Connected components of groups

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Around classification theory
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It is work in progress.

General references:

- "*G*-compactness and groups" L. Newelski, J.G., to appear in Archive for Mathematical Logic
- "*G*-compactness and groups II" J.G., preprint

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Problem

Find an example of group for which $G^{*00} \neq G^{*\infty}$, or at least $G^{*00}_A \neq G^{*\infty}_A$ for some A.

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- existence ${G^*}^{\infty}$ $\Rightarrow$ existence ${G^*}^{00}$ $\Rightarrow$ existence ${G^*}^0$
- (Wagner) If $\mathrm{Th}(G, \cdot, \dots)$ is supersimple, then ${G^*}^{\infty} = {G^*}^{00} = {G^*}^0$
- If $\mathrm{Th}(G, \cdot, \dots)$ has *NIP*, then ${G^*}^{\infty}$ exists (the proof is based on the similar proof of Shelah, when additionally G is abelian)

Other description of $G^{*\infty}_{A}$

Definition

The set $P \subseteq G$ is *thick* if $P = P^{-1}$ and there is a natural number N such that for any sequence $g_0, \ldots, g_{N-1} \in G$ there exist $i < j < N$ such that

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$$G^{*\infty}_A = \langle X_{\Theta_A} \rangle$$

The strategy

Lemma (Newelski)

$G^{*\infty}_A$ is type definable ($G^{*\infty}_A = G^{*00}_A$) $\iff$ $G^{*\infty}_A = (X_{\Theta_A})^n$ for some natural n

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Lemma (Newelski)

${G^*}_A^\infty$ is type definable (${G^*}_A^\infty = {G^*}_A^{00}$) $\iff$ ${G^*}_A^\infty = (X_{\Theta_A})^n$ for some natural n

In order to find a group G^*, satisfying ${G^*}_A^{00} \neq {G^*}_A^\infty$, it is enough to find (by compactness), for every natural n, a group G_n, such that $X_{\Theta_{A_n}}$ generates ${G^*}_{A_n}^\infty$ in at least n steps.

Two-step cases

Remark

If G satisfies one the following conditions

- $\mathrm{Th}(G, \cdot, \ldots)$ is simple,
- G is definably compact, definable in an o-minimal expansions of RCF,
- $(G, <, +, \ldots)$ is an o-minimal expansion of an ordered group $(G, <, +)$,
- there exist $p \in S(G^*)$ such that the sets $G^* \cdot p$ and $\mathrm{Aut}(G^*) \cdot p$ are bounded,

then

$$G^{*\infty}_A = (X_{\Theta_A})^2 = G^{*00}_A.$$

Almost strongly simple groups

Trivial observation: if G^* is a simple (in the algebraic sense) group, then $G^{*\infty}$ exists and is equal to G^* (e.g. $PSL_n(K)$, when K is a saturated field)

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Remark

G is N-almost simple, then if G^* is saturated extension (a monster model) of an arbitrary first order expansion $(G, \cdot, \ldots)$ of G, then $G^{*\infty}$ exists and

$$G^* = G^{*\infty} = (X_{\Theta_A})^{4N}.$$

Absolutely connected groups

Definition

G is *N-absolutely connected* (N-a.c.) when for every G^* - saturated extension of an arbitrary first order expansion $(G, \cdot, \ldots)$ of G, $G^{*\infty}$ exists and

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Let $f\colon G \twoheadrightarrow H$ be an epimorphism of groups.

- G is N-a.c., then H is also N-a.c.
- H is N_1-a.c. and $\ker(f)$ is N_2-a.c., then G is $(N_1 + N_2)$-a.c.

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- extension of a.c. group by a a.c. group and image of a a.c. group

Question

Is there any abelian a.c. group? i.e. is there any a.c. group G such that $[G, G] \neq G$ (non-perfect group a.c. group)

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Therefore, if for some thick subset $P \subseteq G$

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then in the expansion $(G, \cdot, P)$, the set $X_{\Theta_\emptyset}$ generates $G^{*\infty}$ in **at least** n-steps.