Dependent Fields have no Artin-Schreier Extensions

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Introduction

In [7], Macintyre showed that an infinite ω-stable field is algebraically closed; this was subsequently generalized by Cherlin and Shelah to the superstable case, and commutativity need not be assumed but follows [3].
It is known [8] that separably closed fields are stable; the converse has been conjectured early on, but little progress has been made.
In 1992, Thomas Scanlon noted that an infinite stable field of characteristic p at least has no Artin-Schreier extensions, and hence has no finite Galois extension of degree divisible by p.
Here we show the same for dependent (NIP) infinite fields.

Dependent theories

Definition

A theory T has the independence property if there is a formula $\phi(\bar{x}, \bar{y})$ and tuples $(\bar{a}_i : i \in \omega)$ and $(\bar{b}_A : A \subset \omega)$ such that $\models \phi(\bar{a}_i, \bar{b}_A)$ if and only if $i \in A$.

Definition

A theory T is dependent if it does not have the independence property (also known as NIP).

Example

Every stable theory is NIP as well as any o-minimal, c-minimal (and many others).

Dependent groups

Definition

A group G is dependent if $Th(G)$ is dependent.

We use only the following theorem about dependent groups:

Theorem (Baldwin-Saxl)

[1] Let G be a dependent group. Given a family of uniformly defined subgroups, there is a number $n < \omega$ such that any finite intersection of groups from this family is already an intersection of n of them.

Artin-Schreier theory

Let k be a field of characteristic p. Let $\wp$ be the polynomial $X^p - X$.

Theorem (Artin-Schreier)

1. *Given $a \in k$, either the polynomial $\wp - a$ has a root in k, in which case all its root are in k, or it is irreducible. In the latter case, if α is a root then $k(\alpha)$ is cyclic of degree p over k.*
2. *Conversely, let K be a cyclic extension of k of degree p. Then there exists $\alpha \in K$ such that $K = k(\alpha)$ and for some $a \in k$, $\wp(\alpha) = a$.*

Such extensions are called Artin-Schreier extensions.

Main Theorem

Theorem

Let K be an infinite dependent field of characteristic $p > 0$. Then K is Artin-Schreier closed - i.e. $\wp$ is onto.

Corollary

(also appeared in [4]) The field $\mathbb{F}_p((t))$ has the independence property - 1 does not have an Artin Schreier root.

Corollary

If L/K is a finite separable extension, then p does not divide $[L : K]$.

Corollary

K contains $\mathbb{F}_p^{alg}$.

Proof of last corollary

Let $k = K \cap \mathbb{F}_p^a$. $\mathbb{F}_p \subseteq k$. Suppose now that $L \subseteq k$ and that M/L is a Galois extension of degree p. Then it is Artin-Schreier, and so there is some α which is an Artin-Schreier root, that generates M over L. As K is Artin-Schreier closed, $\alpha \in K$, in addition, $\alpha \in \mathbb{F}_p^a$, so $\alpha \in k$, i.e. $M \subseteq k$. So it follows that for all $n < \omega$, $\mathbb{F}_{p^{p^n}} \subseteq k$, and so k is infinite. By the estimates of Lang-Weil, k is PAC. But by a theorem of Duret [5, 6.4], as k is relatively algebraically closed in K, it must be also algebraically closed, i.e. $k = \mathbb{F}_p^a$.

Beginning of proof

- Let F be an algebraically closed field containing K.
- For a number $n < \omega$ and $\bar{b} \in F^{n+1}$, we define the algebraic group

$$G_{\bar{b}} = \{(t, x_1, \ldots, x_n) | t = b_i(x_i^p - x_i) \text{ for } 1 \leq i \leq n\}.$$

- $G_{\bar{b}}$ is an algebraic subgroup of $(F, +)^{n+1}$.
- If $\bar{b} \in K$, then by Baldwin-Saxl, for some number n_0, for every finite tuple $\bar{b}$, there is a sub n_0 tuple $\bar{b}'$ such that the projection $\pi : G_{\bar{b}}(K) \to G_{\bar{b}'}(K)$ is onto. (Consider the family of subgroups of $(K, +)$ of the form $\{t | \exists x[t = a(x^p - x)]\}$).

First claim

Claim

Let F be an algebraically closed field. Suppose $\bar{b} \in F^{\times}$ is algebraically independent, then $G_{\bar{b}}$ is a connected group.

Proof.

(sketch) By induction on $n := length(\bar{b})$. For $n = 1$ it's easy. Assume the claim for n, and let $length(\bar{b}) = n + 1$, $\bar{b}' = \bar{b} \upharpoonright n$. $\pi : G_{\bar{b}} \to G_{\bar{b}'}$ is onto. Let $H = G_{\bar{b}}^0$ (the connected component of 0).
$[G_{\bar{b}'} : \pi(H)] \leq [G_{\bar{b}} : H] < \infty$, so $\pi(H) = G_{\bar{b}'}$ by induction. Assume that $H \neq G_{\bar{b}}$. Then for every $(t, \bar{x}) \in G_{\bar{b}'}$ there is exactly one x_{n+1} such that $(t, \bar{x}, x_{n+1}) \in H$.

First claim (cont)

Proof.

So H is a graph of a definable function over $\bar{b}$. Now choose x_i such that $b_i \cdot (x_i^p - x_i) = 1$ for $i = 1, \ldots, n$. Then we can find a solution to $b_{n+1} \cdot (x^p - x) = 1$ in $dcl(b_{n+1}, x_1, \ldots, x_n) = \mathbb{F}_p(b_{n+1}, x_1, \ldots, x_n)_{ins}$. (for a field L, $L_{ins} = \bigcup_{n<\omega} L^{p^{-n}}$). Denote $L = \mathbb{F}_p(b_{n+1}, x_1, \ldots, x_n)$. As x is separable over L, it follows that $x \in L$. By our assumption, b_{n+1} is transcendental over $x_1, \ldots, x_n$. From this one can easily derive a contradiction. □

Second claim

Claim

Let F be an algebraically closed field of characteristic p, and let $f : F \to F$ be an additive polynomial (i.e. $f(x+y) = f(x) + f(y)$). Then f is of the form $\sum a_i x^{p^i}$.
Moreover, if $ker(f) = \mathbb{F}_p$ then $f = (a \cdot (x^p - x))^{p^n}$ for some $n < \omega$, $a \in k^\times$.

Proof.

(sketch) By induction on $deg(f)$. First show that $f'(x)$ is constant (a_0), and prove the first part. The induction base for the moreover: if $a_0 \neq 0$, then $(f, f') = 1$, hence f has no multiple factors, hence $deg(f) = p$, hence $f = a_0 x + a_1 x^p$, but letting $x = 1$ we get $a_0 = -a_1$. □

Important fact

We use the following:

Fact

Let k be a perfect field, and G a closed 1-dimensional connected algebraic subgroup of $(k^{alg}, +)^n$ defined over k, for some $n < \omega$. Then G is isomorphic over k to $(k^{alg}, +)$.

The proof of this fact uses Galois cohomology, and the following, from [6]

Fact

A closed connected subgroup of a $(k^{alg}, +)^n$ group is isomorphic to $(k^{alg}, +)^{dim(G)}$.

Proof continues

Let us use the fact and the 2 claims to finish the proof.

- We may assume that K is $\aleph_0$ saturated.
- Let $k = \bigcap_n K^{p^n}$. k is an infinite perfect field.
- choose $\bar{b} \in k^{n_0+1}$ that are algebraically independent.
- By Baldwin-Saxl, there is some sub n_0 tuple $\bar{b}'$ such that the projection $\pi : G_{\bar{b}}(K) \to G_{\bar{b}'}(K)$ is onto.
- By the first claim, both $G_{\bar{b}}$ and $G'_{\bar{b}}$ are connected. Of course, their dimension is 1.

Proof continues

- By the fact, we know that both these groups are isomorphic over k to $(K^{alg}, +)$.
- So we have some $\nu \in k[X]$ such that

$$\begin{array}{ccc} G_{\bar{b}}(K^{alg}) & \xrightarrow{\pi} & G_{\bar{b}'}(K^{alg}) \\ \downarrow & & \downarrow \\ (K^{alg}, +) & \xrightarrow{\nu} & (K^{alg}, +) \end{array}$$

  commutes.
- As the sides are isomorphisms, defined over $k \subseteq K$, we can restrict them to K. As $\pi \restriction G_{\bar{b}}(K)$ is onto $G_{\bar{b}'}(K)$, then so is $\nu \restriction K$.
- $|ker(\nu)| = p = |ker(\pi)|$ (even when restricted to k).

Proof ends

- Suppose that $0 \neq c \in ker(\nu) \subseteq k$. let $\nu' = \nu \circ m_c$ $(m_c(x) = c \cdot x)$.
- ν' is an additive polynomials over K whose kernel is $\mathbb{F}_p$. So WLOG $ker(\nu) = \mathbb{F}_p$.
- By the second claim, ν is of the form $a \cdot (x^p - x)^{p^n}$ for $a \in K^\times$.
- But ν is onto, hence so is $\wp$ and we are done (given $y \in K$, there is some $x \in K$ such that $a \cdot (x^p - x)^{p^n} = a \cdot y^{p^n}$)

On Simple fields

It is known that bounded (i.e. with only finitely many extensions of each degree) PAC (pseudo-algebraically closed: every absolutely irreducible variety has a rational point) fields are simple [2] and again the converse is conjectured. In 2006, Frank Wagner adapted Scanlon's argument to the simple case and showed that simple fields have only finitely many Artin-Schreier extensions. He even showed this is true when the field is type definable in a simple theory.

As of yet, I do not know the situation with type definable fields in a dependent theory. What is known is that

- For a field type definable in a dependent theory there are either unboundedly many Artin-Schreier extensions, or none.

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