

# Strong theories, burden and weight

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# Outline

Weight in a simple theory

Independence relations

Thorn-weight in a rosy theory

Independent partitions

An *independent forking system* for  $a$  over  $C$  is a set  $\mathcal{B}$  of tuples such that

- ▶  $\mathcal{B}$  is independent over  $C$ , and
- ▶  $a \perp_C b$  for all  $b \in \mathcal{B}$ .

$$\begin{aligned} \text{pwt}(a/C) &= \sup\{|\mathcal{B}| : \mathcal{B} \text{ indep. fork. syst. for } a \text{ over } C\} \\ \text{wt}(a/C) &= \sup\{\text{pwt}(a/C') \mid C' \supseteq C, a \perp_C C'\} \end{aligned}$$

### Fact

If  $\text{wt}(a/C)$  is a successor cardinal or  $\aleph_0$ , the supremum is attained.

But how about  $\text{wt}(a/C) = \aleph_\omega$ ?

## Theorem

If  $\text{wt}(a/C)$  is finite, then there are  $C' \supseteq C$  s.t.  $a \downarrow_C C'$  and  $b_0, \dots, b_{n-1}$ , independent over  $C'$ , such that  $\text{wt}(b_i/C') = 1$  and  $a$  is domination equivalent to  $b_0 \dots b_{n-1}$  over  $C$ .

In supersimple theories we can choose  $U(b_i/C') = \omega^\alpha$ . In simple theories without dense forking chains we can choose  $\text{tp}(b_i/C')$  regular.

## Fact

If  $X$  is a set of tuples of weight 1 over  $C$ , then the following operator defines a matroid ('pregeometry') on  $X$ :

$$b \in \text{cl}(A) \iff \exists A_0 \subseteq A, A_0 \text{ indep. over } C, A_0 \not\downarrow_C b.$$

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To speak of independence, one needs a ternary relation that has at least the following properties:

(invariance)

If  $A \perp_C B$  and  $(A', B', C') \equiv (A, B, C)$ , then  $A' \perp_{C'} B'$ .

(monotonicity)

If  $A \perp_C B$ ,  $A' \subseteq A$  and  $B' \subseteq B$ , then  $A' \perp_C B'$ .

(transitivity)

Suppose  $D \subseteq C \subseteq B$ . If  $B \perp_C A$  and  $C \perp_D A$ , then  $B \perp_D A$ .

(strong finite character)

If  $A \not\perp_C B$ , then there is a finite tuple  $\bar{a} \in A$  and a formula  $\varphi(\bar{x}, \bar{b}\bar{c})$  with parameters in  $BC$  such that

- ▶  $\models \varphi(\bar{a}, \bar{b}\bar{c})$ , and
- ▶  $\bar{a}' \not\perp_C B$  for all  $\bar{a}'$  satisfying  $\models \varphi(\bar{a}', \bar{b}\bar{c})$ .

We call it a *preindependence relation* if it also satisfies the following:

(base monotonicity)

Suppose  $D \subseteq C \subseteq B$ . If  $A \downarrow_D B$ , then  $A \downarrow_C B$ .

(normality)

$A \downarrow_C B$  implies  $AC \downarrow_C B$ .

It is *extensible* if it also satisfies:

(extension)

If  $A \downarrow_C B$  and  $\hat{B} \supseteq B$ , then there is  $A' \equiv_{BC} A$  such that  $A' \downarrow_C \hat{B}$ .

**Fact**

*In any complete first-order theory, dividing, thorn-dividing and  $M$ -dividing are preindependence relations, and therefore(!) forking and thorn-forking are extensible preindependence relations.*

An extensible preindependence relation is an *independence relation* if it also satisfies the following axioms:

(local character)

For every  $A$  there is a cardinal  $\kappa(A)$  with the following property:

For any set  $B$  there is a subset  $C \subseteq B$  of cardinality  $|C| < \kappa(A)$  such that  $A \perp_C B$ .

(symmetry)

$$A \perp_C B \iff B \perp_C A.$$

(existence)

For all  $A, B, C$  there is  $B' \equiv_C B$  such that  $A \perp_C B'$ .

**Theorem**

*It's enough to check extension and local character, or to check symmetry and existence.*

Note: The easy direction is not true without *strong* finite character.

A theory is simple iff forking independence is an independence relation. Forking independence in a simple theory is the only independence relation that satisfies the independence theorem over models, or equivalently the *chain condition*:

**(chain condition)** If  $(\bar{b}_i)_{i < \kappa}$  sequence of  $C$ -indiscernibles,  $\bar{a} \perp_C \bar{b}_0$  ( $\bar{a}$  and  $\bar{b}_i$  may be infinite), then there is  $\bar{a}' \equiv_{\bar{b}_0 C} \bar{a}$  such that  $(\bar{b}_i)_{i < \kappa}$  is  $\bar{a}'$ - $C$ -indiscernible and  $\bar{a}' \perp_C (\bar{b}_i)_{i < \kappa}$ .

(The official definition of the chain condition is different, but equivalent.)

A theory is rosy iff thorn-forking independence is an independence relation, iff there is an independence relation on  $\mathbb{M}^{\text{eq}}$  that satisfies  $a \perp_B a \implies a \in \text{acl}^{\text{eq}} B$ . Thorn-forking in a rosy theory is the weakest such independence relation.

### Theorem

*Forking and thorn-forking agree in every simple theory with elimination of hyperimaginaries.*

Is there any complete first-order theory with two such independence relations?

### Fact

*Simple and o-minimal theories are rosy.*

*C-minimal theories are not rosy.*

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An  $\mathfrak{p}$ -independent  $\mathfrak{p}$ -forking system for  $a$  over  $C$  is a set  $\mathcal{B}$  of tuples such that

- ▶  $\mathcal{B}$  is  $\mathfrak{p}$ -independent over  $C$ , and
- ▶  $a \not\downarrow_C^{\mathfrak{p}} b$  for all  $b \in \mathcal{B}$ .

$$\text{pwt}^{\mathfrak{p}}(a/C) = \sup\{|\mathcal{B}| : \mathcal{B} \text{ indep. fork. syst. for } a \text{ over } C\}$$

$$\text{wt}^{\mathfrak{p}}(a/C) = \sup\{\text{pwt}^{\mathfrak{p}}(a/C') \mid C' \supseteq C, a \downarrow_C^{\mathfrak{p}} C'\}$$

## Fact

If  $\text{wt}^{\mathfrak{p}}(a/C)$  is a successor cardinal or  $\aleph_0$ , the supremum is attained.

How about  $\text{wt}^{\mathfrak{p}}(a/C) = \aleph_{\omega}$ ?

## Theorem

If  $\text{wt}^{\beta}(a/C)$  is finite, then there are  $C' \supseteq C$  s.t.  $a \not\perp_C^{\beta} C'$  and  $b_0, \dots, b_{n-1}$ ,  $\beta$ -independent over  $C'$ , such that  $\text{wt}^{\beta}(b_i/C') = 1$  and  $a$  is domination equivalent to  $b_0 \dots b_{n-1}$  over  $C$ .

In superrosy theories we can choose  $U(b_i/C') = \omega^{\alpha}$ . In rosy theories without dense forking chains we can choose  $\text{tp}(b_i/C')$  regular.

## Fact

If  $X$  is a set of tuples of  $\beta$ -weight 1 over  $C$ , then the following operator defines a matroid ('pregeometry') on  $X$ :

$$b \in \text{cl}(A) \iff \exists A_0 \subseteq A, A_0 \beta\text{-indep. over } C, A_0 \not\perp_C^{\beta} b.$$

- ▶ Rosiness and thorn-forking are **good** notions, because they allow us to continue essentially as before. As with the generalisation from stability to simplicity, many (but not all) definitions and arguments carry over to the new context with little change.

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- ▶ Rosiness and thorn-forking are **good** notions, because they allow us to continue essentially as before. As with the generalisation from stability to simplicity, many (but not all) definitions and arguments carry over to the new context with little change.
- ▶ Rosiness and thorn-forking are **not** good notions because they were not first found by Shelah. In other words, they are (apparently) not directly related to 'dividing lines'.

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An *inp-pattern* for a partial type  $p(x)$  is a sequence

$$(\varphi(x, b^0), k^0), (\varphi(x, b^1), k^1), (\varphi(x, b^2), k^2), \dots$$

for which there are mutually (over  $\text{dom } p$ ) indiscernible sequences

$$b^0 = b_0^0, \quad b_1^0, \quad b_2^0, \quad \dots$$

$$b^1 = b_0^1, \quad b_1^1, \quad b_2^1, \quad \dots$$

$$b^2 = b_0^2, \quad b_1^2, \quad b_2^2, \quad \dots$$

such that for the associated array of formulas

$$\varphi(x, b_0^0), \quad \varphi(x, b_1^0), \quad \varphi(x, b_2^0), \quad \dots$$

$$\varphi(x, b_0^1), \quad \varphi(x, b_1^1), \quad \varphi(x, b_2^1), \quad \dots$$

$$\varphi(x, b_0^2), \quad \varphi(x, b_1^2), \quad \varphi(x, b_2^2), \quad \dots$$

the following holds:

- ▶ each row  $i$  is  $k^i$ -inconsistent, and
- ▶ each 'path' is consistent with  $p(x)$ .

$\kappa_{inp}(T)$  is the smallest cardinal  $\kappa$  such that there is no inp-pattern of length  $\kappa$  for any  $\bar{x} = \bar{x}$  ( $\bar{x}$  finite).

$T$  has  $TP_2$ , the *tree property of the second kind*, if  $\kappa_{inp}(T) = \infty$ .

## Fact

$T$  has the tree property (i.e. is not simple) if and only if

- ▶  $T$  has the  $TP_2$ , or
- ▶  $T$  has the  $SOP_2$  (formerly known as  $TP_1$ ).

Burden of a type:  $\text{bdn}(p)$  is the supremum of the lengths of inp-patterns of  $p$ . (May or may not be attained!)

## Theorem

*If  $T$  is simple, then*

$$\text{bdn}(p) = \sup_{p' \supseteq p} \text{pwt}(p) = \sup_{p' \supseteq p} \text{wt}(p).$$

*If  $T$  is rosy, then*

$$\text{bdn}(p) \geq \sup_{p' \supseteq p} \text{pwt}^b(p) = \sup_{p' \supseteq p} \text{wt}^b(p).$$

$T$  is *strong* if  $\kappa_{inp}(T) = \aleph_0$ .

$T$  is strong iff for every (finitary) type  $p(x)$ , either  $\text{bdn}(p)$  is finite, or  $\text{bdn}(p) = \aleph_0$  and the supremum is not attained.

### Theorem

*A simple theory is strong iff every type has finite weight.*

*A dependent (i.e. NIP) theory is strong iff it is strongly dependent.*

### Corollary

*A stable theory is strongly dependent if and only if every type has finite weight.*

$T$  is *inp-minimal* if every 1-type has burden  $\leq 1$ .

$T$  is *dp-minimal* if  $T$  is NIP and inp-minimal.

Examples of dp-minimal theories:

- ▶ stable theories in which every 1-type has weight  $\leq 1$ ; e.g. strongly minimal theories
- ▶ (weakly) o-minimal theories
- ▶ C-minimal theories
- ▶ p-minimal theories
- ▶ (weakly) D-minimal theories, i.e. theories controlled by a family of Vapnik-Chervonenkis codimension 1.