Superrosy lovely pairs

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This document contains the slides and something similar to what I wrote on the blackboard for my talk at the meeting Around Classification Theory which took place at the University of Leeds from 27 to 30 June 2008. My talk followed the related talks of E. Vassiliev and A. Berenstein.

1 Statement of result

Let T be a (one-sorted) þ-rank one theory which eliminates quantifiers and $\exists^\infty$. We prove the following:

Theorem: T_P is superrosy with þ-rank at most ω.

2 Notation

The language of T is L.

The language of T_P is $L_P = L \cup \{P\}$, where P is a unary predicate.

$(M, P(M)) \models T_P$ is sufficiently saturated (and sufficiently homogeneous).

M is the reduct of $(M, P(M))$ to L.

M is the universe of both $(M, P(M))$ and M.

X, Y, Z are definable sets.

$\ulcorner X \urcorner$ is a code for X in $(M, P(M))^{eq}$.

A, B, C, D are sufficiently small sets of parameters.

The language L_P shall be assumed unless we specify L.

We work in $(M, P(M))^{eq}$.

3 Recall (from the work of Ealy and Onshuus)

The material in this section may be found, in perhaps a slightly different form, in *Characterizing Rosy Theories* by C. Ealy and A. Onshuus.

X þ-divides over A if there is some $D \supseteq A$ such that $\ulcorner X \urcorner \notin acl(D)$ and the D-conjugates of X are k-inconsistent for some $k < \omega$.

þ-*rank* is the least function assigning to each (non-empty) definable set either an ordinal or ∞ such that:
(1) þ-$rank(X) \geq \alpha + 1$ if there is some $Y \subseteq X$ such that Y þ-divides over $\ulcorner X \urcorner$ and þ-$rank(Y) \geq \alpha$,
(2) for any limit ordinal λ, þ-$rank(X) \geq \lambda$ if þ-$rank(X) \geq \beta$ for all $\beta < \lambda$.

T_P is superrosy if for any $tp(e/A)$ there is a finite $A_0 \subseteq A$ such that $e \downarrow^{þ}_{A_0} A$.

T_P is superrosy if þ-$rank(T_P) =$ þ-$rank(M^1) < \infty$.

If þ-$rank(X) = 1$ then þ-$rank(X^n) = n$.

If $\varphi(x, \bar{y})$ is algebraic in x for every $\bar{y}$ and X is defined by $\exists \bar{y} \varphi(x, \bar{y})$ then þ-$rank(X) \leq$ þ-$rank(Y)$ where Y is defined by $\exists x \varphi(x, \bar{y})$. (Note that $\varphi(x, \bar{y})$ may have additional parameters.)

4 Recall (from the previous two talks)

The material in this section may be found or easily deduced from that presented in *Lovely pairs and dense pairs of o-minimal structures* by A. Berenstein, although some of the definitions are slightly different.

$P(M) \prec M$.

Density: for any infinite L-definable $X \subseteq M^1$, $X \cap P(M) \neq \emptyset$.

Extension: any infinite L-definable $X \subseteq M^1$ is large.

A partial type $p(x)$ in one real variable is said to be large if it has an infinite sequence of realisations which is L-algebraically independent over $P(M)$.

If $X \subseteq M^1$ is not large then there is a formula $\varphi(x, \bar{y})$ which is algebraic in x for every $\bar{y}$ and such that X is defined by $\exists \bar{y} \varphi(x, \bar{y})$ and $P(M)^n$ (for some $n < \omega$) is defined by $\exists x \varphi(x, \bar{y})$.

A set $B \subseteq M$ is said to be P-independent if $B \downarrow^{þ}_{B \cap P(M)} P(M)$ in T.

If $\bar{a}$ and $\bar{b}$ are both P-independent and $qftp(\bar{a}) = qftp(\bar{b})$ then $tp(\bar{a}) = tp(\bar{b})$.

For any $X \subseteq P(M)^1$ there is an L-definable Y such that $X = Y \cap P(M)^1$.

For any $X \subseteq M^1$ there is an L-definable Y such that $X \triangle Y$ is not large.

If $B \subseteq M$ is P-independent then $acl(B) \cap M^{eq} = acl_L(B) \cap M^{eq}$.

5 Proof of Theorem

Proposition 1: Let $X \subseteq P(M)^1$ be infinite. Then X does not þ-divide over $\emptyset$.

Proposition 2: Let $X \subseteq M^1$ be large. Then X does not þ-divide over $\emptyset$.

It follows from Proposition 1 that þ-$rank(P(M)^1) = 1$. Therefore þ-$rank(P(M)^n) = n$. It follows that any non-large $X \subseteq M^1$ has finite þ-$rank$. By Proposition 2 then þ-$rank(M^1) \leq \omega$.

6 Proof of Proposition 1

Proof: Suppose X does thorn-divide over $\emptyset$. Let D be such that $\ulcorner X \urcorner$ is not algebraic over D and, for some k, the D-conjugates of X are k-inconsistent. We may choose $D \subseteq M$. We may also choose D to be P-independent. There is then some $b \in X$ such that $b \notin acl(D\ulcorner X \urcorner)$ and $\ulcorner X \urcorner \in acl(Db) \setminus acl(D)$. So $(M, P(M))^{eq} \models \varphi(b, \ulcorner X \urcorner)$ for some formula $\varphi(z, x)$ over D which implies "$x \in acl(Dz)$ and $P(z)$". Let $Z = \{z : \varphi(z, \ulcorner X \urcorner)\}$. Clearly $\ulcorner X \urcorner$ is interalgebraic with $\ulcorner Z \urcorner$ over D.

Claim: $\ulcorner X \urcorner$ is interalgebraic over D with some $e \in M^{eq}$ (so e is a code for an L-definable set).

Proof of Claim: Let Y be L-definable and such that $Z = Y \cap P(M)^1$. Suppose $Y = \{z : \psi(z, \bar{c})\}$ where $\psi \in L$ and $\bar{c} \in M$. Consider the equivalence relation defined by $\neg(\exists^\infty z)[\psi(z, \bar{u}) \triangle \psi(z, \bar{v})]$, recalling that T eliminates $\exists^\infty$.

From the density property of T_P it follows that if $\psi(z, \bar{c})$ and $\psi(z, \bar{c}')$ agree on $P(M)$ then $(\bar{c}, \bar{c}') \in E$. So the E-class of $\bar{c}$ is definable over $\ulcorner Z \urcorner$. Let e be this E-class.

Let m be the maximum number of realisations of $\psi(z, \bar{c}) \triangle \psi(z, \bar{c}')$ for $(\bar{c}, \bar{c}') \in E$. The algebraicity of $\ulcorner X \urcorner$ over $D\ulcorner Z \urcorner$ is witnessed by the following: "$\varphi(z, \ulcorner X \urcorner)$ for all but m members z of Z". The previous sentence remains true if we replace $\ulcorner Z \urcorner$ with $\ulcorner Z' \urcorner$ for any e-conjugate Z' of Z. Therefore $\ulcorner X \urcorner \in acl(De)$. This proves the claim.

So $e \in acl(Db) \setminus acl(D)$ but $b \notin acl(De)$. If we could replace acl with acl_L then this would contradict the fact that T has þ-$rank$ one. We can make this replacement because Db is P-independent and acl always contains acl_L. Therefore we have a contradiction and the proof is complete.

The proof of Proposition 2 follows very similar lines. It makes use of the extension property and the fact that for any $X \subseteq M^1$ there is an L-definable Y such that $X \triangle Y$ is not large.