

# Dense pairs and Lovely pairs of O-minimal structures

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Let  $T$  be a complete theory with quantifier elimination and the exchange property in a countable language  $L$ .

Let  $P$  be a new unary predicate and let  $L_P = L \cup \{P\}$ .

Let  $T'$  be the  $L_P$ -theory of all structures  $(M, P(M))$ , where  $M \models T$  and  $P(M)$  is an  $L$ -algebraically closed subset of  $M$ .

## Definition

We say that a structure  $(M, P(M))$  is a lovely pair of models of  $T$  if

1.  $(M, P(M)) \models T'$
2. (Density property) If  $A \subset M$  is algebraically closed and finite dimensional and  $q \in S_1(A)$  is non-algebraic, there is  $a \in P(M)$  such that  $a \models q$ .
3. (Extension property) If  $A \subset M$  is algebraically closed and finite dimensional and  $q \in S_1(A)$  is non-algebraic, there is  $a \in M$ ,  $a \models q$  and  $a \notin \text{acl}(A \cup P(M))$ .

# Examples of lovely pairs

$$T = \exists^\infty x \quad (\mathbb{Z}, \mathbb{N}).$$

$$T = DLO \quad (\mathbb{R}, <, \mathbb{Q})$$

$$T = ACF_0 \quad (\mathbb{C}, +, \times, 0, 1, \overline{\mathbb{Q}(e_0, e_1, \dots)})$$

## Basic Facts

Let  $(M, P(M)), (N, P(N))$  be lovely pairs of a theory  $T$ , then  $(M, P(M)) \equiv (N, P(N))$ . We write  $T_P$  for the common theory.

Let  $T$  be a theory with the exchange property that eliminates  $\exists^\infty$ . Then whenever  $(M, P(M)) \models T_P$  is  $\omega$ -saturated, then  $(M, P(M))$  is a lovely pair.

### Theorem

*Let  $(M, P(M))$  be a lovely pair of a theory  $T$ . Then every  $L_P$  formula is equivalent to a boolean combination of formulas of the form:*

$$\exists y_1 \in P \dots \exists y_k \in P \varphi(\bar{y}, x)$$

*where  $\varphi(\bar{y}, x)$  is an  $L$ -formula.*

*If  $\psi(x)$  defines a subset of  $P$ , then there is  $\varphi(x)$   $L$ -definable such that  $\psi(x) = P(x) \cap \varphi(x)$ .*

Assume that  $T$  is *Dependent*. That is, given a saturated model  $M$  of  $T$  and a formula  $\varphi(x, \vec{y})$  there does not exist an indiscernible sequence  $\{a_i : i \in \omega\}$  and  $\vec{b} \in M^n$  such that  $M \models \varphi(a_i, \vec{b})$  iff  $i$  is even.

## Theorem

(B., Dolich, Onshuus; Boxall) If  $T$  is dependent and eliminates  $\exists^\infty$ , then  $T_P$  is dependent.

Let  $T$  be a complete theory and let  $M = M^{eq}$  be a universal domain for  $T$ , let  $A \subset M$  be a set.

## Definition

*A formula  $\delta(x, \bar{a})$  thorn divides over  $A$  if there is a tuple  $\bar{c}$  such that  $\text{tp}(\bar{a}/A\bar{c})$  is non-algebraic and  $\{\delta(x, \bar{a}') : \bar{a}' \models \text{tp}(\bar{a}/A)\}$  is  $k$ -inconsistent.*

*We say  $\delta(x, \bar{a})$  thorn forks over  $A$  if it implies a finite disjunction of formulas each of which thorn divides over  $A$ .*

$T$  is rosy if thorn-independence has local character.

## Definition

*thorn-rk* is the least function taking values in  $On \cup \{\infty\}$  satisfying the following:

(1) *thorn-rk* $(\varphi(\vec{x}, \vec{b})) \geq 0$  if  $\varphi(\vec{x}, \vec{b})$  is consistent.

(2) *thorn-rk* $(\varphi(\vec{x}, \vec{b})) \geq \alpha + 1$  if there is  $\psi(\vec{x}, \vec{c})$  that thorn-divides over  $\vec{b}$ , such that  $\psi(\vec{x}, \vec{c}) \vdash \varphi(\vec{x}, \vec{b})$  and *thorn-rk* $(\psi(\vec{x}, \vec{c})) \geq \alpha$ .

(3) For  $\lambda$  a limit ordinal, *thorn-rk* $(\varphi(\vec{x}, \vec{b})) \geq \lambda$  if *thorn-rk* $(\varphi(\vec{x}, \vec{b})) \geq \alpha$  for all  $\alpha < \lambda$ .

# The O-minimal case

Let  $T$  be O-minimal expansion of  $Th(\mathbb{R}, +, <, 0, 1)$ , where  $1$  stands for a positive constant.

## Definition

*A dense pair of models of  $T$  is a pair  $(M, P(M))$  of models of  $T$  such that  $P(M) \preceq M, P(M) \neq M$  and  $P(M)$  is dense in  $M$ .*

Clearly a lovely pair of models of  $T$  is a dense pair.

van den Dries: the theory of dense pairs is complete.

Thus, for such  $T$ , *dense pairs are the models of the theory of lovely pairs!*

## Theorem

(van den Dries; B.) Let  $T$  be an O-minimal theory extending DLO and let  $(M, P)$  be a lovely pair of models of  $T$ . If  $A \subset M$  is  $L_P$ -definable, then there is a partition

$$-\infty = b_0 < b_1 < \cdots < b_{k+1} = \infty$$

of  $M$  such that for each  $i = 0, \dots, k$ , either:

$$A \cap (b_i, b_{i+1}) = \emptyset,$$

$$A \cap (b_i, b_{i+1}) = (b_i, b_{i+1}), \text{ or}$$

$A \cap (b_i, b_{i+1})$  is dense and codense in  $(b_i, b_{i+1})$ .

## Theorem

*(B., Ealy, Günaydin) Let  $T$  be an O-minimal theory extending DLO and let  $(M, P)$  be a lovely pair of models of  $T$ . Then  $T_P$  has O-minimal open core and it is super-rosy of rank  $\leq \omega$ .*

Examples:

$$(\mathbb{R}, <, \mathbb{Q}), \text{ thorn-rk}(x = x) = 1.$$

$$(\mathbb{R}, <, +, 0, \overline{\mathbb{Q}}), \text{ thorn-rk}(x = x) = 2.$$

$$(\mathbb{R}, <, +, \times, 0, 1, \dots, \overline{\mathbb{Q}}), \text{ thorn-rk}(x = x) = \omega.$$

# The O-minimal case

Examples:

If  $(R, <, \dots, P(R))$  O-minimal,  $a \in R$  and the  $L$ -structure is trivial around  $a$ ,  $U(tp_P(a)) \leq 1$ .

If  $(R, <, \dots, P(R))$  O-minimal,  $a \in R$  and the  $L$ -structure has a definable field in a neighborhood of  $a$  and  $a$  is generic, then  $U(tp_P(a)) = \omega$ .