Simplicity in finitary abstract elementary classes

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Question: Can we build an analogue of the F.O. independence calculus in (some subclass of) Abstract Elementary Classes?

Hence, in the title, *simplicity* refers to the existence of a well-behaved notion of independence, which works over *sets*.

To start with, we might also assume stability, and even that will not be enough.

Independence in non-elementary classes

- Stable **homogeneous classes** admit a notion of independence over strongly $\kappa(\mathfrak{M})$-saturated models, and with simplicity also over sets.

- (ω-stable) **excellent classes** admit a notion of independence over (ω-saturated) models, and with simplicity also over sets.

For these classes there is a categoricity transfer theorem and a Main Gap theorem.

Authors: Keisler, Shelah, Grossberg, Hart, Hyttinen, Lessmann etc

Also:

Hyttinen, Lessmann, Shelah: *Interpreting groups and fields in some nonelementary classes*, Journal of Mathematical Logic, 2005.

Conclusion: A lot can be done without compactness.

A few words about homogeneous and excellent classes

- There is no compactness

- Both motivated by the study of sentences in $L_{\omega_1\omega}$ (or $L_{\kappa^+\omega}$), but we only restrict to sentences where the class of structures has 'good' properties

- We study F.O. types, but not all types are allowed. There is a monster model.

- The two contexts are incomparable, for example, in the h.c. we assume more amalgamation and in e.c. more stability

We study an **abstract elementary class** $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ with AP, JEP and arbitrarily large models.

There $\mathbb{K}$ is a class of structures and $\preccurlyeq_{\mathbb{K}}$ is 'elementary substructure' -relation with certain properties.

(closed under chains, downward Löwenheim-Skolem etc)

For $\mathcal{A}, \mathcal{B} \in \mathbb{K}$ a $\mathbb{K}$-**embedding** is an embedding $f : \mathcal{A} \to \mathcal{B}$ with

$$f(\mathcal{A}) \preccurlyeq_{\mathbb{K}} \mathcal{B}.$$

In this context we have a κ-model-homogeneous **monster model**, that is,

- For any $\mathcal{A} \in \mathbb{K}, |A| < \kappa$, there is a $\mathbb{K}$-embedding $f : \mathcal{A} \to \mathfrak{M}$.

- For $\mathcal{A} \preccurlyeq_{\mathbb{K}} \mathfrak{M}, |\mathcal{A}| < \kappa$, any K-embedding

$$f : \mathcal{A} \to \mathfrak{M}$$

  extends to an automorphism of $\mathfrak{M}$.

Definition: We say that a set $A \subset \mathfrak{M}$ is *bounded*, if $|A| < \kappa$.

We say that $\mathcal{A}$ is a *model* if it's bounded, $\mathcal{A} \in \mathbb{K}$ and $\mathcal{A} \preccurlyeq_{\mathbb{K}} \mathfrak{M}$.

We also assume countable Löwenheim-Skolem number, that is,

- For any subset $A \subset \mathfrak{M}$ there is $\mathcal{A} \preccurlyeq_{\mathbb{K}} \mathfrak{M}$ with

$$|\mathcal{A}| \leq |A| + \aleph_0.$$

Galois type:

$$\mathrm{tp}^g(\bar{a}/A) = \mathrm{tp}^g(\bar{b}/A) \text{ iff}$$

$$\text{there is } f \in \mathrm{Aut}(\mathfrak{M}/A) \text{ such that } f(\bar{a}) = \bar{b}.$$

Note: Types over *sets* make only sense for subsets of the monster model, since we don't have amalgamation for sets.

We also define a notion of type with *finite character*, so called **weak type**:

$$\mathrm{tp}^w(\bar{a}/A) = \mathrm{tp}^w(\bar{b}/A) \text{ iff}$$

$$\mathrm{tp}^g(\bar{a}/A_0) = \mathrm{tp}^g(\bar{b}/A_0) \text{ for all } \textit{finite } A_0 \subset A.$$

Note: In excellent and homogeneous classes these two notions agree over *models*.

We say that a sequence $(\bar{a}_i)_{i<\lambda}$ is *strongly A-indiscernible* if

- it can be extended to $(\bar{a}_i)_{i<\lambda'}$ for any bounded $\lambda' > \lambda$.

- Any order-preserving partial $f : \lambda \to \lambda$ extends to $F \in \mathrm{Aut}(\mathfrak{M}/A)$ mapping each $\bar{a}_i$ to $\bar{a}_{f(i)}$.

Lemma 1 *For any bounded $A \subset \mathfrak{M}$ and $X \subset \mathfrak{M}$ s.t.*

$$|X| \geq \beth_{2^{(|A|+LS(\mathbb{K}))^+}}$$

and any finite n there is a strongly A-indiscernible $(a_i)_{i<\omega}$ s.t. $(a_0, ..., a_n) \subset X$.

We say that $\text{tp}^w(\bar{a}/A)$ **Lascar splits** over finite $E \subset \mathcal{A}$ if there is a strongly E-indiscernible sequence $(\bar{a}_i)_{i<\omega}$ such that $\bar{a}_0, \bar{a}_1 \in A$ and

$$\text{tp}^g(\bar{a}_0/E \cup \bar{a}) \neq \text{tp}^g(\bar{a}_1/E \cup \bar{a}).$$

We write that

$$\bar{a} \downarrow_A B$$

if there is finite $E \subset A$ such that for each $D \supseteq B$ there is $\bar{b}$ realizing $\text{tp}^w(\bar{a}/A \cup B)$ such that $\text{tp}^w(\bar{b}/D)$ does not Lascar split over E.

We say that $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is **simple** if for each tuple $\bar{a}$ and *finite set* A we have that

$$\bar{a} \downarrow_A A$$

.

Note that requiring this for *models* A is a different thing.

We define that $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is (weakly) λ-stable, if the number of *weak types* over a set of size $\leq \lambda$ is $\leq \lambda$.

We define that that $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is **superstable** if it is weakly λ-stable for some λ and there are no $\bar{a}$ and finite A_n, $n < \omega$ such that

- $\bigcup_{n<\omega} A_n$ is a model
- $\bar{a} \downarrow_{A_n} A_{n+1}$ for all $n < \omega$.

A practise Lemma: Assume that $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is simple and weakly λ-stable for some λ. Then $\downarrow$ satisfies

- invariance, monotonicity, extension (by definition)
- symmetry over finite sets
- transitivity
- stationarity for *Lascar strong types*

We "almost" get results such that

- simplicity and ω-stability imply superstability

- simplicity and superstability imply all the usual properties of independence, stationarity for Lascar strong types

- ω-stability, ‘tameness’ for types and finite U-rank imply simplicity

- simplicity and ‘tameness’ for types imply a categoricity transfer theorem

What "almost" means: We also need to assume

1. *Finite character* of $\preccurlyeq_{\mathbb{K}}$.

2. *Tarski-Vaught property* for $\preccurlyeq_{\mathbb{K}}$
   (This also follows from 1. and weak ω-stability.)

Note: These hold if $\preccurlyeq_{\mathbb{K}}$ is given by a countable fragment of $L_{\omega_1\omega}$, these hold in homogeneous and excellent classes

Questions: Are these the 'right' generalizations of such notions? Are there other notions, are there many different notions for different 'applications'?

But: There is a $L_{\omega_1\omega}$-definable class which is **not simple**, although it is homogeneous, excellent and *totally categorical* (hence also ω-stable)

Also: If $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ would admit *any* notion of independence which would satisfy

- invariance, monotonicity, extension, transitivity
- local character (for any $\bar{a}$,B there is *finite* $E \subset B$ s.t. $\bar{a} \downarrow_E B$)
- bounded number of free extensions of a type (over a finite set)

then it would be simple.