

On SOP_n and triangle maximality properties

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Introduction

SOP_n

Universal models

Another external
property

SOP_2 and SOP_1

Examples

Hrushovski
constructions

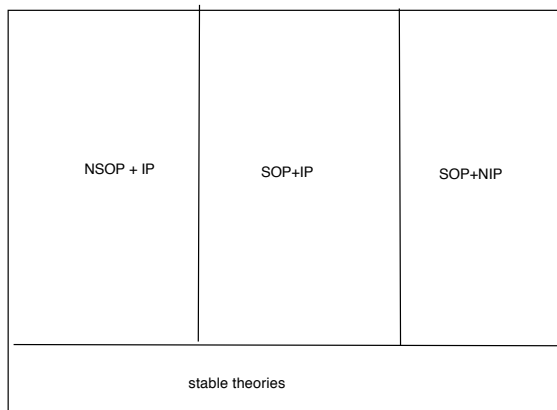
Remarks

Applications

The universe of models and theories

All theories here are complete and written in a countable language.

First order theories



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We shall concentrate on the NSOP+IP region.

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Divide it into smaller regions by **meaningful** dividing lines
=Shelah programme.

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Meaningful: defined both 'internally' and 'externally'.

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We shall concentrate on the $NSOP+IP$ region.

Divide it into smaller regions by **meaningful** dividing lines
=Shelah programme.

Meaningful: defined both ‘internally’ and ‘externally’.

- Internally: a syntactic property of a theory T (e.g. a formula is realised in the monster model \mathfrak{C})
- Externally: a semantic property (e.g. all models of the theory have some property)

Definition

- Let $n \geq 3$. Then T has S(trong)O(rder)P(roperty)_n if some formula $\varphi(\bar{x}, \bar{y})$ defines on \mathcal{C} a directed graph with infinite indiscernible chains and no cycles of length n .
- T has the strict order property if a formula $\varphi(\bar{x}, \bar{y})$ defines on \mathcal{C} a partial order with infinite indiscernible chains.

\implies not simple.

¹S. Shelah, "Toward classifying unstable theories", APAL vol. 80 (1996), 229-255.

Definition

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- T has the strict order property if a formula $\varphi(\bar{x}, \bar{y})$ defines on \mathcal{C} a partial order with infinite indiscernible chains.

Observation/Fact Strict order property \implies
 $SOP_{n+1} \implies SOP_n \implies$ not simple.

¹S. Shelah, "Toward classifying unstable theories", APAL vol. 80 (1996), 229-255.

So we have an internal (sub)division.

Question

Do SOP_n properties correspond to an external property?

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Two external candidates put forward by Sh 500:

- The existence of a universal model.
- \triangleleft^* -maximality.

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Question

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Two external candidates put forward by Sh 500:

- The existence of a universal model.
- \triangleleft^* -maximality.

Idea: Having a universal model says T is somehow 'simple'. Being \triangleleft^* -maximal says it is 'complicated'.

On universal models

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A model of T of size λ is said to be *universal* if all other models of size λ elementarily embed in it.

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On universal models

A model of T of size λ is said to be *universal* if all other models of size λ elementarily embed in it.

Examples: Any of the following imply that T has a universal model in λ :

- T is λ -categorical;
- T has a saturated model of size λ ;
- T has a special model of size λ .

On universal models

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Examples: Any of the following imply that T has a universal model in λ :

- T is λ -categorical;
- T has a saturated model of size λ ;
- T has a special model of size λ .

Corollary If *GCH* holds every T has a universal model in every $\lambda > \aleph_0$.

However:

Fact If T is unstable, and κ is regular in a model of GCH then in the forcing extension adding κ^{++} Cohen subsets of κ there is no universal model of T of size κ^+ .

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However:

Fact If T is unstable, and κ is regular in a model of GCH then in the forcing extension adding κ^{++} Cohen subsets of κ there is no universal model of T of size κ^+ .

Hence having or not having a universal model depends on the model of set theory one is in.

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However:

Fact If T is unstable, and κ is regular in a model of GCH then in the forcing extension adding κ^{++} Cohen subsets of κ there is no universal model of T of size κ^+ .

Hence having or not having a universal model depends on the model of set theory one is in. Therefore this property cannot correspond to an ‘internal’ property such as $NSOP_n$.

The right formulation is: T is *amenable* if it is possible to force the existence of a universal model of T at κ^+ for regular κ over any model of GCH by a forcing which makes 2^κ as large as desired, and preserves cardinals. ².

²Details in M. Džamonja and S. Shelah, "On properties of theories which preclude the existence of universal models", APAL vol. 139, (1-3), (2006), pg. 280-302.

³M. Džamonja and S. Shelah, "On the existence of universal models", Arch. Math. Logic, vol. 43, no.7 (2004), pg. 901-936.

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Theorem

(Shelah in Sh 500) If T has SOP_4 , then it is not amenable.

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Theorem

(Shelah in Sh 500) If T has SOP_4 , then it is not amenable.

Theorem

(Shelah in Sh 500/ Dž.+Shelah ³) If T is the theory of a random triangle-free graph then it is amenable.

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³M. Džamonja and S. Shelah, "On the existence of universal models", Arch. Math. Logic, vol. 43, no.7 (2004), pg. 901-936.

There are further theorems which support a connection between universality and SOP_n .

However, it is known that the correspondence is not perfect. For example:
the Oak Property defined by Dž. and Shelah does not fit the SOP_n hierarchy but behaves like SOP_4 in the universality questions.

Many fine model-theoretic and set-theoretic questions remain.

\triangleleft^* -maximality

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The SOP_n s give a hierarchy so it is reasonable to compare them with some 'external' order.

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The SOP_n s give a hierarchy so it is reasonable to compare them with some 'external' order.

Definition

For a cardinal λ and theories T_0, T_1 , we say $T_0 \triangleleft_\lambda^* T_1$ if there is a theory T of size $< \lambda$ which interprets both T_0 and T_1 and such that in every model M of T , if the reduct to the interpretation of T_1 is λ -saturated, then so is the reduct to the interpretation of T_0 .

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Informally, it is harder for T_1 to have a λ -saturated model, than it is for T_0 . Note: this is not a strict order.

Definition

T^* is $\triangleleft_{\lambda}^*$ maximal if $T \triangleleft_{\lambda}^* T^*$ holds for all T .

⁴S. Shelah and A. Usvyatsov, "More on SOP_1 and SOP_2 , to appear in APAL.

Definition

T^* is $\triangleleft_{\lambda}^*$ maximal if $T \triangleleft_{\lambda}^* T^*$ holds for all T .

Theorem

(Shelah in Sh 500, Shelah-Usvyatsov⁴) If T is SOP_3 then it is $\triangleleft_{\lambda}^*$ -maximal for all $\lambda > \aleph_0$.

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What about the converse?

⁴S. Shelah and A. Usvyatsov, "More on SOP_1 and SOP_2 , to appear in APAL.

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Theorem

(Shelah in Sh 500, Shelah-Usvyatsov⁴) If T is SOP_3 then it is $\triangleleft_{\lambda}^*$ -maximal for all $\lambda > \aleph_0$.

What about the converse? If you want to know, let us first solve a puzzle ...

⁴S. Shelah and A. Usvyatsov, "More on SOP_1 and SOP_2 , to appear in APAL.

Recall:

Definition

For $n \geq 3$, T has SOP_n if some formula $\varphi(\bar{x}, \bar{y})$ defines on \mathcal{C} a directed graph with infinite indiscernible chains and no cycles of length $\leq n$.

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For $n \geq 3$, T has SOP_n if some formula $\varphi(\bar{x}, \bar{y})$ defines on \mathcal{C} a directed graph with infinite indiscernible chains and no cycles of length $\leq n$.

Puzzle: Can we define a property SOP_2 which would fit rightly, so

$$SOP_3 \implies SOP_2 \implies \text{not simple?}$$

Similarly for SOP_1 .

SOP₂ and SOP₁

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We think in terms of trees rather than graphs:

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We think in terms of trees rather than graphs:

Definition

(1) T is SOP₂ if for some formula $\varphi(\bar{x}, \bar{y})$ and some $\langle \bar{a}_\rho : \rho \in {}^\omega 2 \rangle$ in \mathfrak{C} we have

- (i) For every $\eta \in {}^\omega 2$, the set $\{\varphi(\bar{x}, \bar{a}_{\eta \upharpoonright n}) : n < \omega\}$ is consistent, but
- (ii) For any incomparable ρ and ρ' , the set $\{\varphi(\bar{x}, \bar{a}_\rho), \varphi(\bar{x}, \bar{a}_{\rho'})\}$ is inconsistent.

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- (ii) For any incomparable ρ and ρ' , the set $\{\varphi(\bar{x}, \bar{a}_\rho), \varphi(\bar{x}, \bar{a}_{\rho'})\}$ is inconsistent.

(2) For SOP₁, the same definition but require (ii) only for ρ, ρ' such that $\rho = \nu \frown \langle 1 \rangle$ for some ν such that $\nu \frown \langle 0 \rangle \subseteq \rho'$ of any node in ${}^\omega 2$.

Theorem

(Dž.+ Shelah⁵) $SOP_3 \implies SOP_2 \implies SOP_1 \implies$ *not simple*.

⁵M. Džamonja and S. Shelah, "On $\triangleleft_{\lambda}^*$ -maximality", APAL, vol. 125, (1-3), (2004), pg. 119-158.

Theorem

(Dž.+ Shelah⁵) $SOP_3 \implies SOP_2 \implies SOP_1 \implies$ not simple.

Theorem

(Dž.+ Shelah 2004, Shelah+Usvyatsov to appear)
Assume that $2^\lambda = \lambda^+$ holds for all large enough regular λ ,
and that T is \triangleleft_λ^* -maximal for all large enough λ . Then T
has SOP_2 .

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Assume that $2^\lambda = \lambda^+$ holds for all large enough regular λ ,
and that T is \triangleleft_λ^* -maximal for all large enough λ . Then T
has SOP_2 .

Note: It is not known if any of the implications $SOP_3 \implies SOP_2 \implies SOP_1$ is reversible!!

⁵M. Džamonja and S. Shelah, "On \triangleleft_λ^* -maximality", APAL, vol. 125, (1-3), (2004), pg. 119-158.

Examples

First, a no brainer: the model completion $T_{\text{dcf}(3)}^*$ of the theory of directed 3-cycle free graphs has SOP_3 .

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First, a no brainer: the model completion $T_{\text{dcf}(3)}^*$ of the theory of directed 3-cycle free graphs has SOP_3 .

Theorem

(Shelah, Sh 500)

- (1) *There is a theory $T_{\text{dcf}(n)}^*$ which has SOP_n but does not have SOP_{n+1} .*
- (2) *The model completion T_{trf}^* of triangle free graphs has SOP_3 and not SOP_4 .*

Some further examples.

Theorem

(Shelah-Usvyatsov, to appear) The model completion T_{feq}^ of the theory of infinitely many independent parametrized equivalence relations does not have SOP_1 but is not simple (the latter was proved by Shelah in Sh 500).*

⁶G. Cherlin, S. Shelah and N. Shi, "Universal graphs with forbidden subgraphs and algebraic closure", Adv. Appl. Math. 22 (1999), 454-491.

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An example by Casonavas and Peláes! Just heard of it

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Notation

For graphs G and H let $G + H$ be the free join of copies of G and H sharing a common vertex. K_n =complete graph on n vertices.

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Notation

For graphs G and H let $G + H$ be the free join of copies of G and H sharing a common vertex. K_n = complete graph on n vertices.

Theorem

(Cherlin-Shelah-Shi⁶) For $n \geq 3$, there is an existentially complete (so model complete) countable universal $K_n + K_3$ graph \mathfrak{A}_n .

⁶G. Cherlin, S. Shelah and N. Shi, "Universal graphs with forbidden subgraphs and algebraic closure", Adv. Appl. Math. 22 (1999), 454-491.

Theorem

(Patel⁷) *The complete theory of \mathfrak{A}_n is SOP_3 and not SOP_4 .*

⁷R. Patel, "A Family of Countably Universal Graphs without SOP_4 "
to appear in Notre Dame Jour.

Theorem

(Patel⁷) *The complete theory of \mathfrak{A}_n is SOP_3 and not SOP_4 .*

Unfortunately it does not seem clear how one could extend the above results to SOP_2 and SOP_1 , because of the graph-theoretic element of the definition. So, let's try something else ...

⁷R. Patel, "A Family of Countably Universal Graphs without SOP_4 "
to appear in Notre Dame Jour.

Predimension constructions

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A special class of Fräisse constructions invented by Hrushovski in which a functional parameter 'a predimension' is used to control 'a dimension'. Provides a rich class of examples of theories.

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⁸D. Evans and M. Wong, "Some remarks on generic structures", preprint.

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Question

Is it possible to distinguish SOP_n for $n \leq 3$ by a predimension-style construction?

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Is it possible to distinguish SOP_n for $n \leq 3$ by a predimension-style construction?

Theorem

(Evans and Wong⁸) If the theory of a generic structure defined by a predimension is \aleph_0 -categorical, then it is either simple or $SOP_3 + NSOP_4$.

⁸D. Evans and M. Wong, "Some remarks on generic structures", preprint.

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Remarks, Conjectures, etc.

Recall: it is known that simple theories are amenable and SOP_4 are not.

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Recall: it is known that simple theories are amenable and SOP_4 are not.

Question

Are SOP_n for $n \leq 3$ theories amenable?

Conjecture

If the theory of a generic structure defined by a predimension is \aleph_0 -categorical, then it is 'highly' amenable.

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Harder than I thought. I do think it is an interesting project to study structures obtained by forcing with strongly embedded submodels of a generic structure.

⁹A. Mekler, "Universal Structures in Power \aleph_1 ", JSL vol.(55), no.2, 1990, pg. 466-477.

Harder than I thought. I do think it is an interesting project to study structures obtained by forcing with strongly embedded submodels of a generic structure.

Remark: There are various theorems to connect amenability with amalgamation:

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Harder than I thought. I do think it is an interesting project to study structures obtained by forcing with strongly embedded submodels of a generic structure.

Remark: There are various theorems to connect amenability with amalgamation:

Theorem

(1) (Mekler⁹) *If T is relational and has*

$\mathcal{P}(3)^-$ -amalgamation then it is amenable.

(2) (Dž. + Shelah 2004) *An amalgamation-like condition called workability implies amenability.*

⁹A. Mekler, "Universal Structures in Power \aleph_1 ", JSL vol.(55), no.2, 1990, pg. 466-477.

NSOP₁ theories are nice

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Theorem

(Shelah-Usvyatsov, to appear) T is NSOP₁ iff
 $rk_{\varphi}^1(\bar{x} = \bar{x}, \bar{y} = \bar{y})$ is finite for every $\varphi(\bar{x}, \bar{y})$.

Some of the ideas gained can be applied quite far from the first order theories:

¹⁰Y.Benyamini, M. E. Rudin, and M.Wage "Continuous images of weakly compact subsets of Banach spaces", *Pacific Journ. of Mathematics*, 70:309324, (1977)

Some of the ideas gained can be applied quite far from the first order theories:

Definition

A compact topological space is a uniform Eberlein compact (UEF) if it is homeomorphic to a weakly compact subspace of a Hilbert space.

¹⁰Y.Benyamini, M. E. Rudin, and M.Wage "Continuous images of weakly compact subsets of Banach spaces", *Pacific Journ. of Mathematics*, 70:309324, (1977)

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Question

(Benyamini, M. Rudin, Wage¹⁰) Is there for every weight κ a UEF of that weight that maps continuously onto all others of the same weight?

¹⁰Y. Benyamini, M. E. Rudin, and M. Wage "Continuous images of weakly compact subsets of Banach spaces", Pacific Journ. of Mathematics, 70:309324, (1977)

Theorem

It is consistent that the answer is no (Bell¹¹), it follows from GCH that the answer is yes (Bell 2000) and it is consistent with the failure of GCH that the answer is yes for $\kappa \geq \aleph_2$ (Dž.¹²).

¹¹M. Bell "Universal uniform Eberlein compact spaces" PAMS 128, 7:21912197 (2000).

¹²M. Džamonja, "Universality of uniform Eberlein compacta", PAMS, vol. 134, no. 8 (2006), pg. 2427-2435.