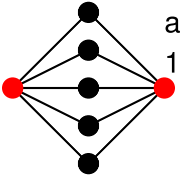
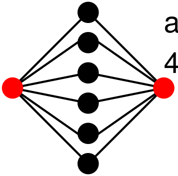


a proper 1-component

$5 \cdot 9/5 - 8 = 1$



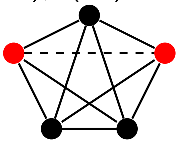
a proper 4/5-component

$1 + 9/5 - 2 = 4/5$

A Case of Binary Graphs ($\alpha \leq 1/2$)

Consider the case of binary graphs with $\delta(X) = |X| - \frac{3}{7}e(X)$ ($\alpha = 3/7$).
1-components and 6/7-components ($6/7 = 1 - 1/7$) will play rolls for 1-components and 0-components in the case $\alpha = 1$.

$\delta(K_2), \delta(K_3), \delta(K_4) > 1, \delta(K_5) = 5 - (3/7) \cdot 10 = 5/7 < 1.$



a proper 8/7-component.

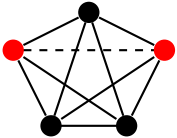
$\delta(K_5) + \alpha = 5/7 + 3/7 = 8/7$

There is a proper 1-component by $7 \cdot 8/7 - 7 = 1$.
There is a proper 6/7-component by $6 \cdot 8/7 - 6 = 6/7$.

$(1+\alpha_R)$-Components

Once you get a 1-component with a single relation R with coefficient α_R, you get a proper $(1+\alpha_R)$-component by removing one “edge”.

Example. $\delta(X) = |X| - \frac{2}{5}e(X)$. K_5 is a 1-component.
$\delta(K_5) = 5 - \frac{2}{5} \cdot 10 = 1$.



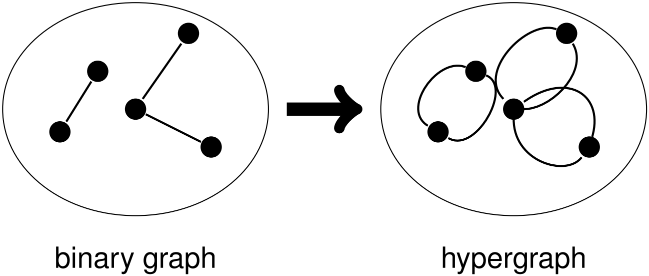
a proper (1+2/5)-component.

$\delta(K_5) + \alpha = 1 + 2/5$

Components in Hypergraphs

An s-component in binary graph can be transformed into an s-component in hypergraph with the same domain if the domain is big enough.

Transform each edge in the binary graph to a m-hyperedge by a 1-1 map $f : [D]^2 \to [D]^m$ such that $x, y \in f(\{x, y\})$ for any $\{x, y\} \in [D]^2$.



General Case with Rational Coefficient

Suppose $\mathcal{L} = \{R_1, R_2, \ldots, R_l\}$ and $\alpha_{R_i} = n_i/d_i$ with relatively prime natural numbers n_i, d_i for each i.

For each i, there is a proper $(1 + n_i/d_i)$-component with only R_i-edges, and thus there is a proper $(1 + 1/d_i)$-component.

Let

$$d = \text{the least common multiple of } d_1, d_2, \ldots, d_l$$

Then we have a proper $(1 + 1/d)$-component and thus a $(1 - 1/d)$-component in $\mathbf{K}_\alpha$.

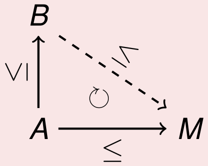
General Case with Rational Coefficient

Lemma (Extension Property)

Assume α consists of rational numbers. Suppose $A, B \in \mathbf{K}_\alpha$ and $A \leq B$. Then for any natural number n, there is $C \in \mathbf{K}_\alpha$ such that $B \leq_n C$, $A \leq C$ and $\delta(C/A) = 0$.

Proposition

Suppose α consists of rational numbers. If M is a countably saturated model of Axioms 1, 2 for $\mathbf{K}_\alpha$ then for any A, B in $\mathbf{K}_\alpha$ with $A \leq B$,



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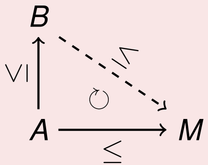


Theorem

Suppose α consists of rational numbers.
The set of Axioms 1, 2 for $\mathbf{K}_\alpha$ is complete.

General Case

When α may have some irrational element, we can prove the following:

Lemma (Approximating Extension Property)

Suppose $A, B \in \mathbf{K}_\alpha$ and $A \leq B$. Then for any natural number n and for any real number $\varepsilon > 0$, there is $C \in \mathbf{K}_\alpha$ such that $B \leq_n C$, $A \leq C$ and $\delta(C/A) < \varepsilon$.

If α has irrational element, then there are proper $(1 + t)$-component and proper $(1 - t')$-component in $\mathbf{K}_\alpha$ with $t, t' > 0$ arbitrarily close to 0.

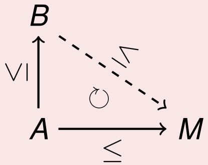
General Case

Definition

For any $\mathcal{L}$-structures A, B with $A \subseteq B$,
$A \leq B$ if $\delta(A \cap X) \leq \delta(X)$ for any $X \subseteq_{\text{fin}} B$.

Proposition

If M is an $\aleph_1$-saturated model of Axioms 1, 2 for $\mathbf{K}_\alpha$ then for any countable structures A, B in $\overline{\mathbf{K}}_\alpha$ with $A \leq B$,



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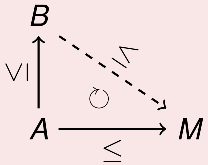

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The Paper Available

The paper is available at

http://www2.kobe-u.ac.jp/~kikyo/

There will 10th Asian Logic Conference at Kobe, Japan from 1st – 6th September, 2008.
Please visit:

http://kurt.scitec.kobe-u.ac.jp/ALC10/