Forking and dividing in dependent theories

(and NTP_2 in there)

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“**NIP**” and “**dependent**” are synonyms during this talk

Tree property of second kind (TP_2)

$\phi(x, y)$ has **$\mathbf{TP_2}$** if there is an array $\{a_i^j\}_{i,j<\omega}$ and k s. t.

$$\begin{array}{cccc} \phi(x, a_0^0) & \phi(x, a_1^0) & \phi(x, a_2^0) & \ldots \\ \phi(x, a_0^1) & \phi(x, a_1^1) & \phi(x, a_2^1) & \ldots \\ \phi(x, a_0^1) & \phi(x, a_1^1) & \phi(x, a_2^1) & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{array}$$

rows are k-inconsistent:

$\forall j < \omega \ \forall i_0 < i_1 < \ldots < i_k < \omega$

$\phi(x, a_{i_0}^j) \wedge \phi(x, a_{i_1}^j) \wedge \ldots \wedge \phi(x, a_{i_k}^j) = \emptyset$

vertical paths are consistent:

$\forall f : \omega \to \omega \ \exists c_f \models \bigwedge_{j<\omega} \phi(x, a_{f(j)}^j)$

Classification

How do all these classes of theories relate?

- stable
- simple
- nice dependent: stable, o-minimal, C-minimal (D-minimal?)
- dependent
- NTP_2

Forking / dividing / quasi-dividing

Artem Chernikov, Itay Kaplan

$\phi(x, a)$ forks
$\phi(x, a)$ divides
$\phi(x, a)$ quasi-divides (some people wrongly say weakly divides instead)

Shoulders of giants

Shelah: introduced NTP_2
Poizat/Shelah: classical theory of *NIP* theories
Kim: in simple theories forking = dividing
Dolich: forking = quasi-dividing in nice o-minimal theories (+goodness machinery)
Adler, **Hrushovski/Peterzil/Pillay**
Tressl: heirs and coheirs in o-minimal theories

Main theorem

- (NTP_2) $\phi(x,a)$ forks $/M \iff \phi(x,a)$ divides $/M$ (NIP) $\iff \phi(x,a)$ quasi-divides $/M$
- (T is nice dependent) $\phi(x,a)$ forks $/A \iff \phi(x,a)$ divides $/A \iff \phi(x,a)$ quasi-divides $/A$

NTP_2: First improvement

Observation: Suppose $\phi(x, a)$ divides over M. Then **exists a global coheir** of $tp(a/M)$, which witnesses it

Proof(very imprecise sketch):
Suppose not, let I be an M-indiscernible witnessing dividing.
Take $N \supseteq M$, $|M|^+$-saturated, and I is an indicsernible with the same EM type, very long w.r.t. N.
Take its type over M, expand to N - infinitely often it is the same coheir.
Generate sequence in it - this is our array.

Strict non-forking and non-forking heirs

Lifting of indiscernibles:

Definition (Shelah): Type p strictly does not fork if it does not fork and lifts indiscernibles.

Example: Non-forking heir

Analog of Kim's lemma in dependent context?

Lemma (T dependent): Let $\phi(x, a)$ divide over M. Let $p \in S(\mathbb{M})$ be a global type strictly non-forking $/M$, $tp(a/M) \subseteq p$. Then any sequence generated by it witnesses dividing.

But do non-forking heirs always exist?

Broom lemma

Corollary 1 (NTP_2): Forking implies quasi-dividing over models
Why? Can always arrange assumption of the broom lemma for a forking formula, using existence of global coheirs witnessing dividing.
Corrolary 2 (NTP_2): Every type over model has a global non-forking heir

Nice dependent theories

Why proofs work over arbitrary sets instead of models?
Hint: broom lemma works with non-forking instead of coheirs.

Corrolaries

- (T dependent) Forking is type-definable, so dependent theories are low
- (T is NTP_2) Non-forking satisfies left extension
- (T is NTP_2) T is dependent iff non-forking is bounded

Optimality of results?

Martin Ziegler (several days ago):

- T with NTP_2 s.t. forking $\neq$ dividing **over model**
- T with NTP_2 s.t. forking = dividing always

First is a “dense independent” family of circular orderings, second is a “dense independent” family of linear ones.

Questions

- Does “T has TP_2” imply “forking is not type-definable”?
- Does every type over model has a global non-forking heir? (without any assumptions on theory)
- Characterize dependence of a theory by behaviour of forking / ...