Measures in model theory

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Introduction I

- I will discuss the growing use and role of measures in "pure" model theory, with an emphasis on extensions of stability theory outside the realm of stable theories.
- The talk is related to current work with Ehud Hrushovski and Pierre Simon (building on earlier work with Hrushovski and Peterzil).
- I will be concerned mainly, but not exclusively, with "tame" rather than "foundational" first order theories.
- T will denote a complete first order theory, 1-sorted for convenience, in language L.
- There are canonical objects attached to T such as $B_n(T)$, the Boolean algebra of formulas in free variables $x_1, .., x_n$ up to equivalence modulo T, and the type spaces $S_n(T)$ of complete n-types (ultrafilters on $B_n(T)$).

Introduction II

- Everything I say could be expressed in terms of the category of type spaces (including $S_I(T)$ for I an infinite index set).
- However it has become standard to work in a fixed saturated model $\bar{M}$ of T, and to study the category $Def(\bar{M})$ of sets $X \subseteq \bar{M}^n$ definable, possibly with parameters, in $\bar{M}$, as well as solution sets X of types $p \in S_n(A)$ over small sets A of parameters.
- Let us remark that the structure $(\mathbb{C}, +, \cdot)$ is a saturated model of ACF_0, but $(\mathbb{R}, +, \cdot)$ is not a saturated model of RCF.
- The subtext is the attempt to find a meaningful classification of first order theories.

Stable theories I

- The *stable* theories are the "logically perfect" theories (to coin a phrase of Zilber). They came to prominence through Shelah's work on classifying theories T according to the number $I(\kappa, T)$ of models of T of cardinality κ, as κ varies.
- The class of stable theories is rather small, with mathematically interesting examples being (the theories of) abelian groups, separably closed fields, differentially closed fields, and (more recently) the free group on ≥ 2 generators.
- A formal definition of stability (of a theory T) is that there do not exist a formula $\phi(x, y) \in L$ and $a_i, b_i \in \bar{M}$ such that $\bar{M} \models \phi(a_i, b_j)$ if and only if $i < j$.
- But a characteristic property of stable theories is the existence of a canonical $\{0, 1\}$-valued measure on "definable sets" (which underlies all the machinery behind Shelah's counting models).

Stable theories II

- Let us start with an example for those familiar with naive algebraic geometry, the (archetypical) stable theory being ACF_0, which has quantifier elimination in the language $\{+, -, \cdot, 0, 1\}$, and where our saturated model is $\mathbb{C}$.
- Let $X \subseteq \mathbb{C}^n$ be an irreducible algebraic variety (any algebraic variety is a finite union of such), and Y a definable subset of X.
- Call Y *large* in X, if it contains a Zariski open (and small otherwise). Then by the definition of irreducible (+ QE) one sees that precisely one of $Y, X \setminus Y$ is large, giving the required $\{0, 1\}$ valued measure on definable subsets of X, a complete type over $\mathbb{C}$ and the "generic point" of X in the sense of algebraic geometry.
- Note that this fails for X a real algebraic variety, Y a definable (in $(\mathbb{R}, +, \cdot)$) subset of X and with the Euclidean topology in place of the Zariski topology.

Stable theories III

- Shelah found (in hindsight) a general model-theoretic substitute for "Zariski open", but now with X being the set of realizations of a complete type $p(x)$ over a small elementary submodel M_0 of $\bar{M}$, to smoothen things out.
- Let Y be a relatively definable subset of X (i.e. of the form $\phi(x,b)^{\bar{M}} \cap X$, for some formula $\phi(x,b)$ with witnessed parameters b).
- Call Y *small* in X if $p(x) \cup \{\phi(x,b)\}$ *forks over* M_0, namely for some indiscernible over M_0 sequence $(b = b_0, b_1, b_2, \ldots.)$, $p(x) \cup \{\phi(x,b_i) : i < \omega\}$ is inconsistent, that is to say, $X \cap \bigcap\{Y_i : i < \omega\} = \emptyset$.
- By definition Y is *large* in X if it is not small in X.

Stable theories IV

With this notation we have

Theorem 0.1

(T stable.)
(i) For any relatively definable subset Y of X, either Y or $X \setminus Y$ is large in X, giving rise to a unique *global "nonforking extension" $p'(x)$ of $p(x)$ (a certain complete type over $\bar{M}$, or $\{0,1\}$-valued measure on definable sets).*
(ii) $p'(x)$ is both definable over, and finitely satisfiable in, M_0.
(iii) A technical condition on Morley sequences*: if $(a_1, a_2, ...)$ is any "Morley sequence" in p' over M_0, then for any formula $\phi(x,y)$ with parameters from M_0, and $b \in \bar{M}$, $\phi(x,b) \in p'$ (i.e. defines a "large" subset of X) if and only if $\bar{M} \models \phi(a_i, b)$ for all but finitely many i.*

Stable theories V

- There is an "equivariant" version (i.e. in the presence of a group operation). Let G be a definable group, which we assume to be "connected" (no proper definable subgroup of finite index).
- For $Y \subseteq G$ definable, call Y generic if finitely many left translates of Y cover G.
- Then, assuming stability of T, the family of nongenerics is a proper ideal (in the Boolean algebra $Def(G)$ of definable subsets of G), and for $Y \subseteq G$ definable, exactly one of Y, $G \setminus Y$ is generic.
- This gives rise to a $\{0, 1\}$-valued measure on the $Def(G)$ (the global generic type of G), which is moreover the unique left (right) invariant such measure on $Def(G)$. Note the formal analogy with uniqueness of Haar measure on compact groups.

NIP theories.

- T is stable if and only if T is simple AND T has NIP (not the independence property).
- Simple theories are those without the "tree property" which I will not define, but simple unstable theories include the random graph, nonprincipal ultraproducts of finite fields, as well as the model companion $ACFA$ of ACF + "σ is an automorphism".
- T has NIP if whenever $(a_i : i < \omega)$ is an indiscernible sequence (over some base set) and $\phi(x, b)$ any formula, then either for eventually all i, $\models \phi(a_i, b)$ or for eventually all i, $\models \neg\phi(a_i, b)$. Unstable NIP theories include RCF, algebraically closed valued fields ($ACVF$), the p-adics, Presburger.
- To what extent are general NIP theories informed by stability?

Generically stable types.

- For any theory T and $p(x) \in S(M_0)$, we call p *generically stable* if it satisfies (i), (ii), (iii) of Theorem 0.1.
- A "generically stable" group G is by definition a connected definable group with a left G-invariant generically stable type (necessarily unique).
- If T has NIP and G is a definably amenable definable group without any proper, nontrivial, "type-definable" subgroups, then G is generically stable.
- If T has NIP then the family of (global) generically stable types (in a given sort S) has the structure of a $*$-definable (or pro-definable) set, and for $T = ACVF$ and the sort S an algebraic variety V, this set, equipped with a certain topology, is a version of *Berkovich space* over V (in rigid algebraic/analytic geometry).

Keisler measures I

- However in o-minimal theories like RCF there are NO (nontrivial) generically stable types (or groups).
- But we can recover, even in the o-minimal case, stable-like behaviour if we pass from complete types ($\{0,1\}$-valued measures on definable sets) to $[0,1]$-valued measures on definable sets.
- A Keisler measure $\mu(x)$ over M_0 is a finitely additive probability measure on formulas $\phi(x)$ over M_0 (and identifies with a regular Borel probability measure on the Stone space $S_x(M_0)$).
- When $M_0 = \bar{M}$ we speak of a global Keisler measure. A special case of a Keisler measure $\mu(x)$ over M_0 is a complete type $p(x)$ over M_0.
- In NIP theories Keisler measures have "small support".

Keisler measures II

- Assume T has NIP.
- A Keisler measure $\mu(x)$ over (small) M_0 is said to be *generically stable* if it satisfies the analogues (i)', (ii)', (iii)', of (i), (ii), (iii) of Theorem 0.1.
- (i)' says that $\mu(x)$ has a unique global nonforking extension $\mu'(x)$ over $\bar{M}$ (nonforking meaning that every formula $\phi(x)$ over $\bar{M}$ such that $\mu'(\phi(x)) > 0$ does not fork over M_0).
- (ii)' is uncontroversial: Definability of μ' over M_0 means that for any $\phi(x,y) \in L$ the map taking $tp(b/M_0)$ to $\mu'(\phi(x,b)$ is continuous. Finite satisfiability in M_0 means that any formula with positive μ' measure is satisfied by an element from M_0.
- (iii)' is a bit subtle, and I won't give it, but in any case it is a nontrivial theorem that both (i)' and (iii)' follow from (ii)'.

Examples I

- Lebesgue measure on the real unit interval $[0,1]$ induces a Keisler measure $\mu(x)$ over $\mathbb{R}$ with support the definable set $0 \leq x \leq 1$ in the theory RCF.
- μ is not only generically stable, but is *smooth*, namely has a *unique* extension to a Keisler measure over a saturated model.
- In fact the same holds for the Keisler measure over $\mathbb{R}$ induced by *any* Borel probability measure on a real semialgebraic set. Likewise for $\mathbb{Q}_p$ and $Th(\mathbb{Q}_p, +, \cdot)$ in place of $\mathbb{R}$ and RCF.

Examples II

- There is a general conjecture around that any NIP theory can be decomposed into a stable part and a "purely unstable" part.
- Pierre Simon has suggested the following definition of a "purely unstable" NIP theory: every generically stable measure is smooth, in particular there are no nonrealized (nonalgebraic) generically stable types.
- Moreover he has confirmed that some basic NIP theories such as o-minimal theories and $Th(\mathbb{Q}_p)$ are purely unstable in this sense.

Examples III

- Note that by definition of an NIP theory, for any indiscernible sequence $(a_i : i \in I)$ (where $(I, <)$ is a totally ordered index set) and parameter set B, the average type $Avtp(I/B)$ of I over B is well-defined: the collection of formulas $\phi(x)$ over B which are eventually true of the a_i.
- For $I \subseteq M_0$, $Avtp(I/M_0)$ need not be generically stable.
- However if $(a_i : i \in I)$ is an indiscernible segment, i.e. I is the real unit interval, then we can form the average *measure* $Avms(I/M_0)$ which will be a generically stable measure.
- Where by definition the measure of a formula $\phi(x)$ over M_0 is the usual measure of $\{i \in [0,1] : \models \phi(a_i)\}$ (a finite union of intervals and points, by NIP).

Examples IV

- A definable group G is fsg (finitely satisfiable generics) if for some global type $p(x) \in S_G(\bar{M})$ every left translate of p is finitely satisfiable in some fixed small elementary substructure M_0.
- This is an abstract notion of "definably compact", agrees with it in familiar examples (o-minimal, $ACVF$, p-adics..), and also includes arbitrary definable groups in stable theories.
- A rather satisfying common generalization of the uniqueness of generic types in stable groups, and uniqueness of Haar measure in compact groups is:

Examples V

Theorem 0.2

(Assume T has NIP.) Let G be a definable group with fsg. Then G is generically stable for measure. Namely there is a global left G-invariant Keisler measure μ on $Def(G)$ which is generically stable. Moreover μ is the unique left (right) G-invariant global Keisler measure on $Def(G)$.

Examples VI

- Here we go outside the "tame" environment.
- Let L be a first order language with among other things a predicate P.
- Let $\{M_i : i \in I\}$ be a family of L-structures, with $X_i =_{def} P^{M_i}$ finite.
- Let M be an ultraproduct of the M_i, and $X = P^M$.
- Then the counting measures on the X_i give rise to a Keisler measure on $Def(X)$ over M, which, after tinkering a bit with the language L can be assumed to be *definable* over $\emptyset$ (but not generically stable).
- Borrowing ideas from the theory of definable groups in simple (rather than NIP) theories, Hrushovski recently used such Keisler measures to give partial answers to conjectures of Ben Green on finite approximate subgroups of arbitrary groups.