Schemes for rings with a Hasse derivation

Franck Benoist (University of Freiburg)

June 8, 2007
Definition 1 A Hasse derivation on a ring $A$ is a sequence $D = (D_i)_{i \in N}$ of maps $A \rightarrow A$ such that:

- $D_0 = id_A$
- $D_i$ is additive for all $i$
- $D_i(xy) = \sum_{m+n=i} D_m(x)D_n(y)$ (generalized Leibniz rule)
- $D_i \circ D_j = \binom{i+j}{i} D_{i+j}$ (iteration rule)
D-ring (or Hasse ring): commutative unitary ring with a Hasse derivation.
D-field (or Hasse field): commutative field with a Hasse derivation.
D-homomorphism: homomorphism which commutes with the Hasse derivation.
D-ideal: ideal stable by the application of the Hasse derivation.
\{B\}:= D-ideal generated by a subset \( B \) of a D-ring=ideal generated by the \( D_i(b) \)'s, \( b \in B, i \geq 0 \).
A model-theoretic motivation

We want an analogue of algebraically closed fields as universal domain among Hasse fields.

**Definition 2** A Hasse field $K$ is existentially closed if every finite system of equations and inequations which has a solution in a Hasse field extension of $K$ has a solution in $K$.

**Theorem 1 (Robinson-Ziegler)** For any $p$ (0 or prime number), existentially closed Hasse fields of characteristic $p$ are axiomatizable.

For $p = 0$, existentially closed Hasse fields are differentially closed fields.

For $p > 0$, existentially closed fields are non perfect separably closed Hasse fields which are strict, i.e. $D_1(x) = 0$ iff $\exists y, x = y^p$. 
An analogue of the Weil approach of algebraic varieties

D-algebraic (affine) variety: set of zeros of D-polynomials in a (saturated) closed Hasse field $K$

Some results analogue to the case of algebraic varieties:

- Analogue of Hilbert’s Nullstellensatz

- Analogue of a theorem of Weil about constructible groups in an algebraically closed field: the category of connected infinitely definable groups in $K$ is equivalent to the category of connected D-algebraic groups over $K$
Affine D-schemes

$A$ a D-ring.

$V = Spec_D(A) :=$ set of prime D-ideals of $A$.

Topology on $V$: trace of the Zariski topology of $Spec(A)$ via the inclusion $Spec_D(A) \subseteq Spec(A)$.

Closed sets: $\mathcal{V}_D(B) := \{ I \in Spec_D(A) \mid B \subseteq I \}$ for every $B \subseteq A$.

Basis of open sets: $\mathcal{D}(b) := \{ I \in Spec_D(A) \mid b \notin I \}$ for $b \in A$.

**Fact** For $B \subseteq A$, $\sqrt{\{B\}} =$intersection of all prime D-ideals containing $B$.

**Corollary**

- $\mathcal{V}_D(B) = \mathcal{V}_D(C)$ iff $\sqrt{\{B\}} = \sqrt{\{C\}}$
- $\mathcal{D}(b) \subseteq \bigcup_i \mathcal{D}(b_i)$ iff $b \in \sqrt{\{b_i\}_i}$
- each $\mathcal{D}(b)$ is compact for the induced topology
Structure sheaf

The structure sheaf $\mathcal{O}_V^D$ on $V$ is induced by the structure sheaf of $\text{Spec}(A)$ via the inclusion $\text{Spec}_D(A) \subseteq \text{Spec}(A)$. For $I \in V$, $A_I$ has a natural structure of $D$-ring. It gives a natural structure of sheaf of $D$-rings to $\mathcal{O}_V^D$, with $\mathcal{O}_{V,I}^D \simeq A_I$.

$\hat{A} := \mathcal{O}_V^D(V)$ is the $D$-ring of global sections. Each $f \in \hat{A}$ is given by a finite tuple $(a_i, b_i)_i$ such that:
- $V = \bigcup_i D(b_i)$
- for $I \in D(b_i)$, $f(I) = (a_i/b_i)_I$
An affine D-scheme is a ringed space in local D-rings of the form 
\((V, \mathcal{O}_V^D)\) for some \(V = \text{Spec}_D(A)\).
A morphism of D-schemes is a morphism of ringed spaces in local D-rings.

\(\text{Spec}_D\) is a contravariant functor from the category of D-rings into the category of affine D-schemes:
if \(\phi : A \rightarrow B\) is a D-homomorphism, we define

\[\text{Spec}_D(\phi) := (t_\phi, s_\phi) : \text{Spec}_D(B) \rightarrow \text{Spec}_D(A) :\]

- for \(I \in \text{Spec}_D(B)\), \(t_\phi(I) = \phi^{-1}(I) \in \text{Spec}_D(A)\)
- for \(I \in \text{Spec}_D(B)\),

\[s_\phi_I : A_{t_\phi(I)} \rightarrow B_I \]
\[(a/b) \mapsto (\phi(a)/\phi(b))\]
$Spec_D(A)$ vs $Spec_D(\hat{A})$

The natural $D$-homomorphism

$$\iota_A : A \rightarrow \hat{A}$$

is neither injective nor surjective in general.

However, does it induce an isomorphism

$$Spec_D(\iota_A) : Spec_D(\hat{A}) \rightarrow Spec_D(A)$$?
The “well mixed” case

Definition 3 Let $A$ be a $D$-ring. A proper $D$-ideal $I$ of $A$ is said to be well mixed if

$$ab \in I \Rightarrow aD_i(b) \in I, \forall i \geq 0.$$ 

The $D$-ring $A$ is said to be well mixed if the 0 ideal is well mixed.

Remarks
- $A$ is well mixed if every annihilators in $A$ are $D$-ideal
- any radical $D$-ideal is well mixed
- there is a smallest well mixed $D$-ideal in $A$, denoted by $0_{wm}$. We denote by

$$\pi_A : A \longrightarrow A_{wm}$$

the projection of $A$ onto the well mixed $D$-ring $A_{wm} := A/0_{wm}$. 
Assume that $A$ is well mixed. Then:

- the $D$-homomorphism $\iota_A$ is injective

- it is still true locally: for $b \in A$, the $D$-homomorphism 
  
  $A_b \rightarrow O^D_V(D(b))$

  induced by $\iota_A$ is injective

- $\iota_A$ is almost surjective: for each prime $D$-ideal $I$ of $A$, and $f \in \hat{A}$, there are $a, b \in A$, with $b \notin I$, such that $\iota_A(b)f = \iota(a)$
Theorem 2 (Kovacic) Assume $A$ is well mixed. Then $Spec_D(\iota_A)$ is an isomorphism.

Sketch of the proof
1. For $I \in Spec_D(A)$, we define
   \[ \mathcal{N}_A(I) := (eval_I)^{-1}(IA_I) \in Spec_D(\hat{A}). \]
   Then $t_{\iota_A} \circ \mathcal{N}_A = id_V$ without any assumption on $A$.
2. Since $\iota_A$ is almost surjective, $t_{\iota_A}$ is injective. Hence $t_{\iota_A}$ is an homeomorphism.
3. For $J \in Spec_D(\hat{A})$, $(s_{\iota_A})_J$ is injective because of the description of the inverse $\mathcal{N}_A$ of $t_{\iota_A}$. It is surjective because $\iota_A$ is almost surjective.
A partial result in the general case

**Proposition 1** There is a commutative diagram of $D$-homomorphisms

\[
\begin{array}{ccc}
A & \xrightarrow{\iota_A} & \hat{A} \\
\downarrow{\pi_A} & & \downarrow{\pi_{loc}} \\
A_{wm} & \xrightarrow{\iota_{A_{wm}}} & \hat{A}_{wm}
\end{array}
\]

It induces a commutative diagram of homeomorphisms

\[
\begin{array}{ccc}
Spec_D(A) & \xrightarrow{t\iota_A} & Spec_D(\hat{A}) \\
\downarrow{t\pi_a} & & \downarrow{t\pi_{loc}} \\
Spec_D(A_{wm}) & \xrightarrow{t\iota_{A_{wm}}} & Spec_D(\hat{A}_{wm})
\end{array}
\]
In particular, $\text{Spec}_D(A)$ and $\text{Spec}_D(\hat{A})$ are homeomorphic as topological spaces.

Are they isomorphic as $D$-schemes?

We know that the morphism of sheaves $s_{\iota A}$ is injective without any assumption on $A$.

It is surjective if and only if, for every $J \in \text{Spec}_D(\hat{A})$ and $I = t \iota(J)$, the evaluation map $\text{eval}_I : \hat{A}_J \rightarrow A_I$ is injective. A sufficient condition is that $\iota_A$ is almost surjective. We do not know any counter-examples for this property.