Dense pairs and Lovely pairs of O-minimal structures

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Let $T$ be a complete theory with quantifier elimination and the exchange property in a countable language $L$. Let $P$ be a new unary predicate and let $L_P = L \cup \{P\}$.

Let $T'$ be the $L_P$-theory of all structures $(M, P(M))$, where $M \models T$ and $P(M)$ is an $L$-algebraically closed subset of $M$. 
Lovely Pairs

Definition
We say that a structure \((M, P(M))\) is a lovely pair of models of \(T\) if

1. \((M, P(M)) \models T'\)
2. (Density property) If \(A \subseteq M\) is algebraically closed and finite dimensional and \(q \in S_1(A)\) is non-algebraic, there is \(a \in P(M)\) such that \(a \models q\).
3. (Extension property) If \(A \subseteq M\) is algebraically closed and finite dimensional and \(q \in S_1(A)\) is non-algebraic, there is \(a \in M\), \(a \models q\) and \(a \notin \text{acl}(A \cup P(M))\).
Examples of lovely pairs

\[ T = \exists^\infty x \quad (\mathbb{Z}, \mathbb{N}). \]

\[ T = \text{DLO} \quad (\mathbb{R}, <, \mathbb{Q}) \]

\[ T = \text{ACF}_0 \quad (\mathbb{C}, +, \times, 0, 1, \mathbb{Q}(e_0, e_1, \ldots)) \]
Basic Facts

Let \((M, P(M)), (N, P(N))\) be lovely pairs of a theory \(T\), then \((M, P(M)) \equiv (N, P(N))\). We write \(T_P\) for the common theory.

Let \(T\) be a theory with the exchange property that eliminates \(\exists^\infty\). Then whenever \((M, P(M)) \models T_P\) is \(\omega\)-saturated, then \((M, P(M))\) is a lovely pair.

Theorem

Let \((M, P(M))\) be a lovely pair of a theory \(T\). Then every \(L_P\) formula is equivalent to a boolean combination of formulas of the form:

\[
\exists y_1 \in P \ldots \exists y_k \in P \varphi(\bar{y}, x)
\]

where \(\varphi(\bar{y}, x)\) is an \(L\)-formula.

If \(\psi(x)\) defines a subset of \(P\), then there is \(\varphi(x)\) \(L\)-definable such that \(\psi(x) = P(x) \cap \varphi(x)\).
Dependent Theories

Assume that $T$ is Dependent. That is, given a saturated model $M$ of $T$ and a formula $\varphi(x, \vec{y})$ there does not exist an indiscernible sequence $\{a_i : i \in \omega\}$ and $\vec{b} \in M^n$ such that $M \models \varphi(a_i, \vec{b})$ iff $i$ is even.

**Theorem**

(B., Dolich, Onshuus; Boxall) If $T$ is dependent and eliminates $\exists^\infty$, then $T_P$ is dependent.
Thorn forking

Let $T$ be a complete theory and let $M = M^{eq}$ be a universal domain for $T$, let $A \subset M$ be a set.

**Definition**

A formula $\delta(x, \bar{a})$ thorn divides over $A$ if there is a tuple $\bar{c}$ such that $\text{tp}(\bar{a}/A\bar{c})$ is non-algebraic and

$$\{\delta(x, \bar{a}') : \bar{a}' \models \text{tp}(\bar{a}/A)\}$$

is $k$-inconsistent.

We say $\delta(x, \bar{a})$ thorn forks over $A$ if it implies a finite disjunction of formulas each of which thorn divides over $A$.

$T$ is rosy if thorn-independence has local character.
Thorn rank

Definition

\( \text{thorn-rk} \) is the least function taking values in \( \text{On} \cup \{ \infty \} \) satisfying the following:

1. \( \text{thorn-rk}(\varphi(\vec{x}, \vec{b})) \geq 0 \) if \( \varphi(\vec{x}, \vec{b}) \) is consistent.
2. \( \text{thorn-rk}(\varphi(\vec{x}, \vec{b})) \geq \alpha + 1 \) if there is \( \psi(\vec{x}, \vec{c}) \) that thorn-divides over \( \vec{b} \), such that \( \psi(\vec{x}, \vec{c}) \vdash \varphi(\vec{x}, \vec{b}) \) and \( \text{thorn-rk}(\psi(\vec{x}, \vec{c})) \geq \alpha \).
3. For \( \lambda \) a limit ordinal, \( \text{thorn-rk}(\varphi(\vec{x}, \vec{b})) \geq \lambda \) if \( \text{thorn-rk}(\varphi(\vec{x}, \vec{b})) \geq \alpha \) for all \( \alpha < \lambda \).
The O-minimal case

Let $T$ be O-minimal expansion of $\text{Th}(\mathbb{R}, +, <, 0, 1)$, where 1 stands for a positive constant.

**Definition**

A dense pair of models of $T$ is a pair $(M, P(M))$ of models of $T$ such that $P(M) \leq M, P(M) \neq M$ and $P(M)$ is dense in $M$.

Clearly a lovely pair of models of $T$ is a dense pair. van den Dries: the theory of dense pairs is complete. Thus, for such $T$, dense pairs are the models of the theory of lovely pairs!
The O-minimal case

Theorem

(van den Dries; B.) Let $T$ be an O-minimal theory extending DLO and let $(M, P)$ be a lovely pair of models of $T$. If $A \subset M$ is $L_P$-definable, then there is a partition $-\infty = b_0 < b_1 < \cdots < b_{k+1} = \infty$ of $M$ such that for each $i = 0, \ldots, k$, either:

- $A \cap (b_i, b_{i+1}) = \emptyset$,
- $A \cap (b_i, b_{i+1}) = (b_i, b_{i+1})$, or
- $A \cap (b_i, b_{i+1})$ is dense and codense in $(b_i, b_{i+1})$. 


The O-minimal case

Theorem

(B., Ealy, Günaydin) Let $T$ be an O-minimal theory extending DLO and let $(M, P)$ be a lovely pair of models of $T$. Then $T_P$ has O-minimal open core and it is super-rosy of rank $\leq \omega$.

Examples:

$(\mathbb{R}, <, \mathbb{Q})$, thorn-rk$(x = x) = 1$.

$(\mathbb{R}, <, +, 0, \mathbb{Q})$, thorn-rk$(x = x) = 2$.

$(\mathbb{R}, <, +, \times, 0, 1, \ldots, \mathbb{Q})$, thorn-rk$(x = x) = \omega$. 
The O-minimal case

Examples:

If \((R, <, \ldots, P(R))\) O-minimal, \(a \in R\) and the \(L\)-structure is trivial around \(a\), \(U(tp_P(a)) \leq 1\).

If \((R, <, \ldots, P(R))\) O-minimal, \(a \in R\) and the \(L\)-structure has a definable field in a neighborhood of \(a\) and \(a\) is generic, then \(U(tp_P(a)) = \omega\).