Semiabelian varieties over separably closed fields, maximal divisible subgroups, and exact sequences

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Abstract

Given a separably closed field K of characteristic $p > 0$ and degree of imperfection finite (often 1) we study the $\sharp$-functor which takes a semiabelian variety G over K to the maximal divisible subgroup of $G(K)$. We show that the $\sharp$-functor need not preserve exact sequences. We relate preservation of exactness to issues of descent as well as to model-theoretic properties of $G^\sharp$, and give an example where $G^\sharp$ does not have "relative Morley rank". We also mention characteristic 0 versions of our results, where differential algebraic methods are more prominent.

1 Introduction

For a semiabelian variety G over a separably closed field K of characteristic $p > 0$ and finite degree of imperfection, the group $p^\infty G(K) = \cap_n p^n(G(K))$

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played a big role in Hrushovski's proof of the function field Mordell-Lang conjecture in positive characteristic. The group $p^\infty G(K)$ which we also sometimes call $G^\sharp$, is *type-definable* in the structure $(K, +, \cdot)$. (Strictly speaking K should be taken to be "saturated" for this to be meaningful, and this will be assumed below). It was claimed in [Hr1] that $p^\infty G(K)$ always has *finite relative Morley rank* (see section 2.3 for the definition). One of the reasons or motivations for writing the current paper is to show that this is not the case: there are G such that $p^\infty G(K)$ does not even have relative Morley rank. (However $p^\infty G(K)$ *does* have finite U-rank which suffices for results such as Proposition 4.3 of [Hr1] to go through, hence the validity of the main results of [Hr1] is unaffected.)

As the second author noticed some time ago, the "relative Morley rank" problem is related in various ways to whether the p^∞ (or $\sharp$)-functor preserves exact sequences. So another theme of the current paper is to give conditions on an exact sequence $0 \to G_1 \to G_2 \to G_3 \to 0$ of semiabelian varieties over K which imply exactness of the sequence $0 \to G_1^\sharp \to G_2^\sharp \to G_3^\sharp \to 0$, as well as giving situations where the sequence of $G_i^\sharp$ is NOT exact.

A third theme relates the preservation of exactness by $\sharp$ to the issue of descent of a semiabelian variety G over K to the field of "constants" $K^{p^\infty} = \cap_n K^{p^n}$ of K.

If K has degree of imperfection e (meaning that K has dimension p^e as a vector space over its *pth* powers K^p), then K can be equipped naturally with e commuting iterative Hasse derivations. We will, for simplicity, mainly consider the case where $e = 1$ (so for example where $K = \mathbb{F}_p(t)^{sep}$), in which case we have a single iterative Hasse derivation $(\partial_n)_n$ whose field of absolute constants is K^{p^∞}. This differential structure on K will play a role in some proofs, by virtue of so-called D-structures on varieties over K. However p-torsion and Tate modules will be our central technical tools in the positive characteristic case.

The analogue in characteristic 0 of the differential field $(K, (\partial_n)_n)$ is simply a differentially closed field (K, ∂) (of characteristic zero). And for an abelian variety G over our characteristic 0 differentially closed field K we have what is often called the "Manin kernel" for G, the smallest Zariski-dense "differential algebraic" subgroup of $G(K)$, which we denote again by $G^\sharp$. The issues of preservation of exactness by $\sharp$ and the relationship to descent to the field $\mathcal{C}$ of constants, make sense in characteristic 0 too, and where possible we give uniform results and proofs.

Our paper builds on earlier work by the second author and Françoise Delon [BoDe2] where among other things, the groups $G^\sharp$ (in positive characteristic) are characterized as precisely the commutative divisible type-definable groups in separably closed fields. Our results, especially in characteristic 0, are also influenced by and closely related to themes in the third author's joint paper with Daniel Bertrand [BePi].

Let us now describe the content and results of the paper.

Section 2 recalls key notions and facts about differential fields, and semiabelian varieties over separably closed fields. We also discuss relative Morley rank, preservation of descent under isogeny, and some properties of $p^\infty G(K)$.

In section 3 we introduce the $\sharp$-functor in all characteristics and begin relating relative Morley rank to exactness. We also make some observations about descent of semiabelian varieties, D-structures, p-torsion, and Tate modules, proving for example that in positive characteristic the semiabelian variety G descends to the constants if and only if G has a D-group structure if and only if, in the ordinary case, all of the (power of p)-torsion of G is K-rational.

Section 4 contains the main results of the paper. The key result, Proposition 4.2, characterizes the obstruction to preservation of exactness by the $\sharp$-functor, and is proved in all characteristics. Proposition 4.3 concludes that if $0 \to G_1 \to G_2 \to G_3 \to 0$ is an exact sequence of semiabelian varieties (ordinary in characteristic p) such that G_1 and G_3 are defined over the constants, $\mathcal{C}$, then the sequence of $G_i^\sharp$'s is exact if and only if G_2 descends to $\mathcal{C}$. Together with results from section 3 we are then able to present our example (in positive characteristic) of a semiabelian variety G such that $G^\sharp$ does not have relative Morley rank (in fact the example is simply any nonconstant extension of a constant ordinary abelian variety by an algebraic torus). The remainder of section 4 contains both positive and negative results about preservation of exactness by $\sharp$ in various situations. For example in characteristic 0, the $\sharp$-functor applied to any exact sequence of *abelian varieties* preserves exactness, whereas there is a counterexample in positive characteristic.

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2 Preliminaries

2.1 Hasse fields

We summarise here basic facts and notation about the fields K that concern us. More details can be found in [BeDe], [Zie1] for the characteristic p case and [Mar] for the characteristic zero case.

If K is a separably closed field of characteristic $p > 0$ then the dimension of K as a vector space over the field K^p of p^{th} powers is infinite or a power p^e of p. In the second case, e is called the degree of imperfection (we will just say the "invariant") of K and we will be interested in the case when $e \geq 1$ (and often when $e = 1$). For e finite, a p-basis of K is a set $a_1, .., a_e$ of elements of K such that $\{a_1^{n_1} a_2^{n_2} ... a_e^{n_e} : 0 \leq n_i < p^e\}$ form a basis of K over K^p.

The first order theory of separably closed fields of characteristic $p > 0$ and invariant e (in the language of rings) is complete (and model complete). We call the theory $SCF_{p,e}$. It is also stable (but not superstable) and certain natural (inessential) expansions that we mention below, have quantifier elimination.

For R an arbitrary ring (commutative with a 1), an *iterative Hasse derivation* ∂ on R is a sequence $(\partial_n : n = 0, 1, ...)$ of additive maps from R to R such that
(i) ∂_0 is the identity,
(ii) for each n, $\partial_n(xy) = \sum_{i+j=n} \partial_i(x)\partial_j(y)$, and
(iii) for all i, j, $\partial_i \circ \partial_j = \binom{i+j}{i} \partial_{i+j}$.

Note that ∂_1 is a derivation, and that when R has characteristic 0, $\partial_n = \partial_1^n / n!$ (So in the characteristic 0 case the whole sequence $(\partial_n)_n$ is determined by ∂_1.)

By the *constants* of $(R, (\partial_n)_{n \geq 0})$ one usually means $\{r \in R : \partial_1(r) = 0\}$ and by the *absolute constants* $\{r \in R : \partial_n(r) = 0 \text{ for all } n > 0\}$. In this paper, we will mainly consider the field of absolute constants, denoted $\mathcal{C}$, and refer to them in the sequel as "the constants".

If ∂^1 and ∂^2 are iterative Hasse derivations on R we say that they commute if each ∂_i^1 commutes with each ∂_j^2.

Fact 2.1 *(i) If K is a separably closed field of invariant $e \geq 1$, then there are commuting iterative Hasse derivations $\partial^1, .., \partial^e$ on K such that the common constants of $\partial^1_1, .., \partial^e_1$ is K^p. In this case the common (absolute) constants of $\partial^1, .., \partial^e$ is the field $K^{p^\infty} = \cap_n K^{p^n}$.*
(ii) Moreover in (i), if $a_1, .., a_e$ is a p-basis of K, then each ∂^i_j is definable in the field K over parameters consisting of the $a_1, .., a_e$ and their images under the maps ∂^n_m ($n = 1, .., e$, $m \geq 0$).
(iii) The theory $CHF_{p,e}$ of separably closed fields of degree e, equipped with e commuting iterative Hasse derivations $\partial^1, .., \partial^e$, whose common field of constants is K^p, is complete, stable, with quantifier elimination (in the language of rings together with unary function symbols for each ∂^i_n, $i = 1, .., e$, $n > 0$).

Note that after adding names for a p-basis $a_1, .., a_e$ of the separably closed field K, we obtain for each n a basis $1, d_1, .., d_{p^n-1}$ of K over K^{p^n}, and the functions $\lambda_{n,i}$ such that $x = \sum_i (\lambda_{n,i}(x))^{p^n} d_i$ for all x in K, are definable with parameters $a_1, .., a_e$ in the field K. The theory of separably closed fields also has quantifier elimination in the language with symbols for a p-basis and for each $\lambda_{n,i}$. The relation between the λ-functions and the ∂^i_j is given in section 2 of [BeDe].

In the current paper we concentrate on the iterative Hasse derivation formalism. In fact when we mention separably closed fields K with an iterative Hasse structure, we will usually assume that $e = 1$ and so K is equipped with a single iterative Hasse derivation $\partial = (\partial_n)_n$. The basic example is $\mathbb{F}_p(t)^{sep}$ (where ^{sep} denotes separable closure) with $\partial_1(t) = 1$ and $\partial_i(t) = 0$ for all $i > 1$. The assumption that $e = 1$ is made here for the sake of simplicty, as some of the results we will be quoting are only explicitely written out for this case, but it will be no real restriction, thanks to:

Fact 2.2 *(see for example [BeDe]) Let K_0 be an algebraically closed field of characteristic p, and K_1 a finitely generated extension of K_0. Then there is a separably closed field K of degree of imperfection 1, extending, K_1 and such that $K_0 = K^{p^\infty}$.*

Our characteristic 0 analogue is simply a differentially closed field (K, ∂) of characteristic 0, where now ∂ is the single distinguished derivation (rather than a sequence). The corresponding first order theory is DCF_0, in the language of rings together with a symbol for ∂. The theory DCF_0 is complete with quantifier elimination, but is now ω-stable.

2.2 Characteristic p

Let K be a separably closed field of characteristic p and finite degree of imperfection $e \geq 1$, and let $\overline{K}$ denote an algebraic closure of K.

2.2.1 Separability and related issues

We first make some simple remarks about morphisms and varieties which are essential when working in characteristic p over non perfect fields.

Recall that if V and W are two irreducible varieties over K, and f is a dominant K-morphism from V to W, f is said to be *separable* if the field extension $K(W) < K(V)$ is separable.

If V is a variety defined over K, $V(K)$ denotes the set of K-rational points of V. Recall that when K is separably closed, $V(K)$ is Zariski dense in V.

Proposition 2.3 *Let G, H be two connected algebraic groups defined over K and f a surjective separable morphism from G to H . Then f takes $G(K)$ onto $H(K)$.*

Proof : **Claim 1** We can suppose without loss of generality that K is sufficiently saturated: Let $K_1 > K$ be saturated. Then f extends uniquely to a surjective separable morphism f_1 from $G \times_K K_1$ to $H \times_K K_1$. Suppose we have proved that $f(G(K_1)) = H(K_1)$. This is a first order statement about K_1 :

$$\forall y (y \in H \rightarrow \exists x (x \in G \wedge f(x) = y).$$

with parameters in K , hence as $K < K_1$, it is also true in K. □

So we can suppose that G, H and f are all defined over some small $K_0 < K$ and that K is $|K_0|^+$-saturated.

Claim 2. If $h \in H(K)$ is a generic point of H over K_0 (in the sense of algebraic geometry) then $h \in f(G(K))$.

Proof: We can find a generic point g of $G(\overline{K})$ over K_0 such that $f(g) = h$. By separability of f, $K_0(g)$ is a separable extension of $K_0(h)$, so contained in a separable closure of $K_0(h)(a_1, .., a_n)$ for some a_i which are algebraically independent over $K_0(h)$. Choosing, by saturation of K, $b_1, .., b_n \in K$, algebraically independent over $K_0(h)$, and an isomorphism taking the separable closure of $K_0(h)(a_1, .., a_n)$ to the separable closure of $K_0(h)(b_1, .., b_n)$, we find $g' \in G(K)$ such that $f(g') = h$. □

Now let $h \in H(K)$ be arbitrary. By Zariski-denseness of $H(K)$ and saturation of K we can find $h_1 \in H(K)$, generic over $K_0(h)$ (in the sense of algebraic groups). Let $h_2 = h_1^{-1}h$ which is also in $H(K)$ and also a generic point of H over $K_0(h)$. By Claim 2, both h_1 and h_2 are in the image of $G(K)$ under f. Hence h is too. □

When we say that an *exact sequence of algebraic groups*

$$0 \to G_1 \xrightarrow{g} G_2 \xrightarrow{f} G_3 \to 0$$

is defined over a field K, we mean that the algebraic groups G_1, G_2, G_3 are defined over K, that f, g are morphisms of algebraic groups which are defined over K *and separable*. Then G_3 is isomorphic (as an algebraic group) to $G_2/g(G_1)$ and we will often suppose that G_1 is a closed subgroup of G_2.

2.2.2 Semiabelian varieties

We now recall some very basic facts about semiabelian varieties (see for example [Mu]). We will be particularly interested in rationality issues, that is in the groups of K-rational points of some basic subgroups of $G(K)$.

Recall that a *semiabelian* variety G (over K) is an extension of an abelian variety by a torus, i.e.

$$0 \to T \to G \to A \to 0$$

where T is a torus defined over K, A is an abelian variety defined over K and the two morphisms are separable and defined over K (G is then also defined over K in the usual sense as an algebraic group).

The following facts hold when K is separably closed:

Fact 2.4 *(i) Let T be a torus defined over K. Then T is K-split, that is T is isomorphic* over *K to some product of the multiplicative group, $(\mathbb{G}_m)^{\times n}$. Any closed subgroup of T is then also defined over K.*
(ii) Semiabelian varieties are commutative and divisible, i.e. $G(\overline{K})$, the group of $\overline{K}$-rational points of G is a commutative divisible group.
(iii) Let G be a semiabelian variety defined over K, then any closed connected subgroup of G is defined over K.

Remark 2.5 *Over a separably closed field K of characteristic $p > 0$, the semiabelian varieties defined over K are exactly the commutative divisible algebraic groups defined over K.*

The behaviour of the torsion elements of G is particularly important. The next classical facts will enable us to fix some notation for the rest of the paper.

Fact 2.6 *Let G be a semiabelian variety over K, written additively, and*

$$0 \to T \to G \to A \to 0,$$

with $dim(A) = a$ and $dim(T) = t$
1. *If n is prime to p, then $[n] : G \mapsto G$, $x \mapsto nx$ is a separable isogeny of degree (= separable degree) n^{2a+t}. We denote by $G[n]$ the kernel of $[n]$, the points of n-torsion: $G[n] \cong (\mathbb{Z}/n\mathbb{Z})^{2a+t}$. By separability, $G[n] = G[n](K)$.*
2. *$[p] : G \mapsto G$ is an inseparable isogeny of degree p^{2a+t}, and of inseparable degree at least $p^{(a+t)}$. Hence there is some r, $0 \leq r \leq a$ such that, for every n,*

$$Ker[p^n] = G[p^n] \cong (\mathbb{Z}/p^n\mathbb{Z})^r.$$

We say that G is ordinary *if $r = a$.*
As $G[p^n]$ is finite, it is contained in $G(\overline{K})$, but not necessarily in $G(K)$.
3. *Let $G[p^\infty]$ or $G[p^\infty](\overline{K})$ denote the elements of G with order a power of p, and $G[p']$ or $G[p'](\overline{K})$ denote the elements of G with order prime to p. Then $G[p'] = G[p'](K)$ is Zariski dense in G.*
Note that, even for G ordinary, we may well have that $G[p^\infty](K) = \{0\}$.

We will also need the following easy observations:

Lemma 2.7 *Let $0 \to G_1 \to G_2 \xrightarrow{f} G_3 \to 0$ be an exact sequence of semiabelian varieties over K. Then*
1. The restriction of f to prime-to-p torsion remains exact, i.e.

$$0 \to G_1[p'] \to G_2[p'] \xrightarrow{f} G_3[p'] \to 0,$$

2. The restriction of f to the p^∞-torsion remains exact, i.e.

$$0 \to G_1[p^\infty] \to G_2[p^\infty] \xrightarrow{f} G_3[p^\infty] \to 0,$$

3. It follows that

$$0 \to TorG_1 \to TorG_2 \xrightarrow{f} TorG_3 \to 0,$$

where $TorG$ denotes the group of all torsion elements of G.

Proof : One need only check that for any n, if $a \in G_3[n]$, there is some $g \in G_2[n]$ such that $f(g) = a$: Let $h \in G_2(\overline{K})$ be such that $f(h) = a$. Then $nf(h) = f(nh) = 0$, hence $nh \in G_1(\overline{K})$. If $nh \neq 0$, by divisibility of $G_1(\overline{K})$, let $t \in G_1(\overline{K})$ be such that $nt = nh$. then $f(h - t) = a$ and $h - t \in G[n]$. □

Divisibility by p also behaves quite differently in $G(\overline{K})$ and in $G(K)$. Let

$$p^\infty G(K) := \bigcap_{n \geq 1} [p^n]G(K).$$

Lemma 2.8 *1. $G(K)$ is n-divisible for any n prime to p.*
2. For n prime to p, for every k, $G[n] \subset [p^k]G(K)$.
3. $G[p']$ is a divisible subgroup of $G(K)$.
4. $p^\infty G(K)$ is n-divisible for any n prime to p.
5. $p^\infty G(K)$ is infinite and Zariski dense in G.
6. $p^\infty G(K)$ is the biggest divisible subgroup of $G(K)$.

Proof : 1. Because $[n]$ is separable, $[n]$ induces a surjection from $G(K)$ onto $G(K)$ (by 2.3).
2. Let $g \in G[n]$, n prime to p. Let a, b be integers such that $an + bp^k = 1$; then $ang + bp^k g = g = p^k(bg)$ and $bg \in G(K)$. Furthermore note that bg has finite order prime to p.
3. Clear from the above.
4. For every k, $p^k G(K)$, being a homomorphic image of the group $G(K)$ is also n-divisible. It follows that $p^\infty G(K)$ is n-divisible.
5. By 2. $p^\infty G(K)$ contains $G[p']$, which is infinite and Zariski dense in G.
6. It suffices to show that $p^\infty G(K)$ is p-divisible. This will follow from the finiteness of p^n-torsion for every n. Let g be any element in $G(K)$, consider the following tree, $T(g)$, indexed by finite sequences of elements of $\mathbb{N}$: $g_\emptyset = g$, for any g_s in the tree, the successors of g_s are the finite number of elements $g_{s\frown i}$ in $G(K)$ such that $[p](g_{s\frown i}) = g_s$ (by finiteness of p-torsion). Then:

– for any s of length $n > 0$, $[p]g_{s\frown i} = g_s$, in particular for any s of length n, $[p^n]g_s = g$. Conversely, if, for some n, $[p^n]h = g$ and $h \in G(K)$, then $h = g_s$ for some s of length n,

– If $g \in p^\infty G(K)$, the tree $T(g)$ is infinite. It is a finitely branching tree, so by Koenig's Lemma, it must have an infinite branch.

– Conversely, suppose that there is an infinite branch in $T(g)$, then $g \in p^\infty G(K)$.

So let $g \in p^\infty G(K)$, Let $X = \{g_s\}$ be the elements along an infinite branch. Then for any k, $[p^k]g_s = g$ where s has length k. And $T(g_s)$ has an infinite branch, $X_s = \{g_t \in X; t \supset s\}$, so $g_s \in p^\infty G(K)$. □

Various equivalent characterizations of $p^\infty G(K)$ were given in [BoDe2]. But the following one was omitted at the time. Recall that an *infinitely definable* set in K^n, denoted $\bigwedge\!\!\bigwedge$-definable set, is a subset of K^n which is the intersection of a small (size strictly smaller than the cardinality of K) collection of definable subsets of K^n.

Proposition 2.9 *Suppose that K is ω_1-saturated. Let G be a semiabelian variety defined over K. Then $p^\infty G(K)$ is the smallest $\bigwedge\!\!\bigwedge$-definable group of $G(K)$ which is Zariski dense in G.*

Proof : Let H be any $\bigwedge\!\!\bigwedge$-definable subgroup of $G(K)$, also Zariski dense in G. By stability, H is a decreasing intersection of definable subgroups of $G(K)$, $(H_i)_{i \in I}$. Certainly each H_i is itself Zariski dense in H. By [BoDe1], the connected component of H_i, C_i is also definable in $G(K)$ and has finite index in H_i. It follows that it is also Zariski dense in G.

Now, for every $r \geq 1$ the (definable) subgroup $[p^r]C_i$ is also Zariski dense in G. It follows by compactness, that $\cap_{n\geq 1}[p^n]C_i$ is also Zariski dense in G. But $\cap_{n\geq 1}[p^n]C_i$ is a divisible group, and $p^\infty G(K)$ is the unique divisible subgroup of $G(K)$ which is Zariski dense in G ([BoDe2], Prop. 3.6). So $p^\infty G(K) = \cap_{n\geq 1}[p^n]C_i$ for every i and is hence contained in H. □

2.2.3 Isogenies and descent in char.p

We will not necessarily directly use all the classical facts about isogenies recalled below, but they give a picture of the various problems linked to descent questions in characteristic p.

In this section, K is any separably closed field of characteristic $p > 0$, G and H are semiabelian varieties defined over K.

Note first that if f is any morphism (= morphism of algebraic groups) from G to H, both defined over K, then f is also defined over K: by 2.4, the graph of f, which is a closed connected subgroup of $G \times H$ is also defined over K.

Recall that an *isogeny* is a surjective morphism of algebraic groups with finite kernel.

It is classical that if A is a semiabelian variety over K, for every n the n^{th}-Frobenius isogeny $Fr^n : A \longrightarrow Fr^n A$ ($Fr^n A$ is then defined over K^{p^n}) admits a dual isogeny, the n^{th}-Verschiebung, denoted $V_n : Fr^n A \longrightarrow A$, such that $V_n \circ Fr^n = [p^n]_A$ and $Fr^n \circ V_n = [p^n]_{Fr^n A}$. It is easily seen, counting degrees, that:

Fact 2.10 *If G is ordinary, then for every n, the Verschiebung V_n is separable.*

Lemma 2.11 *Let G be a semiabelian variety defined over K. Then if $a \in p^n G(K)$, there exists $b \in G(K)$ such that $a \in K(b^{p^n})$. So if G is defined over K^{p^n}, then $[p^n]G(K) \subset G(K^{p^n})$ and in particular $p^\infty G(K) = p^\infty G(K^{p^n})$.*

Proof: Consider the n^{th}-Verschiebung V_n, described above. If $a \in p^n G(K)$, then $a = p^n b$, for some $b \in G(K)$, and $a = V_n(b^{p^n})$. If G is defined over K^{p^n}, then the Verschiebung is also defined over K^{p^n} and $a \in K^{p^n}(b^{p^n}) = K^{p^n}$. □

Abelian varieties have one specific very important property:

Fact 2.12 *Let A be an abelian variety defined over K. Then A is isogenous over K to a finite product of simple (i.e. which have no proper nontrivial closed connected subgroup) abelian varieties.*

Let $K_0 < K_1$, with K_0 algebraically closed, and let G_1 be a semiabelian variety defined over K_1. We will say that G_1 *descends* to K_0 if there is a semiabelian variety G_0, defined over K_0 and an **isomorphism** f between G_1 and $G_0 \times_{K_0} K_1$.

In characteristic 0, any semiabelian variety which is isogenous to one defined over some algebraically closed K_0 descends, in the sense above, to K_0 (the proof is identical to that of the following lemma). The situation is more complicated in characteristic p.

Lemma 2.13 *Let f be a separable isogeny from G_1 to H_1, both being semiabelian varieties. If G_1 is defined over some algebraically closed field K_0, then H_1 descends to K_0.*

Proof: As f is a separable isogeny, the kernel of f is a finite closed subgroup of $G_1(K_0)$, H, of cardinality the degree (= separable degree) of f. Then $G' := G_1/H$ is a semiabelian variety defined over K_0, and f induces an isomorphism from H_1 onto G'. □

The following is more complicated but also classical.

Proposition 2.14 *Let $K_0 \subset K_1$, with K_0 algebraically closed. Let A be an abelian variety defined over K_1, B an abelian variety defined over K_0 and f a separable isogeny from A onto B. Then A is isomorphic to $A' \times_{K_0} K$ for some abelian variety A' over K_0.*

Proof: This is a particularly simple case of the "Proper base change theorem" (see for example in [SGA1] or [Mil]). Consider N the kernel of f, which is a finite subgroup of $A(K_1)$, The set of abelian varieties over K_1 which contain N and are isomorphic to $B \times_{K_0} K_1$ are parametrized by a certain cohomology group $H^1(B \times_{K_0} K_1, N)$. Now let N' be an algebraic group (finite of course) defined over K_0 which is isomorphic to N. The base change theorem says that $H^1(B \times_{K_0} K_1, N)$ is isomorphic to $H^1(B, N')$, and through this isomorphism, A will be isomorphic to some $A_0 \times_{K_0} K_1$, for A_0 defined over K_0. □

In the case of dimension one, one does not need the assumption that f is separable:

Proposition 2.15 *Let $K_0 < K$, with K_0 algebraically closed. Let A be an elliptic curve defined over K, B an elliptic curve defined over K_0, and f an isogeny from A onto B. Then A is isomorphic to $B' \times_{K_0} K$ for some elliptic curve B' over K_0.*

Proof: First go up to $\overline{K}$, the algebraic closure of K, and consider the situation over $\overline{K}$. By the remark at the beginning of the section, it suffices to show that there exists an isomorphism g, defined over $\overline{K}$ from $A \times_K \overline{K}$ to some $B' \times_K \overline{K}$ where B' is defined over K_0, So we can suppose that K itself is algebraically closed.
Consider the inverse isogeny, h from B onto A, defined over K. As K is perfect, the isogeny h factors through some power of the Frobenius (see for example [Si]):

$$B \to Frob^n B \xrightarrow{g} A$$

where g is now a separable isogeny from $Frob^n B$ onto A, defined over K. As K_0 is algebraically closed, and $Frob^n B$ is also defined over K_0, Lemma 2.13 now applies. □

Remark 2.16 *In section 3.3 we will give some further results about isogenies and descent which seem to be less classical. In fact, we have given here* *the proof for 2.15 as it is particularly simple, but the above result is also a direct consequence of the fact (Corollary 3.19) that if A is an ordinary abelian variety, which is isogenous to one defined over some algebraically closed field K_0, then A descends to K_0. In dimension bigger than 1, the above is no longer true for inseparable isogenies, in the case of non ordinary abelian varieties: For any abelian variety A there is a one-one correspondence between (isomorphism classes of) purely inseparable isogenies and sub p-Lie algebras of Lie A (see [Se] or [Mu]): It follows that for any supersingular elliptic curve E over $\overline{\mathbb{F}_p}$, there is an abelian variety A, isogenous to $E \times E$, which cannot be isomorphic to any abelian variety defined over $\overline{\mathbb{F}_p}$.*

2.3 Relative Morley Rank

In this section T will be a complete theory, and we work in a given very saturated model M of cardinality κ say. We will here define *relative* Morley rank, namely Morley rank inside a given $\bigwedge$-definable set. This was called *internal Morley dimension* in [Hr1]. By an $\bigwedge$-definable set (infinitely definable set) we mean a subset of some M^n which is the intersection of a small (size $< \kappa$) collection of definable subsets of M^n (that is the set of realizations of a partial type over a small set of parameters). We will fix an $\bigwedge$-definable set $X \subseteq M^n$.

If X is an infinitely definable subset of M^n, by a *relatively definable* subset of X we mean a subset of the form $Z = X \cap Y$ for $Y \subseteq M^n$ definable with parameters. Then we can define in the usual way Morley rank for relatively definable subsets Z of X:
(i) $RM_X(Z) \geq 0$ if Z is nonempty.
(ii) $RM_X(Z) \geq \alpha + 1$ if there are $Z_i \subseteq Z$ for $i < \omega$ which are relatively definable subsets of X, such that $Z_i \cap Z_j = \emptyset$ for $i \neq j$ and $RM_X(Z_i) \geq \alpha$ for all i.
(iii) for limit ordinal α, $RM_X(Z) \geq \alpha$ if $RM_X(Z) \geq \delta$ for all $\delta < \alpha$.

As in the absolute case we obtain (relative) Morley degree. Namely suppose that $RM_X(Z) = \alpha < \infty$. Then there is a greatest positive natural number d such that Z can be partitioned into d (relatively in X) definable sets Z_i such that $RM_X(Z_i) = \alpha$ for all i.

We will say that X has relative Morley rank if $RM_X(X) < \infty$.

Remark 2.17 *(i) Suppose that Y is a relatively definable subset of X. Then $RM_X(Y) = RM_Y(Y)$.*
(ii) We can also talk about the relative Morley rank $RM_X(p)$ of a complete type p of an element of X over a set of parameters. It will just be the infimum of the relative Morley ranks of the (relatively) definable subsets of X which are in p.
(iii) Suppose that T is countable and X is $\bigwedge$-definable over a countable set of parameters A_0. Then X has relative Morley rank if and only if for any countable set of parameters $A \supseteq A_0$ there are only countably many complete types over A extending X.

Now suppose that X, Y are $\bigwedge$-definable sets and $f : X \to Y$ is a surjective definable function. By definability of f we mean that f is the restriction to X of some definable function on a definable superset of X. Note that then each fibre $f^{-1}(c)$ of f is a relatively definable subset of X, so we can talk about its relative Morley rank (with respect to X or to itself, which will be the same by Remark 2.2 (i)).

Lemma 2.18 *Suppose X, Y are $\bigwedge$-definable sets and $f : X \to Y$ is surjective and definable.*
(i) Suppose that $RM_Y(Y) = \beta$ and for each $c \in Y$, $RM_X(f^{-1}(c)) \leq \alpha$. Then $RM_X(X) \leq \alpha(\beta + 1)$ if $\alpha > 0$, and $\leq \beta$ if $\alpha = 0$.
(ii) $RM_Y(Y) \leq RM_X(X)$.

Proof : (i) This is proved in the definable (absolute) case by Shelah [Sh] (Chapter V, Theorem 7.8) and Erimbetov [Erimb]. Martin Ziegler [Zie] also gives a self-contained proof which adapts immediately to our more general context.
(ii) is easier, and has the same inductive proof as in the definable (absolute) case, bearing in mind that because f is the restriction to X of a definable function on a definable superset of X, the preimage under f of any relatively definable subset of Y is a relatively definable subset of X. □

If $X = G$ is an $\bigwedge$-definable group with relative Morley rank then the general theory of totally transcendental groups applies, for example giving DCC on (relatively) definable subgroups, theory of generics, stabilizers, connected components, etc. Likewise if G has finite relative Morley rank then the general theory of definable groups of finite Morley rank applies. If one assumes stability of the ambient theory T, some of these facts may be easier to see (using for example the fact that G will be an intersection of definable groups). As our intended application or example is the stable theory $CHF_{p,e}$, there is no harm assuming stability, but we emphasize that it is not required.

We now consider an exact sequence of $\bigwedge\!\!\bigwedge$-definable groups $1 \to G_1 \to G_2 \xrightarrow{h} G_3 \to 1$. We will assume that $G_1 = Ker(h) \subseteq G_2$, and note again that G_1 is then a relatively definable (normal) subgroup of G_2. With this notation we have:

Corollary 2.19 *(i) Suppose G_1 and G_3 have (finite) relative Morley rank. Then so does G_2.*
(ii) Moreover if G_1, G_3 have finite relative Morley ranks k, s respectively, then $RM_{G_2}(G_2) = k + s$.

Proof : (i) follows immediately from Lemma 2.18.
(ii): By part (i) G_2 has finite relative Morley rank. But then the proof that U-rank and Morley rank coincide in definable groups of finite Morley rank (see [Pi-Po], Remark B.2(iii) for example) goes through in the present context to show that for complete types of elements of G_2^{eq}, U-rank coincides with relative Morley rank (as defined in Remark 2.2(ii).). In particular relative Morley rank on types is additive, so if b realizes the generic type of G_2 (over a base set of parameters), then as $tp(h(b))$ realizes the generic type of G_3 and $tp(b/h(b))$ is the generic of a translate of G_1, we see, writing $RM(b)$ for relative Morley rank of $tp(b)$ etc, that $RM(b) = RM(b, h(b)) = RM(b/h(b)) + RM(h(b))$. Hence $RM_{G_2}(G_2) = RM_{G_1}(G_1) + RM_{G_3}(G_3)$.
□

3 The ♯ functor and descent to the constants

3.1 The ♯ functor

Here K will be either a separably closed field of characteristic $p > 0$ and finite degree of imperfection, or a differentially closed field of characteristic 0 (so with distinguished derivation ∂). We distinguish the cases by "characteristic p", "characteristic 0". In the characteristic p case we will take K to be say ω_1-saturated. Definability will mean in the sense of the structure K.

G will be a semiabelian variety defined over K. In the characteristic 0 case, as DCF_0 is ω-stable we have DCC on definable subgroups of a definable group, so any $\bigwedge$-definable group is definable. In the characteristic p case, by stability, any $\bigwedge$-definable subgroup is an intersection of at most countably many definable groups.

Definition 3.1 *$G^\sharp$ is the smallest $\bigwedge$-definable subgroup of $G(K)$ which is Zariski-dense in G.*

Remark 3.2 *By Proposition 2.9, in characteristic p, $G^\sharp$ coincides with $p^\infty G(K)$. In characteristic 0, $G^\sharp$ is sometimes called the "Manin kernel" (see [Mar1]). In any case alternative characterizations and key properties are given in the Lemma following.*

Lemma 3.3 *(i) $G^\sharp$ can also be characterized as the smallest $\bigwedge$-definable subgroup of $G(K)$ which contains the (prime-to-p, in char. p case) torsion of G.*
(ii) $G^\sharp$ is connected (no relatively definable subgroup of finite index), and of finite U-rank in char. p, and finite Morley rank in char. 0.
(iii) If G is defined over the constants $\mathcal{C}$ of K, then $G^\sharp = G(\mathcal{C})$.

Proof : (i) Note first that the (prime-to-p) torsion is contained in $G(K)$. In the characteristic p case, $G^\sharp = p^\infty G(K)$ does contain the prime-to p-torsion. On the other hand as the prime-to p-torsion is Zariski-dense in G any subgroup of G containing the prime-to-p torsion is Zariski-dense. So the lemma is established in characteristic p. The characteristic 0 case is well-known and due originally to Buium. See for example Lemma 4.2 of [Pi] where it is proved that any definable Zariski-dense subgroup of a connected commutative algebraic group G contains $Tor(G)$.
(ii) $G^\sharp$ is connected as any finite index subgroup of a Zariski-dense subgroup is also Zariski-dense. In the characteristic 0 case, Buium [Bu1] showed that $G^\sharp$ has finite Morley rank. An account, using D-groups, appears in [BePi]. In the characteristic p case, finite U-rank of $G^\sharp$ was shown by Hrushovski in [Hr1], and follows easily from Lemma 2.11.
(iii) In characteristic p, this is a direct consequence of Lemma 2.11. In characteristic 0 it can be seen as follows: Assume G to be defined over $\mathcal{C}$. Note that $G(\mathcal{C})$ is definable in the differentially closed field K. As $\mathcal{C}$ is algebraically closed $G(\mathcal{C})$ is Zariski-dense in $G(K)$. (True for any variety defined over $\mathcal{C}$.)

If H is an $\bigwedge$-definable subgroup of $G(K)$, properly contained in $G(\mathcal{C})$, then H will be clearly an algebraic subgroup of $G(\mathcal{C})$, but then $H(K)$ is a proper algebraic subgroup of $G(K)$ containing H, so H could not be Zariski-dense in $G(K)$. $\square$

Lemma 3.4 *Let G, H be semiabelian varieties defined over K, and $f : G \to H$ a (not necessarily separable) rational homomorphism, also defined over K. Then*
(i) $f(G^\sharp) \subseteq H^\sharp$.
(ii) If f is (geometrically) surjective then $f(G^\sharp) = H^\sharp$.

Proof: (i) Let $Tor_{p'}(G)$ be the prime to p torsion (so all the torsion in char. 0). Note that $f(Tor_{p'}(G)) \subseteq Tor_{p'}(H)$. If (i) fails then $C = f(G^\sharp) \cap H^\sharp$ is a proper $\bigwedge$-definable subgroup of $H(K)$ which by Lemma 3.3 contains $f(Tor_{p'}(G))$. But then $f^{-1}(C) \cap G(K)$ is an $\bigwedge$-definable subgroup of $G(K)$ which contains $Tor_{p'}(G)$ and is properly contained in $G^\sharp$, contradicting Lemma 3.3.
(ii) If f is geometrically surjective then (by ω_1-saturation in the characteristic p case) $f(G^\sharp)$ is $\bigwedge$-definable and it must be Zariski-dense in H. By part (i), and the definition of $H^\sharp$, $f(G^\sharp) = H^\sharp$. $\square$

Remark 3.5 *(Characteristic p) Let $f : G \to H$ be as in the hypothesis of Lemma 3.4 (ii). If f is separable (that is induces a separable extension of function fields) then as we remarked in Proposition 2.3 $f|G(K) : G(K) \to H(K)$ is surjective. If f is not separable, f may no longer be surjective at the level of K-rational points, but nevertheless Lemma 3.4(ii) says it is surjective on the $\sharp$-points when K is ω_1-saturated.*

By Lemma 3.4 (i) we can consider $\sharp$ as a functor from the category of semiabelian varieties over K to the category of $\bigwedge$-definable groups in K. It is natural to ask whether $\sharp$ preserves exact sequences, and this is an important theme of the paper.

Recall that by an *exact sequence of algebraic groups* defined over a field K, we mean that the homomorphisms are not only defined over K but also separable. So we will be considering the situation of semiabelian varieties G_2, G_3 defined over K, a separable surjective rational homomorphism $f : G_2 \to G_3$ defined over K, with $Ker(f) = G_1$ connected and thus a semiabelian subvariety of G_2 defined over K. Then the sequence $0 \to G_1(K) \to G_2(K) \to$

$G_3(K) \to 0$ clearly remains exact (in the category of definable groups in K), using say 2.3 in the characteristic p case. By Lemma 3.4 the sequence

$$0 \to G_1^\sharp \to G_2^\sharp \to G_3^\sharp \to 0$$

will be exact if and only if

$$G_1^\sharp = G_1(K) \cap G_2^\sharp.$$

So the group $(G_1(K) \cap G_2^\sharp)/G_1^\sharp$ is the obstruction to exactness.

In the characteristic 0 case this group which is clearly of finite Morley rank, can be seen to be connected and embeddable in a vector group. By Lemma 4.2 of [Pi] for example, $G_1(K)/G_1^\sharp$ (as a group definable in K by elimination of imaginaries) embeds definably in $(K,+)^n$ for some n. Hence $(G_2^\sharp \cap G_1(K))/G_2^\sharp$ also embeds in $(K,+)^n$, and as such is a (finite-dimensional) vector space over the field of constants of K. Hence $(G_2^\sharp \cap G_1(K))/G_1^\sharp$ is connected. Note that, as $G_1^\sharp$ is also connected, it follows that $G_2^\sharp \cap G_1(K)$ itself is also connected.

The characteristic p case is different in an interesting way. Note first, that the group $(G_1(K) \cap G_2^\sharp)/G_1^\sharp$ is not even infinitely definable, it is the quotient of two $\bigwedge\!\!\bigwedge$-definable groups. Such groups are usually called "hyperdefinable".

We will recall the (model theoretic) definition of a connected component. First, if G is an $\bigwedge\!\!\bigwedge$-definable group in a stable theory, then we have DCC on intersections of uniformly relatively definable subgroups (see [Po] or [Wa]). What this means is that if $\phi(x,y)$ is a formula, then the intersection of all subgroups of G relatively defined by some instance of $\phi(x,y)$, is a finite subintersection. It follows that, working in a saturated model say, the intersection of all relatively definable subgroups of G of finite index, is the intersection of at most $|L|$ many (where L is the language). We call this intersection, G^0, the *connected component* of G. It is normal, and type-definable over the same set of parameters that G is. Moreover G/G^0 is naturally a profinite group. In the ω-stable case (or the relative Finite Morley Rank case as in section 2.3), by DCC on relatively definable subgroups, G^0 will itself be relatively definable and of finite index in G .

Lemma 3.6 *(Characteristic p) Let G_1 be a semiabelian subvariety of the semiabelian variety G_2, both defined over K. Then $G_1^\sharp$ is the connected component of $G_1(K) \cap G_2^\sharp$.*

Proof : First by 3.4, $G_1^\sharp$ is a subgroup of $G_1(K) \cap G_2^\sharp$. By Lemma 3.3 $G_1(K) \cap G_2^\sharp$ is $\bigwedge$-definable of finite U-rank. Hence, for any H $\bigwedge$-definable subgroup of $G_1(K) \cap G_2^\sharp$, classical U-rank inequalities for groups give us that $U(H[n]) + U([n]H) = U(H)$, As for each n the n-torsion of H is finite, $U(H[n]) = 0$, hence for any n, $[n]H$ has finite index in H. It follows that any $\bigwedge$-definable subgroup of $G_1(K) \cap G_2^\sharp$ is connected iff it is divisible. But $G_1^\sharp$ is the maximum divisible subgroup of $G_1(K)$. Thus $G_1^\sharp$ must coincide with the connected component of $G_1(K) \cap G_2^\sharp$

□

Remark 3.7 *By Lemma 3.6, the quotient $(G_1(K) \cap G_2^\sharp)/G_1^\sharp$ is a profinite group. If $G_2^\sharp$ had relative Morley rank, the quotient would have to be finite (as remarked before Lemma 3.6). We will see in section 4 an example where the quotient is infinite and give an explicit description of this quotient in terms of suitable Tate modules.*

For the record we now mention cases (in characteristic p) where $G^\sharp$ has (finite) relative Morley rank.

Fact 3.8 *(Characteristic p). Let G be a semiabelian variety over K. Then*
(i) If G descends to K^{p^∞} (in particular if G is an algebraic torus) then $G^\sharp$ has finite relative Morley rank.
(ii) If $G = A$ is an abelian variety then $G^\sharp$ has finite relative Morley rank.

Proof : (i) We may assume that G is defined over K^{p^∞}. Then by 2.11 $G^\sharp = p^\infty G(K) = G(K^{p^\infty})$. As K^{p^∞} is a "pure" algebraically closed field inside K, $G(K^{p^\infty})$ has relative Morley rank equal to the (algebraic) dimension of G.
(ii) The abelian variety A is isogenous to a product of simple abelian varieties. So we may reduce to the case where A is simple. In that case $A^\sharp$ has no proper infinite definable subgroup (2.16 in [Hr1] or Cor.3.8 in [BoDe2]). By stability, $A^\sharp$ has no proper infinite $\bigwedge$-definable subgroup. We will now use an appropriate version of Zilber's indecomposability theorem to see that $A^\sharp$ has finite relative Morley rank. As $A^\sharp$ has finite U-rank, there is some small submodel K_0 (over which $A^\sharp$ is defined) and a complete type $p(x)$ over K_0 extending "$x \in A^\sharp$", which has U-rank 1 (and is of course stationary). Let $Y \subseteq A^\sharp$ be the set of realizations of p. Then Y is an $\bigwedge$-definable subset of $A^\sharp$ which is "minimal", namely Y is infinite and every relatively definable subset of Y is either finite or cofinite. We claim that Y is "indecomposable" in $A^\sharp$,

namely for each relatively definable subgroup H of $A^\sharp$, $|Y/H|$ is 1 or infinite. For if not, then as remarked earlier the intersection of all the images of H under automorphisms fixing K_0 pointwise, will be a finite subintersection H_0, now defined over K_0, and we will have $|Y/H_0| > 1$ and finite, contradicting stationarity (or even completeness) of p. Let now X be a translate of Y which contains the identity 0. Then X is still a minimal $\bigwedge\!\!\!\bigwedge$-definable subset of $A^\sharp$. Moreover Theorem 3.6.11 of [Wa] applies to this situation, to yield that the subgroup B say of $A^\sharp$ which is generated by X is $\bigwedge\!\!\!\bigwedge$-definable and moreover of the form $X+X+...+X$ (m times) for some m. As noted above, it follows that $B = A^\sharp$, and so the function $f : X^m \to A^\sharp$ is a definable surjective function between $\bigwedge\!\!\!\bigwedge$-definable sets, in the sense of section 2.3. But as X is minimal, clearly $RM_X(X) = 1$ and $RM_{X^m}(X^m) = m$. By Lemma 2.18 (ii), $A^\sharp$ has finite relative Morley rank too.

□

3.2 D-structures and descent

Here again, we consider a model (K, ∂) of DCF_0 or of $CHF_{p,1}$, where in the latter case it is convenient to assume ω_1-saturation. In order to relate some properties of $G^\sharp$ with descent to constants, we introduce the tool of prolongations and D-structures.

We first give an *ad hoc* description of the prolongations. A more systematic definition can be found in [Bu2] or [Voj].

If $V \subseteq \mathbb{A}^m$ is a smooth irreducible algebraic variety over K, we define the n-th prolongation of V to be the Zariski-closure of the image of $V(K)$ by $\partial_{\leq n} := (\partial_0, \ldots, \partial_n)$,

$$\Delta_n V := \overline{\{\partial_{\leq n}(x) \colon x \in V(K)\}} \subseteq \mathbb{A}^{mn+1}.$$

This construction has functorial properties which allows us to build $\Delta_n V$ for any smooth irreducible variety over K, with the definable map $\partial_{\leq n} : V(K) \longrightarrow \Delta_n V(K)$ having Zariski-dense image. For $m \geq n \geq 0$, we have a natural projection morphism $\pi_{m,n} : \Delta_m V \longrightarrow \Delta_n V$ such that $\pi_{m,n} \circ \partial_{\leq m} = \partial_{\leq n}$.

In the case where $V = G$ is a connected algebraic group, each $\Delta_n G$ has a natural structure of algebraic group and the maps $\partial_{\leq n}$, $\pi_{m,n}$ are homomorphisms.

Definition 3.9 *Let G be a connected algebraic group defined over K. A D-structure on G is a sequence of homomorphic regular sections $s = (s_n)_{n\in\mathbb{N}}$ for the projective system $(\pi_{m,n} : \Delta_m G \longrightarrow \Delta_n G)_{m\geq n\geq 0}$, i.e. each $s_n : G \longrightarrow \Delta_n G$ is a regular homomorphism defined over K, and these homomorphisms satisfy $\pi_{m,n} \circ s_m = s_n$ and $s_0 = id_G$. For (G, s) an irreducible algebraic group with a D-structure over K, and L an extension of K, we denote by $(G, s)^{\partial}(L)$ the $\bigwedge\!\!\!\bigwedge$-definable subgroup of $G(L)$,*

$$(G, s)^{\partial}(L) = \{x \in G(L) \colon \partial_n(x) = s_n(x) \text{ for all } n \geq 0\}.$$

Remark 3.10 *Let G be a semiabelian variety over K. In order to define a D-structure on G, it suffices that, for some (any) generic point g of $G^\sharp(L)$ over K (L an extension of K), for any $n \geq 0$, $\delta_n(g) \in K(g)$. Indeed, because $G^\sharp$ is Zariski-dense in G, such a property induces a rational map from $G(L)$ to $\Delta_n G(L)$, which can be extended to an homomorphism s_n by a classical stability argument. We obtain in that way a D-structure on G because s_n coincides with $\partial_{\leq n}$ on the Zariski-dense subgroup $G^\sharp$, and the $\delta_{\leq n}$'s give a sequence of definable sections by definition.*
In particular, if G is defined over the constants $\mathcal{C}$, for each $g \in G^\sharp = G(\mathcal{C})$, $\partial_n(g) = 0$ for $n \geq 1$, hence we can define a natural D-structure on G. The two following results are a converse of this observation.

Fact 3.11 *For each $n \geq 0$, the kernel of $\pi_{n,0} : \Delta_n G \longrightarrow G$ is a unipotent group (see [Pi] in characteristic 0 or [Be] in arbitrary characteristic). It follows that G admits at most one D-structure, since the difference between two sections is an homomorphism $G \longrightarrow Ker(\pi_{n,0})$, hence zero.*

Proposition 3.12 *Let G be a semiabelian variety over K with a D-structure. Then G descends to the constants.*

Proof : In the characteristic 0 case, this result appears implicitly in [Bu1], but see Lemma 3.4 in [BePi] for more explanations.
In the characteristic p case, it is proved in [BeDe] (Proof of Theorem 4.4), that such a semiabelian variety G descends to K^{p^n} for every n (this is actually equivalent). Then it is shown, using moduli spaces, that if G is an abelian variety, G descends to K^{p^n} for every n if and only if G descends to $\mathcal{C} = \bigcap_n K^{p^n}$. The general case will follow from the lemma below. □

Lemma 3.13 *(Characteristic p) Let us consider a semiabelian variety*

$$0 \longrightarrow T \longrightarrow G \stackrel{f}{\longrightarrow} A \longrightarrow 0,$$

and suppose that G descends to K^{p^n} for all n. Then the same is true for A, and both descend to $\mathcal{C} = K^{p^\infty}$.

Proof : Let us consider the following commutative diagramm

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T & \longrightarrow & G & \stackrel{f}{\longrightarrow} & A & \longrightarrow & 0 \\ & & & & \uparrow \pi_{n,0} & & \uparrow \pi_{n,0} & & \\ & & & & \Delta_n G & \underset{\Delta_n f}{\longrightarrow} & \Delta_n A & & \end{array}$$

From our hypothesis, there is a D-structure on G, given by sections $s_n : G \longrightarrow \Delta_n G$. We claim that there is an induced D-structure on A. Indeed, $s_n(T)$ has to lie inside the linear part H of $\Delta_n G$, which is given by the exact sequence

$$0 \longrightarrow H \longrightarrow \Delta_n G \stackrel{f \circ \pi_{n,0}}{\longrightarrow} A \longrightarrow 0,$$

and since $f \circ \pi_{n,0} = \pi_{n,0} \circ \Delta_n f$,

$$0 \longrightarrow \mathrm{Ker}(\Delta_n f) \longrightarrow H \stackrel{\Delta_n f}{\longrightarrow} \mathrm{Ker}(\pi_{n,0}) \longrightarrow 0.$$

It follows that $\Delta_n f \circ s_n(T)$ lies in the unipotent group $\mathrm{Ker}(\pi_{n,0})$, hence is 0. The homomorphism $\Delta_n f \circ s_n$ factorizes through f: we find $t_n : A \longrightarrow \Delta_n A$ such that $\Delta_n f \circ s_n = t_n \circ f$. It follows that $\pi_{n,0} \circ t_n \circ f = \pi_{n,0} \circ \Delta_n f \circ s_n = f \circ \pi_{n,0} \circ s_n = f$, and since f is surjective, $\pi_{n,0} \circ t_n = \mathrm{id}_A$. And for $m \geq n$, since $Ker(\pi_{n,0})$ is unipotent, the homomorphism $\pi_{m,n} \circ t_m - t_n : A \longrightarrow \mathrm{Ker}(\pi_{n,0})$ is zero: we have obtained a D-structure on A.
As explained above, this implies ([BeDe]) that A descends to the constants. But we also know from [BeDe] that G descends to K^{p^n} for every n. It is classical that $\mathrm{Ext}(A, T) \simeq (\mathrm{Ext}(A, \mathbb{G}_m))^t \simeq (\hat{A})^t$, where $t = \dim(T)$ and $\hat{A}$ is the dual abelian variety of A, also defined over $\mathcal{C}$ (see for example [Se]). It follows that the isomorphism type of G is parametrized by a point in $\hat{A}(\bigcap_n K^{p^n}) = \hat{A}(\mathcal{C})$, that is G descends to the constants. □

In the following, we will only make explicit use of D-structures in characteristic 0; in characteristic p, we will use more usual objects, namely the Tate modules.
Note that in characteristic 0, since $\partial_i = \frac{1}{i!}\partial_1$, it suffices to have $s = s_1 : G \longrightarrow \Delta_1 G$ in order to define a D-structure; $\Delta_1 G$ is also known as the twisted tangent bundle of G. Let us quote the following from [BePi], section 3.1.

Fact 3.14 *(Characteristic 0) Let G be a semiabelian variety. The universal extension $\tilde{G}$ of G by a vector group (as defined in [Ro]) admits a unique D-structure s. Let us write $\tilde{G}$ as $0 \longrightarrow W_G \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 0$, and consider U_G the maximal D-subgroup of $(\tilde{G}, s)$ which is a subgroup of W_G. We still denote by s the D-structure induced on $\tilde{G}/U_G$. Then $G^\sharp$ is isomorphic to $(\tilde{G}/U_G, s)^\partial$. It follows from Proposition 3.12 that G descends to the constants if and only if $G \simeq \tilde{G}/W_G$ has a D-structure if and only if $U_G = W_G$.*

Furthermore the ∂ functor is exact on the class of algebraic D-groups ([KoPi]). In particular, $\left(\tilde{G}/U_G, s\right)^\partial \cong (\tilde{G}, s)^\partial/(U_G, s)^\partial$.

3.3 Torsion points, Tate modules and descent

We deal now with the characteristic p case; G being a semiabelian variety over any model (K, ∂) of $CHF_{p,1}$, that is any separably closed field of degree of imperfection 1.

Definition 3.15 *We define $\tilde{G}$ as the inverse limit*

$$\tilde{G} := \varprojlim (G \xleftarrow{[p]} G \xleftarrow{[p]} \ldots).$$

In particular, for L an extension of K (we will mainly consider $L = K$ or $L = \overline{K}$),

$$\tilde{G}(L) = \{(x_i)_{i\in\mathbb{N}} \in G(L)^{\mathbb{N}}; \forall i \geq 0, x_i = [p]x_{i+1}\}.$$

Let π_G be the projection on the "left component" G. The kernel of π_G is called the Tate-module of G, denoted by T_pG.
Its L-points in an arbitrary algebraically closed extension L of K coincide with the sequences of torsion points in $\overline{K}$,

$$T_pG(\overline{K}) = \{(x_i)_{i\in\mathbb{N}} \in G(\overline{K})^{\mathbb{N}}; x_0 = 0, \forall i \geq 0, x_i = [p]x_{i+1}\}$$

Let us remark that for a given $g_0 \in G(K)$, there is some $(x_i)_{i\in\mathbb{N}} \in \tilde{G}(K)$ with $g_0 = x_0$ if and only if $g_0 \in G^\sharp$; we deduce from this the relation between the Tate-module of G and $G^\sharp$.

Lemma 3.16 *The morphism π_G induces an exact sequence:*

$$0 \to T_pG(K) \to \tilde{G}(K) \stackrel{\pi_G}{\to} G^\sharp \to 0.$$

Objects such as $\tilde{G}(K)$ and $T_pG(K)$ are what are called "$*$-definable" groups in K, so the exact sequence in Lemma 3.16 is in the category of $*$-definable groups.
In the case of ordinary semiabelian varieties, with dimension of the abelian part a, it is well-known that $T_pG(\overline{K}) \simeq \mathbb{Z}_p^a$ (see [Mu], chapter IV). We relate now the part of the p^∞-torsion lying in K with issues of descent. Most of the following results seem to be well-known, see for example [Vol] for the description of the torsion of G for abelian schemes of maximal Kodaira-Spencer rank. But we have found no systematic exposition which we could quote and furthermore, we choose to give here particularly elementary proofs which are suitable for our purpose.

Proposition 3.17 *Let G be an ordinary semiabelian variety over K. Then for every n, $G[p^n](K) = G[p^n]$ if and only if G descends to K^{p^n}. In particular, G descends to K^{p^∞} if and only if $G[p^\infty](K) = G[p^\infty]$ if and only if $T_pG(K) = T_pG(\overline{K})$.*

Proof : Let us fix $n \geq 1$. If G descends to K^{p^n}, we may assume that G is defined over K^{p^n}, as is the Verschiebung V_n. Since G is ordinary, the kernel of V_n consists of K^{p^n}-rational points, and since $[p^n] = V_n \circ Fr^n$, $G[p^n] = Fr^{-n}(\mathrm{Ker}(V_n)) \subseteq G(K)$.
Conversely, assume that $G[p^n] \subseteq G(K)$. Since V_n is separable, G is isomorphic to the quotient $Fr^nG/\mathrm{Ker}(V_n)$. But $\mathrm{Ker}(V_n) = Fr^n(G[p^n])$ is a finite group of K^{p^n}-rational points, hence $Fr^nG/\mathrm{Ker}(V_n)$ is defined over K^{p^n}.
The "in particular" statement follows from Lemma 3.13. This was proved with the assumptions that K was ω_1-saturated, but we can easily reduce to this situation: Let L be an ω_1-saturated elementary extension of K. Then L is a separable extension of K, of same degree of imperfection and L^{p^∞} and K are linearly disjoint over K^{p^∞}. Applying 3.13, we conclude that $G \times_K L$ descends to L^{p^∞}, and by linear disjointness, that G descends to K^{p^∞}. □

Corollary 3.18 *Let K_0 be an algebraically closed field and $K_1 > K_0$ a finitely generated extension of K_0. Let G be an ordinary semiabelian variety over K_1. If $G[p^\infty](K_1) = G[p^\infty]$, then G descends to K_0.*

Proof : As K_0 is algebraically closed, K_1 is a separable extension of K_0, hence it is contained in the separable closure of $K_0(t_1, \ldots, t_n)$ for $t_1, \ldots, t_n$ algebraically independent. Then (Fact 2.2) there is a separably closed field K of degree of imperfection 1, extending, K_1 and such that $K_0 = K^{p^\infty}$. We can now apply Proposition 3.17 to conclude that G descends to K^{p^∞}. □

This yields easily the following result which we already mentioned in Section 2.2.3.

Corollary 3.19 *Let G be an ordinary semiabelian variety over some algebraically closed field K_0. If H is any semiabelian variety over $K_1 > K_0$ such that there is an isogeny f from G to H, then H descends to K_0.*

Proof : Let $K_2 < K_1$ be a finitely generated extension of K_0 over which H and the isogeny f from G to H are defined. We claim first that any point of p^∞-torsion in H is the image of a point of p^∞-torsion in G: indeed let $h \in H[p^\infty]$, i.e. for some m, $[p^m]h = 0$. Let $g \in G(\overline{K}_2)$, be a preimage of h, $f(g) = h$. Then $[p^m]g \in Kerf$. If f is purely inseparable, then f is injective on $G(\overline{K})$ and hence $g \in G[p^m]$. Otherwise, let n be the order of the finite group $Kerf$ in $G(\overline{K}_2)$. Then $n = p^r d$, where d is prime to p. By Bezout, $1 = ud + vp^m$, $u, v \in \mathbb{Z}$. Then $g = [ud]g + [vp^m]g$, so $f(g) = f([ud]g)$, and $[p^r][ud]g = 0$. Hence $h = f(e)$ for some $e := [ud]g \in G[p^\infty]$.
Now as G is defined over the algebraically closed field K_0, $G[p^\infty] = G(K_0)$ and hence by the above claim $G[p^\infty] = G(K_2)[p^\infty]$. We can now apply Corollary 3.18. □

Corollary 3.20 *Let $0 \longrightarrow C \longrightarrow B \longrightarrow A \longrightarrow 0$, be an exact sequence of ordinary abelian varieties with A and C defined over K_0 some algebraically closed field. Then B descends to K_0.*

Proof : By Poincaré reducibility theorem, B is isogenous to $A \times C$, which is defined over K_0, and we just have to apply Corollary 3.19. □

Remark 3.21 *(Thanks to A. Chambert-Loir and L. Moret-Bailly for pointing this out to us) The example described in 2.16 shows that the assumption* *that the varieties are ordinary in 3.20 is essential. We remarked that there is an abelian variety A isogenous to $E \times E$ for E a supersingular elliptic curve (hence defined over a finite field), which itself does not descend to $\overline{\mathbb{F}_p}$. Such an abelian variety A, which is of course not ordinary yields an example of an element of $EXT(E_1, E_2)$, where E_1, E_2 are elliptic curves over $\overline{\mathbb{F}_p}$, which* does not descend *to $\overline{\mathbb{F}_p}$. To see this, note first that every proper abelian subvariety of A must be isomorphic to an abelian variety defined over $\overline{\mathbb{F}_p}$: let ρ be the isogeny from A onto $E \times E$ and consider $B < A$. Then through ρ, B is isogenous to some proper abelian subvariety C of $E \times E$, which itself is defined over $\overline{\mathbb{F}_p}$ (Fact 2.4). Both C and B must have dimension one, hence by Proposition 2.15, B is isomorphic to some abelian variety defined over $\overline{\mathbb{F}_p}$. Now, pick any $E_1 < A$, of dimension one (there are some, as A is isogenous to $E \times E$), and consider $E_2 := A/E_1$. By Poincaré reducibility theorem E_2 is isogenous to some $C < A$, such that $A = E_1 + C$ and $E_1 \cap C$ is finite. So again by 2.15, E_2 descends to $\overline{\mathbb{F}_p}$.*

We complete this section with some easy remarks on torsion in $G(K)/G^\sharp$ in characteristic p which will immediately enable us to describe the link between the question of relative Morley rank and that of preservation of exactness.

Lemma 3.22 *(Characteristic p) Let G be a semiabelian variety defined over K.*
(i) $G[p^\infty](K)$ (the group of elements of $G(K)$ with order a power of p) is a direct sum of a divisible group and a finite group.
(ii) $G(K)/G^\sharp$ has finite torsion.
(iii) If G descends to K^{p^∞} then $G(K)/G^\sharp$ is torsion-free.
(iv) If $G(K)$ has trivial p-torsion then $G(K)/G^\sharp$ is torsion-free.

Proof : (i) $G[p^\infty](K)$ is a subgroup of $G[p^\infty]$ which is a finite direct sum of copies of the Prüfer group $\mathbb{Z}_{p^\infty}$.
As $G^\sharp$ is divisible, if $g \in G(K)$ and $ng \in G^\sharp$ then there is $h \in G^\sharp$ so that $ng = nh$ whereby $n(g-h) = 0$ so g is congruent mod $G^\sharp$ to an element of order n. We know that $G^\sharp$ contains all the prime-to-p-torsion of G. On the other hand by (i) $G[p^\infty](K)/G^\sharp$ is finite. This gives (ii) immediately.
Similarly, for cases (iii) and (iv), where $G^\sharp$ contains all the torsion of $G(K)$.
□

Proposition 3.23 *(Characteristic p) Suppose that K is ω_1-saturated and let G be a semiabelian variety over K, $0 \to T \to G \to A \to 0$. Then the following are equivalent:*
(i) $G^\sharp$ has relative Morley rank
(ii) the sequence $0 \to T^\sharp \to G^\sharp \to A^\sharp \to 0$ is exact
(iii) $G^\sharp \cap T(K)/T^\sharp$ is finite
(iv) $G^\sharp \cap T(K)$ is divisible.

Proof : By the previous lemma, as T has no p-torsion, $T(K)/T^\sharp$ is torsion free. Also note that $T^\sharp = T(\mathcal{C})$ is divisible and is the connected component of $G^\sharp \cap T$ (3.6). Hence $T(K) \cap G^\sharp/T^\sharp$ is finite iff it is trivial iff the sequence $0 \to T^\sharp \to G^\sharp \to A^\sharp \to 0$ is exact. And moreover these conditions are equivalent to the divisibility of $G^\sharp \cap T$. This gives the equivalence of (ii), (iii), and (iv).
On the other hand if $G^\sharp$ has finite relative Morley rank, then every relatively definable subgroup is connected by finite, so (i) implies (iii). Conversely, we have seen (3.8) that both $T^\sharp$ and $A^\sharp$ have relative Morley rank. By 2.19, the exactness of the sequence implies that $G^\sharp$ also has relative Morley rank. Thus (ii) implies (i).

□

4 Exactness

As before (K, ∂) is a model of DCF_0 or of $CHF_{p,1}$, which we will assume to be ω_1-saturated in the characteristic p case.

We will now see some equivalent criteria for when the $\sharp$ functor preserves exact sequences, in all characteristics, and obtain as corollary a result linking exactness and descent for (ordinary) semiabelian varieties (Section 4.1).

Then we will look more closely at the case of abelian varieties (Section 4.2) and extensions of elliptic curves (Section 4.3).

4.1 Exactness and descent

For the sake of uniformity, we will harmonize the notation introduced in sections 3.2 and 3.3 for characteristics p and 0,
Let K be of characteristic p, let G be a semiabelian variety over K. We will denote $T_pG(K)$ by $(U_G)^\partial$ and $\tilde{G}(K)$ by $\tilde{G}^\partial$. So again we emphasize that these are $*$-definable groups in K.

From section 3.2 we now see that, in all characteristics

$$G^\sharp \text{ is isomorphic to } \tilde{G}^\partial/(U_G)^\partial$$

where of course, by isomorphic here we mean definably isomorphic in the relevant structure.
Notation: If $f : G \longrightarrow H$ is a morphism of semiabelian varieties defined over K, we denote by $\tilde{f}$ the induced morphism from $\tilde{G}$ to $\tilde{H}$.
(Characteristic 0) If H_1, H_2 are algebraic groups with a D-structure, and $h : H_1 \longrightarrow H_2$ is a morphism of algebraic groups which respects the D-structure, we denote by h^∂ the induced definable homomorphism from ${H_1}^\partial$ to ${H_2}^\partial$. When G, H are semiabelian varieties, $\tilde{G}$ and $\tilde{H}$ have unique D-structures, and so for any $f : G \to H$, $\tilde{f}$ respects the D-structures, whereby $\tilde{f}^\partial$ is defined. (See section 3.2).
(Characteristic p) If $\tilde{f} : \tilde{G} \longrightarrow \tilde{H}$, for G, H semiabelian varieties defined over K, we denote by $\tilde{f}^\delta$ the induced map from $\tilde{G}^\partial = \tilde{G}(K)$ to $\tilde{H}^\partial = \tilde{H}(K)$.

Lemma 4.1 *Let* $0 \longrightarrow G_1 \longrightarrow G_2 \xrightarrow{f} G_3 \longrightarrow 0$ *be an exact sequence of semiabelian varieties defined over* K*. Then the sequence* $0 \longrightarrow (\tilde{G}_1)^\partial \longrightarrow (\tilde{G}_2)^\partial \xrightarrow{\tilde{f}^\partial} (\tilde{G}_3)^\partial \longrightarrow 0$ *is also exact.*

Proof : In characteristic 0, $\tilde{G}_i$ is the universal vectorial extension of G_i (see section 3.2) and the sequence

$$0 \longrightarrow \tilde{G}_1 \longrightarrow \tilde{G}_2 \xrightarrow{f} \tilde{G}_3 \longrightarrow 0$$

is also exact. Each $\tilde{G}_i$ admits a (unique) D-structure and the functor $H \mapsto H^\partial$ preserves exact sequences for the category of algebraic groups with a D-structure (section 3.2).

In characteristic p, $(\tilde{G}_i)^\partial = \tilde{G}_i(K) = \{(x_n)_{n\in\mathbb{N}} : \forall n\ x_n \in G_i(K), x_n = [p]\, x_{n+1}\}$. Clearly the kernel of $(\tilde{f})^\partial$ is $\tilde{G}_1(K)$. The surjectivity of $(\tilde{f})^\partial$ is not as obvious. Let K_0 be a countable subfield of K over which everything is defined. Then, for $(h_i)_{i\in\mathbb{N}} \in \tilde{G}_3(K)$, we can realize in K (which is ω_1-saturated), the following type of length ω over $K_0((h_i)_{i\in\mathbb{N}})$):

$$\bigwedge_{i\in\mathbb{N}} (x_i \in G_2 \wedge f(x_i) = h_i \wedge x_i = [p]\, x_{i+1}).$$

Indeed this type can be finitely realized in K: for $i \leq n$, choose some $g_{n+1} \in G_2(K)$ such that $f(g_{n+1}) = h_{n+1}$ and let $g_i = [p^{n+1-i}]\, g_{n+1}$. For a realisation

$(g_i)_{i\in\mathbb{N}}$ of this type, we have $g_0 \in G_1(K)$ (since $f(g_0) = h_0 = 0$), $(g_i)_{i\in\mathbb{N}} \in \tilde{G}_2(K)$, hence $g_0 \in p^\infty G_2(K)$ and $\tilde{f}((g_i)_{i\in\mathbb{N}}) = (h_i)_{i\in\mathbb{N}}$. □

The next proposition gives us a very useful equivalent to the exactness of the $\sharp$ functor. It should be noted that there is no assumption that any of the $U^\partial_{G_i}$'s or, (in characteristic 0) any of the U_{G_i}'s, are non trivial.

Given the exact sequence $0 \longrightarrow G_1 \longrightarrow G_2 \xrightarrow{f} G_3 \longrightarrow 0$, if $(\tilde{f})^\partial$ is the induced map as above, let $(\tilde{f}_U)^\partial$ denote the restriction of $(\tilde{f})^\partial$ to $(U_{G_2})^\partial$ and let $\tilde{f}_\pi$ denote the induced map from $G_2^\sharp$ to $G_3^\sharp$, when we identify $G_i^\sharp$ with $(\tilde{G}_i)^\partial/(U_{G_i})^\partial$.

Proposition 4.2 *Let* $0 \longrightarrow G_1 \longrightarrow G_2 \xrightarrow{f} G_3 \longrightarrow 0$ *be an exact sequence of semiabelian varieties defined over* K*. Then the following are equivalent:*

(i) $0 \longrightarrow {G_1}^\sharp \longrightarrow {G_2}^\sharp \xrightarrow{f_\pi} {G_3}^\sharp \longrightarrow 0$ *is exact*

(ii) $0 \longrightarrow (U_{G_1})^\partial \longrightarrow (U_{G_2})^\partial \xrightarrow{(\tilde{f}_U)^\partial} (U_{G_3})^\partial \longrightarrow 0$ *is exact*

(iii) $(\tilde{f}_U)^\partial : (U_{G_2})^\partial \longrightarrow (U_{G_3})^\partial$ *is surjective*

(iv) (in characteristic 0*)* $0 \longrightarrow U_{G_1} \longrightarrow U_{G_2} \xrightarrow{\tilde{f}_U} U_{G_3} \longrightarrow 0$ *is exact.*

Furthermore $G_1(K) \cap {G_2}^\sharp/{G_1}^\sharp \xrightarrow{\sim} (U_{G_3})^\partial/(\tilde{f}_U)^\partial((U_{G_2})^\partial)$.

Proof: From the previous lemma, one derives the following commutative diagram of exact sequences (*):

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & (\tilde{G}_1)^{\partial} & & & & \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & (U_{G_2})^{\partial} & \longrightarrow & (\tilde{G}_2)^{\partial} & \xrightarrow{\pi_2} & (G_2)^{\sharp} & \longrightarrow & 0 \\
 & & \downarrow (\tilde{f}_U)^{\partial} & & \downarrow (\tilde{f})^{\partial} & & \downarrow \tilde{f}_{\pi} & & \\
0 & \longrightarrow & (U_{G_3})^{\partial} & \longrightarrow & (\tilde{G}_3)^{\partial} & \xrightarrow[\pi_3]{} & (G_3)^{\sharp} & \longrightarrow & 0 \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}
$$

Claim: $Ker(\tilde{f}_U)^{\partial} = (U_{G_1})^{\partial}$. In char. 0: $Ker\tilde{f}_U$ is a (unipotent) subgroup of $U_{G_2} \cap \tilde{G}_1$, hence of W_1, and contains U_{G_1}. It inherits a D-structure from U_{G_2} and so by maximality of U_{G_1}, they must be equal. Now, going back to the definition, $(U_{G_1})^{\partial} = \{x \in (U_{G_1}, s)(K); s(x) = \partial(x)\} = U_{G_2}^{\partial} \cap U_{G_1}(K)$, and so $Ker(\tilde{f}_U)^{\partial} = (U_{G_1})^{\partial}$.
In char. p, $(U_{G_1})^{\partial} = T_pG_1(K) = T_pG_2(K) \cap \tilde{G}_1(K)$. □

Let $S := (U_{G_3})^{\partial}/(\tilde{f}_U)^{\partial}((U_{G_2})^{\partial})$ (the cokernel of $(\tilde{f}_U)^{\partial}$). Then the classical Snake Lemma applied to diagram (*) gives the existence of a homomorphism d from $Ker(\tilde{f}_{\pi})$ to S, such that the sequence $0 \longrightarrow (U_{G_1})^{\partial} \longrightarrow (\tilde{G}_1)^{\partial} \longrightarrow Ker(\tilde{f}_{\pi}) \xrightarrow{d} S \longrightarrow 0 \longrightarrow 0$ is exact in the following commutative diagram:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & (U_{G_1})^{\partial} & \longrightarrow & (\tilde{G}_1)^{\partial} & \longrightarrow & \mathrm{Ker}(\tilde{f}_{\pi}) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (U_{G_2})^{\partial} & \longrightarrow & (\tilde{G}_2)^{\partial} & \xrightarrow{\pi_2} & {G_2}^{\sharp} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle (\tilde{f}_U)^{\partial}} & & \downarrow{\scriptstyle (\tilde{f})^{\partial}} & & \downarrow{\scriptstyle \tilde{f}_{\pi}} & & \\
0 & \longrightarrow & (U_{G_3})^{\partial} & \longrightarrow & (\tilde{G}_3)^{\partial} & \xrightarrow[\pi_3]{} & {G_3}^{\sharp} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & S & \longrightarrow & 0 & \longrightarrow & 0 & &
\end{array}
$$

This says exactly that
$S = (U_{G_3})^{\partial}/(\tilde{f}_U)^{\partial}((U_{G_2})^{\partial})$ is isomorphic to $Ker(\tilde{f}_{\pi})/((\tilde{G}_1)^{\partial}/(U_{G_1})^{\partial})$, that is, to $(G_1(K) \cap {G_2}^{\sharp})/{G_1}^{\sharp}$.
It follows in particular that
$0 \longrightarrow {G_1}^{\sharp} \longrightarrow {G_2}^{\sharp} \xrightarrow{f_{\pi}} {G_3}^{\sharp} \longrightarrow 0$ is exact
if and only if
$0 \longrightarrow (U_{G_1})^{\partial} \longrightarrow (U_{G_2})^{\partial} \xrightarrow{(\tilde{f}_U)^{\partial}} (U_{G_3})^{\partial} \longrightarrow 0$ is exact
if and only if $(\tilde{f}_U)^{\partial}$ is surjective.
In characteristic 0, this is equivalent to the exactness of the sequence $0 \longrightarrow U_{G_1} \longrightarrow U_{G_2} \xrightarrow{(\tilde{f}_U)} U_{G_3} \longrightarrow 0$. One direction follows because the ∂ functor is exact on groups with a D-structure. For the other direction suppose that the sequence of the $(U_{G_i})^{\partial}$'s is exact. For each i, $U_{G_i}^{\partial}$ is Zariski dense in U_{G_i}, and has transcendence degree (or Morley rank) equal to the dimension of the algebraic group U_{G_i}. It follows that $dimU_{G_1} + dimU_{G_3} = dimU_{G_2}$ and hence that $dim\tilde{f}_U(U_{G_2}) = dimU_{G_3}$. Being vector groups, all these groups are connected, and it follows that $\tilde{f}_U$ is surjective. □

We can now give the proof of the main theorem which relates exactness of the $\sharp$ functor to questions of descent, restricted, in char. p to the class of *ordinary* semiabelian varieties. Proposition 4.3 is no longer true without the assumption ordinary (see Remark 4.11).

Proposition 4.3 *Let* $0 \to G_1 \to G_2 \to G_3 \to 0$ *be an exact sequence of (ordinary in char.p) semiabelian varieties defined over* K*. Suppose that* G_1 *and* G_3 *descend to the constants of* K*.*

Then, $G_1(K) \cap G_2^\sharp = G_1^\sharp$ *(i.e. the sequence with the* $\sharp$*'s remains exact) if and only if* G_2 *also descends to the constants.*

Proof : Let K_0 be a countable elementary submodel of K over which everything is defined. By isomorphism, we can suppose that both G_1 and G_2 are actually defined over $\mathcal{C} \cap K_0$, the field of constants of K_0.

If G_2 descends to the constants, then by isomorphism, we can suppose that G_2 is also defined over the constants, so for every i ${G_i}^\sharp = G_i(\mathcal{C})$. And then $G_1(K) \cap G_2^\sharp = G_1(K) \cap G_2(\mathcal{C}) = G_1(\mathcal{C}) = G_1^\sharp$.
For the converse, suppose that $0 \to {G_1}^\sharp \to {G_2}^\sharp \to {G_3}^\sharp \to 0$ is exact.
In characteristic 0, by Propostion 4.2, then $0 \to U_{G_1} \to U_{G_2} \to U_{G_3} \to 0$ is also exact. We know that (see Fact 3.14) as G_1 and G_3 descend to the constants, $U_{G_1} = W_1$ and $U_{G_3} = W_3$. Consider the dimensions, as vector spaces, of the U_{G_i}'s. By exactness, $dimU_{G_2} = dimU_{G_1} + dimU_{G_3}$. But we also have that the dimension of $dimW_2 = dimW_1 + dimW_3$ (this follows from Lemma 4.1). So $dimU_{G_2} = dimW_2$ and $U_{G_2} = W_2$, that is, G_2 descends to the constants.
In characteristic p, our assumption that the G_i's are ordinary ensures that for each i, $T_pG_i(\overline{K}) \cong {\mathbb{Z}_p}^{a_i}$, where a_i is the dimension of the abelian part of G_i. If G_1 and G_3 descend to $\mathcal{C}$, then $T_pG_1(K) = T_pG_1(\mathcal{C}) = T_pG_1(\overline{K})$ and $T_pG_3(K) = T_pG_3(\mathcal{C}) = T_pG_3(\overline{K})$. By 4.2 the sequence

$$0 \longrightarrow T_pG_1(K) \longrightarrow T_pG_2(K) \longrightarrow T_pG_3(K) \longrightarrow 0$$

is exact. It follows that $T_pG_2(K) \cong {\mathbb{Z}_p}^{a_1+a_3}$. As $a_1 + a_3 = a_2$ (by exactness of $0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow 0$), it follows that $T_pG_2(K) = T_pG_2(\overline{K})$, and by Proposition 3.17, that G_2 descends to the constants. □

Corollary 4.4 *For any ordinary abelian variety* A *defined over the constants of* K*, there exists an exact sequence over* K*,*

$$0 \longrightarrow \mathbb{G}_m \longrightarrow H \longrightarrow A \longrightarrow 0$$

such that

$${\mathbb{G}_m}^\sharp \neq H^\sharp \cap \mathbb{G}_m.$$

Proof: As in the proof of 3.13, we use the fact that $EXT(A, \mathbb{G}_m)$ is parametrized (up to isomorphism) by the dual abelian variety of A, say $\hat{A}$, which is also over the constants (see [Se2]). Then H will descend to the constants $\mathcal{C}$ of K if and only if H corresponds to a $\mathcal{C}$-rational point of $\hat{A}$. So just pick some K-rational point of $\hat{A}$ which is not $\mathcal{C}$-rational. □

We have established in Proposition 3.23 the connection between exactness and relative Morley rank, and we can conclude that:

Corollary 4.5 *(Characteristic p) There is an ordinary semiabelian variety G, such that $G^\sharp$ does not have relative Morley rank.*

In fact, as above, for any ordinary abelian variety A defined over K^{p^∞}, there is some semiabelian variety G in $EXT(A, \mathbb{G}_m)$ such that $G^\sharp$ does not have relative Morley rank.

We will finish this section with some easy corollaries, in characteristic p, of Proposition 4.2. Again, $0 \longrightarrow G_1 \longrightarrow G_2 \xrightarrow{f} G_3 \longrightarrow 0$ is an exact sequence of semiabelian varieties defined over K. Recall from Proposition 4.2 that $G_1(K) \cap G_2^\sharp / G_1^\sharp \cong T_p G_3(K)/f(T_p G_2(K))$.

Corollary 4.6 *(Characteristic p) If $G_3[p^\infty](K)$ is finite, then the $\sharp$ sequence is exact.*

Proof: Since $G_3[p^\infty](K)$ is finite, $T_p G_3(K) = 0$. □

If we have the extra assumption that $G_1(K)$ has no p-torsion, then the non exactness can be read directly from the groups of p^∞-torsion. As we will see in the next section (4.12) this is no longer true if $G_1(K)$ has some p-torsion.

Corollary 4.7 *(Characteristic p) We assume now that $G_1(K)$ has no p-torsion. If $f(G_2[p^\infty](K)) = G_3[p^\infty](K)$, then the $\sharp$ sequence is exact.*

Proof: We show that the hypothesis implies that for each n, $f(G_2[p^n](K)) = G_3[p^n](K)$: consider $g \in G_3[p^n](K)$, and $h \in G_2[p^\infty](K)$ such that $f(h) = g$. Let m be such that $[p^m]h = 0$; if $m \leq n$, the claim is proved. If $m > n$ we have that $[p^n]h \in G_1(K)$ and $[p^{m-n}]([p^n]h) = 0$. Since $G_1(K)$ has no p-torsion, $[p^n]h = 0$.

It follows that $\tilde{f}(T_pG_2(K)) = T_pG_3(K)$: let $(g_i)_{i\in\mathbb{N}}$ be in $T_pG_3(K)$ and consider the tree of sequences of size at most ω, $(h_i)_{i<L}$, such that for all i, $h_i \in G_2[p^\infty](K)$, $f(h_i) = g_i$, $[p]h_i = h_{i-1}$ and $h_0 = 0$, ordered by initial segment. This tree has finite branching, since $G_2[p](K)$ is finite, and has branches of arbitrary length: for every n, pick $h_n \in G_2[p^n](K)$ such that $f(h_n) = g_n$ and consider the sequence $(0, [p^{n-1}]h_n, \dots, h_n)$. It follows by Koenig's Lemma that the tree has an infinite branch, which gives $(h_i)_{i\in\mathbb{N}} \in T_pG_2(K)$ such that $\tilde{f}((h_i)) = (g_i)$. □

If we add the assumption that the semiabelian varieties have relative Morley rank, we get the following characterization:

Proposition 4.8 *(Characteristic p) Let* $0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow 0$ *be an exact sequence of semiabelian varieties such that* $G_2^\sharp$ *has relative Morley rank. then the following are equivalent*

(1) the sequence $0 \longrightarrow {G_1}^\sharp \longrightarrow {G_2}^\sharp \longrightarrow {G_3}^\sharp \longrightarrow 0$ *is exact*

(2) $G_1[p^\infty](K) \cap {G_2}^\sharp = G_1[p^\infty](K) \cap {G_1}^\sharp$.

In particular the sequence will be exact when G_1 *descends to the constants , or, more generally, when* $G_1[p^\infty](\overline{K}) = G_1[p^\infty](K)$*, and also when* $G_1[p^\infty](K) = 0$.

Proof : Recall that ${G_i}^\sharp = p^\infty G_i(K)$. We know that (1) holds if and only if ${G_1}^\sharp = {G_2}^\sharp \cap G_1(K)$. So trivially, (1) implies (2). We know that ${G_1}^\sharp$ contains all the p'-torsion of $G_1(K)$. It follows that if (2) holds, then ${G_2}^\sharp \cap G_1(K)/{G_1}^\sharp$ is torsion free. As by assumption ${G_2}^\sharp$ has relative Morley rank, this quotient must be finite, if it is torsion-free, it is trivial.

If $G_1[p^\infty](\overline{K}) = G_1[p^\infty](K)$ then $G_1[p^\infty](K) \subset G_1^\sharp$. If G_1 descends to the constants, then $G_1^\sharp = G_1(\mathcal{C})$ and in particular, $G_1[p^\infty](K) = G_1[p^\infty](\mathcal{C}) = G_1[p^\infty](\overline{K})$.

□

4.2 Abelian varieties

In characteristic 0, the situation is completely different for abelian varieties and follows quickly from Proposition 4.2.

Proposition 4.9 *(Characteristic 0) Let* $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ *be an exact sequence of abelian varieties over* K*. Then the induced sequence* $0 \longrightarrow A^\sharp \longrightarrow B^\sharp \longrightarrow C^\sharp \longrightarrow 0$ *is also exact.*

Proof : By Poincaré complete reducibility, $A \times C$ is isogenous to B, inducing an isogeny of $\widetilde{A \times C} = \tilde{A} \times \tilde{C}$ with $\tilde{B}$. As this is also an isogeny of D-groups it induces an isogeny between $U_{A\times C} = U_A \times U_C$ and U_B. As these are vector groups it follows that the induced sequence $0 \longrightarrow U_A \longrightarrow U_B \longrightarrow U_C \longrightarrow 0$ is exact. Hence by Proposition 4.2, so is $0 \longrightarrow A^\sharp \longrightarrow B^\sharp \longrightarrow C^\sharp \longrightarrow 0$

□

In contrast to the characteristic 0 case, in characteristic p there are counterexamples to the exactness of $\sharp$, even for ordinary abelian varieties. They will have to be quite different from the counterexamples seen in the previous section for semiabelian varieties, as can be seen from the following direct corollary of Proposition 4.8. Recall from Fact 3.8 that for all abelian varieties A, $A^\sharp$ has finite relative Morley rank.

Corollary 4.10 *(Characteristic p) Let* $0 \longrightarrow C \longrightarrow B \longrightarrow A \longrightarrow 0$, *be an exact sequence of abelian varieties over* K. *If* $C(K)$ *has no* p*-torsion, or if* C *descends to the constants, then the sequence* $0 \longrightarrow C^\sharp \longrightarrow B^\sharp \longrightarrow A^\sharp \longrightarrow 0$ *is exact.*

Remark 4.11 *From Corollary 4.10 we see that Proposition 4.3 does not hold for non ordinary (semi)abelian varieties. Indeed, consider again the example described in Remark 3.21 of a (non ordinary) abelian variety* $A \in EXT(E_1, E_2)$, *where* E_1, E_2 *are two elliptic curves over* $\mathbb{F}_p$, *and which itself does not descend to the constants. Nevertheless, by the above corollary, the sequence* $0 \longrightarrow E_1^\sharp \longrightarrow A^\sharp \longrightarrow E_2^\sharp \longrightarrow 0$ *is exact.*

There are still cases, not covered by Corollary 4.10, where one obtains non exactness, even in the ordinary case:

Proposition 4.12 *(Characteristic p) There is an exact sequence of (ordinary) abelian varieties such that the induced* $\sharp$ *sequence is not exact*

Proof : Let A be an ordinary elliptic curve, defined over K^p, which does not descend to K^{p^∞} and C an ordinary elliptic curve defined over K^{p^∞}. Then we know by Proposition 3.17 that $A[p](K) \cong \mathbb{Z}/p\mathbb{Z} \cong C[p](K)$ but $A[p^\infty](K)$ is finite. Pick an isomorphism f between $A[p](K)$ and $C[p](K)$.
Let $H \subset A[p](K) \times C[p](K) := \{(a, -f(a)); a \in A|p][K)\}$, and $B := (A \times$

$C)/H$. Then A is isomorphic to $A_1 := A \times 0 + H \subset B$. Consider the exact sequence:

$$0 \longrightarrow A_1 \longrightarrow B \xrightarrow{g} B/A_1 \longrightarrow 0.$$

Note that $C_1 := B/A_1$ is isogenous to C, hence by 2.15 or 3.19, descends to K^{p^∞}.

We claim that the p^∞ sequence is no longer exact, that is, we claim that $p^\infty A_1(K) \neq p^\infty B(K) \cap A_1(K)$.

Pick some $c \neq 0$, $c \in C[p]$. As C is defined over K^{p^∞}, $c \in C(K^{p^\infty}) = p^\infty C(K)$. It follows that

– $(0, c) + H \in p^\infty B(K)$

– $(0, c) + H \in A_1[p](K) \quad ((0, c) - (f^{-1}(c), 0) \in H)$.

By assumption $A_1[p^\infty](K)$ is finite, so $A_1[p^\infty](K) \cap p^\infty A_1(K) = \{0\}$, and $c \in [C(K^{p^\infty})] \cap A_1(K)] \setminus p^\infty A_1(K)$. □

We can say more about the example described above:

$$0 \longrightarrow A_1 \longrightarrow B \xrightarrow{g} C_1 \longrightarrow 0$$

Let e be any element in $C_1[p^n](K)$, pick some preimage of e in $B(K)$ of the form $(0, y) + H$. Then $[p^n]((0, y) + H) \in Ker g = (A \times 0) + H$, hence $(0, [p^n]y) \in H$ and $[p^n]y \in C[p](K)$. So e is the image of an element in $B[p^{n+1}]$. From this we can conclude:

(i) $g(B[p^\infty](K)) = C_1[p^\infty](K)$, which shows that 4.7 does not hold without the assumption on the torsion.

(ii) By an infinite tree argument, as in the proof of Cor. 4.7, we deduce that $[p]T_pC_1(K) \subset \tilde{g}(T_pB(K))$. We know that $p^\infty B(K) \cap A_1(K)/p^\infty A_1$ is finite but non trivial and (4.2) that it is isomorphic to $T_pC_1(K)/\tilde{g}(T_pB(K))$. It follows that it must isomorphic to $\mathbb{Z}/p\mathbb{Z}$.

4.3 The case when the abelian part of G is an elliptic curve in characteristic p

Here we consider only the case of characteristic p.

Recall the following basic facts about p-torsion in elliptic curves (see for example [Si]):

– If E is ordinary, then for each n, $E[p^n] \cong \mathbb{Z}/p^n\mathbb{Z}$, and $E[p^\infty] \cong \mathbb{Z}_{p^\infty}$.

– If E is not ordinary, then E is supersingular, i.e. $E[p^\infty] = \{0\}$. In that case, E is isomorphic to an elliptic curve defined over a finite field.

From Proposition 3.17 and Corollary 4.6, it is easy to conclude that:

Corollary 4.13 *Let E be an ordinary elliptic curve which does not descend to K^{p^∞}. If G_1, G_2 are semiabelian varieties over K and if the sequence $0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow E \longrightarrow 0$ is exact, then the sequence $0 \longrightarrow G_1^\sharp \longrightarrow G_2^\sharp \longrightarrow E^\sharp \longrightarrow 0$ is exact.*

We can now summarize exactly the situation for a semiabelian variety G whose abelian part is an elliptic curve, $0 \longrightarrow T \longrightarrow G \longrightarrow E \longrightarrow 0$:

Proposition 4.14 *Let G be as above:*
(i) If E is supersingular, then the $\sharp$ sequence remains exact and G has relative Morley rank.
(ii) If E is ordinary and does not descend to the constants then the $\sharp$ sequence remains exact and G has relative Morley rank .
(iii) If E is ordinary and descends to the constants, the following are equivalent
- *the $\sharp$ sequence is exact*
- *G descends to the constants*
- *G has relative Morley rank*
- *$G[p^\infty](K)$ is infinite.*

In the case when G does not descend to the constants, then $(G^\sharp \cap T(K))/T^\sharp$ is isomorphic to the profinite group $\mathbb{Z}_p$.

Proof: Recall first that Proposition 3.23 says that in the present context $G^\sharp$ has relative Morley rank if and only if the $\sharp$ sequence is exact.
(i) If E is supersingular, it has no p-torsion and Corollary 4.6 applies.
(ii) If E does not descend to the constants, Corollary 4.13 applies.
(iii) If E is ordinary and descends to K^{p^∞}, by Proposition 4.3, the $\sharp$ sequence will be exact if and only if G descends to K^{p^∞}. As T has no p-torsion, $G[p^\infty] \cong E[p^\infty] \cong \mathbb{Z}_{p^\infty}$. So if G descends to the constants, then $G[p^\infty](K) = G[p^\infty]$ so is infinite.
If G does not descend to K^{p^∞}, by Proposition 3.17, for some n, $G[p^n](K)$ must be a proper subgroup of $G[p^n] \cong \mathbb{Z}/p^n Z$ of order p^n, which forces it to be trivial. Hence $G[p^\infty](K)$ is finite.

In particular $T_pG(K) = \{0\}$. By Proposition 4.2, $(G^\sharp \cap T(K))/T^\sharp$ is isomorphic to $T_pE(K)/\tilde{f}(T_pG(K)) \cong T_pE(K) = T_pE \cong \mathbb{Z}_p$, completing the proof of (iii). □

5 Additional remarks and questions

1. In characteristic p, the counterexamples to exactness of the induced $\sharp$ sequence arise from the following situation: we have two connected commutative definable groups $H_1 < H_2$ which are not divisible. We consider D_2 the biggest divisible subgroup (which is infinitely definable) of G_2. The counterexamples are exactly the cases when $G_1 \cap D_2$ is *not* divisible. One can ask the same question also for other classes of groups, in particular for commutative algebraic groups: Given $G_1 < G_2$ two commutative connected algebraic groups defined over some algebraically closed field K of characteristic p, consider $D < G_2$, the biggest divisible subgroup of G_2. It is easy to check that D is a closed subgroup of G_2, also defined over K.

Using the characterizations of the groups $p^\infty G(K)$, given in terms of the Weil restrictions $\Pi_{K/K^{p^n}} G$ in [BeDe], one can deduce easily from our examples that the same phenomenon occurs for commutative algebraic groups.

2. We will finish by mentioning a rather intriguing question, which, as far as we know remains open. Let A be an abelian variety defined over $\mathbb{F}_p(t)$ and let K_0 denote the separable closure of $\mathbb{F}_p(t)$. We can consider $A(K_0)$ and $p^\infty A(K_0)$. As we recalled in section 2.2.2, $p^\infty A(K_0)$ is the biggest divisible subgroup of $A(K_0)$ and contains all the torsion of A which is prime to p. We do not know if $p^\infty A(K_0)$ can contain any element which is not torsion. Note that if A is defined over ${K_0}^{p^\infty} = \overline{\mathbb{F}_p}$, then $p^\infty A(K_0) = A(\overline{\mathbb{F}_p})$, where indeed every element is torsion. Note that, from the beginning of section 3, in characteristic p, when dealing with $A^\sharp = p^\infty A(K)$, we suppose that K is ω_1-saturated, which ensures that $A^\sharp$ contains elements which are not torsion.

In characteristic 0 there are results along these lines, sometimes going under the expression "Manin's theorem of the kernel". A formal statement and proof (depending on results of Manin, Chai,..) appears in [BePi] (Corollary K.3 of the Appendix), and says that if A is an abelian variety over the algebraic closure K_0 say of $\mathcal{C}(t)$, equipped with a derivation with field of constants $\mathcal{C}$, and A has $\mathcal{C}$-trace 0, then $A^\sharp(K_0)$ is precisely the group of torsion points of A.

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