

Differential algebraic groups

Examples as signposts along the way

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Throughout its development, the theory of differential algebraic groups has received its impetus from interesting examples that both instruct and point to new paths to explore. We present two of them.

Commuting derivation operators: $\Delta = \{\partial_1, \dots, \partial_m\}$

$\mathcal{F}^\dagger =$ differentially closed Δ -field (char 0).

$$\mathcal{C} = (\mathcal{F}^\dagger)^\Delta$$

Affine n -space $\mathcal{F}^{\dagger^n} = \mathbb{A}^n$.

$y = (y_1, \dots, y_n)$: family of Δ -indeterminates.

$y^{(r)}$ = family of derivatives of the y_i of order $\leq r$.

$\mathcal{F}^\dagger \{y\} = \mathcal{F}^\dagger \left[y^{(r)} \right]_{r \in \mathbb{N}}$: Δ -polynomial ring.

Definition

$X \subseteq \mathbb{A}^n$ is *Kolchin closed* (Δ -variety) if it is the set of zeros of a finite set of Δ -polynomials.

Let $X \subseteq \mathbb{A}^k$ and $Y \subseteq \mathbb{A}^\ell$ be Δ -varieties. The product $X \times Y$ is a Kolchin closed subset of $\mathbb{A}^{k+\ell}$.

\mathcal{F}^\dagger differentially closed \implies A system of Δ -polynomial equations that has a zero rational over an extension Δ -field of \mathcal{F}^\dagger has a zero rational over \mathcal{F}^\dagger .

The expression “The solutions of a system of differential equations must satisfy all its differential consequences” is still in use.

The interpretation in commutative differential algebra: “defining differential ideal.”

Let F_1, \dots, F_r be in $\mathcal{F}^\dagger\{y\}$. The Δ -ideal $[F_1, \dots, F_r]$ generated by F_1, \dots, F_r is the ideal generated by F_1, \dots, F_r and their derivatives of all orders. Both $[F_1, \dots, F_r]$ and its radical have the same set X of zeros. $\sqrt{[F_1, \dots, F_r]}$ is called the *defining Δ -ideal* of X . X is *irreducible* iff its defining Δ -ideal is prime. All varieties in this talk are irreducible.

Definitions

Let X be a Δ -variety in \mathbb{A}^n , and let \mathfrak{p} be its defining Δ -ideal. Let $\bar{y} = (\bar{y}_1, \dots, \bar{y}_n)$, \bar{y}_j the residue class of $y_j \bmod \mathfrak{p}$. $\mathcal{F}^+ \{\bar{y}\}$ is the ring of Δ -polynomial functions on X . Its quotient field $F \langle X \rangle$ is the field of Δ -rational functions on X .

Definition

Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^p$ be Δ -varieties. A map

$$T : X \longrightarrow Y$$

is a *morphism* if it is a p -tuple of everywhere defined Δ -rational functions on X .

$\Delta = \{\partial_x\}$. The *second Painlevé equation* $P_{II}(c)$ is the second order differential equation

$$\partial_x^2 y = 2y^3 + xy + c, \quad c \in \mathbb{C}.$$

It defines a Δ -subvariety $X(c)$ of the affine line \mathbb{A}^1 . Suppose $c \neq \frac{1}{2}$. The *Bäcklund transformation*

$$T(y) = -y - \frac{c - \frac{1}{2}}{\partial_x y - y^2 - \frac{x}{2}}$$

maps $X(c)$ onto $X(c - 1)$. It is a morphism of Δ -varieties.

Definition

An *affine Δ -group* is a group G , with underlying set a Δ -subvariety of \mathbb{A}^n , whose group laws are morphisms of Δ -varieties.

$GL(n, \mathcal{F}^+)$ is an affine Δ -group. Its ring of Δ -polynomial functions is $\mathcal{F}^+ \left\{ y, \frac{1}{\det y} \right\}$, where y is a matrix of Δ -indeterminates.

$$\mathbb{G}_m := GL(1, \mathcal{F}^+), \quad \mathbb{G}_a := (\mathcal{F}^+, +).$$

Definition

A group homomorphism is a Δ -*homomorphism* if it is a morphism of Δ -varieties. G is *linear* if there exists a positive integer n and an injective Δ -homomorphism from G into $GL(n, \mathcal{F}^+)$.

How does one measure the size of a solution set of a system of differential equations? This question was one of the central topics in the correspondence (1929-1932) between Albert Einstein and Elie Cartan.

In the theory of differentially closed fields there are several such measurements, including *differential dimension*, Kolchin's *dimension polynomial*, and the *Morley rank* and *U-rank* measures of Model Theory.

$X := \Delta$ -subvariety of \mathbb{A}^n , with defining Δ -ideal \mathfrak{p} , and field $\mathcal{F}^+ \langle X \rangle$ of Δ -rational functions.

$$\Delta\text{-dim}(X) = \Delta\text{-tr deg}_{\mathcal{F}^+}(\mathcal{F}^+ \langle X \rangle)$$

Classical differential equations theory:

Δ -dimension = the number of “arbitrary functions” of m independent variables on which X depends, generically.

Δ -dimension is a rough measure that does not distinguish when a Δ -subvariety is proper.

Kolchin sought a finer measure.

$\mathcal{F}^+ \langle X \rangle = \varinjlim_{r \in \mathbb{N}} \mathcal{F}^+ \left(\bar{y}^{(r)} \right)$, $\bar{y}^{(r)}$ the family of derivatives of \bar{y} of order $\leq r$. It is an inductive limit of finitely generated fields.

The dim polynomial d_X is a numerical polynomial that measures the transcendence degree over \mathcal{F}^+ of $\mathcal{F}^+ \left(\bar{y}^{(r)} \right)$ for $r \gg 0$.

$$d_X = \sum_{j=0}^m a_m \binom{X + m}{m}, a_m \in \mathbb{Z}.$$

$$d_X = \sum_{j=0}^m a_m \binom{X + m}{m}, a_m \in \mathbb{Z}.$$

- 1 $a_m = \Delta\text{-dim}(X)$.
- 2 $d_X < d_Y$ if for all sufficiently large r , $d_X(r) < d_Y(r)$.
 $X \subsetneq Y \Rightarrow d_X < d_Y$.
- 3 d_X is not a differential birational invariant.
- 4 To compute d_X , we *must* choose an orderly ranking of the differential indeterminates.

Two diff birat invariants

Two important differential birational invariants: Δ -type, and *typical Δ -dimension*.

$$\tau = \deg(d_X) = \Delta\text{-type}(X).$$

$$a_\tau = \text{typical } \Delta\text{-dim}(X).$$

X depends generically on a_τ arbitrary functions of τ independent variables. If $\tau = 0$, X depends generically on a_τ arbitrary constants.

The heat equation

$$\Delta = \{\partial_x, \partial_t\}$$

$$H = \partial_x^2 y - \partial_t y.$$

card $\Delta = 2$.

ord $H = 2$.

Let G be the Δ -subgroup of \mathbb{G}_a with defining differential polynomial H .

Choose an orderly ranking of the Δ -indeterminates: the leader of H is $\partial_x^2 y$.

$$H = \partial_x^2 y - \partial_t y.$$

$$d_G = 2 \binom{X+1}{1} - 1 = 2X + 1.$$

Algorithm: Kolchin, *Differential algebra and algebraic groups*, Lemma 16, p. 51 and Theorem 6, p. 115.

The type of G is 1.

The typical dimension of G is 2.

Generically, the solutions of the heat equation depend on 2 arbitrary functions of the independent variable t .

A power series expansion

Near the origin, choose arbitrary functions $f(t)$ and $g(t)$, which are restrictions to $x = 0$ of y and $\partial_x y$.

$$\begin{aligned}y(x, t) &= y(t, 0) + \partial_x y(t, 0)x + \partial_x^2 y(t, 0)\frac{x^2}{2!} + \partial_x^3 y(t, 0)\frac{x^3}{3!} + \dots \\ &= f(t) + g(t)x + f'(t)\frac{x^2}{2!} + g'(t)\frac{x^3}{3!} + \dots\end{aligned}$$

Fact

Every proper Δ -subgroup of G has Δ -type 0 (is a finite-dimensional \mathbb{C} -vector space). (Alexander Levin–email, 2007).

"Les groupes de transformations continus, infinis, simples," 1909.

$$\Delta = \{\partial_x, \partial_t\}$$

Let $G_1 \subset G_a$ be the group defined by

$$H = \partial_x^2 y - \partial_t y,$$

$$L = \partial_x y - x \partial_t y.$$

$$\mathfrak{p} := [H, L].$$

Choose an orderly ranking with $\partial_x > \partial_t$. $\partial_x^2 y$ is the leader of H , and $\partial_x y$ of L .

Transform H, L to a *characteristic set* of differential ideal generators of \mathfrak{p} .

(Ritt-Kolchin-Rosenfeld algorithm)

Solving the equations

$$\begin{aligned} \mathfrak{p} &= [\partial_x^2 y - \partial_t y, \partial_x y - x \partial_t y] \\ &= [\partial_x y - x \partial_t y, \partial_t^2 y] \end{aligned}$$

We read off the dim poly from the characteristic set $\partial_x y - x \partial_t y$, $\partial_t^2 y$.

$$d_{G_1} = 2.$$

G_1 depends on 2 arbitrary constants.

The hidden equation is

$$\partial_t^2 y = 0.$$

It helps solve the equations.

$$G_1 = \{a(2t + x^2) + b, \quad a, b \in \mathcal{C}\}.$$

Definition

An infinite Δ -group G is *simple* if every normal Δ -subgroup is finite (its Lie algebra is simple).

A Jordan-Hölder decomposition of a differential algebraic group G should have simple quotients.

Known:

Let G_1 be a normal Δ -subgroup of G .

- G/G_1 is a Δ -group, with the usual universal properties.
- G linear $\implies G/G_1$ is linear.
- The ring of invariants of G_1 in $\mathcal{F}^+(G)$, acting by the regular representation, is finitely Δ -generated.

The infinite simple non-commutative Δ -groups have been classified.

(First proved in 1989; then by Pillay, 1992, 1997; Buium, 1993).

An *impediment* to the existence of Jordan-Hölder decompositions: \mathbb{G}_a and its subgroups.

The *algebraic* group \mathbb{G}_a is simple.

A subgroup G of \mathbb{G}_a is Kolchin closed \iff it is defined by homogeneous linear differential equations.

The Δ -group \mathbb{G}_a is not simple.

Definition

(Cartan 1909) A Δ -subgroup of \mathbb{G}_a is *simple improperly speaking* if every non-trivial homomorphic image is isomorphic to G .

A new definition of Jordan-Hölder sequence for \mathbb{G}_a :
Successive quotient groups are simple improperly speaking.

\mathbb{G}_a has 0 torsion subgroup.

$\mathbb{G}_a(\mathbb{C})$ is the only *simple* Δ -subgroup of \mathbb{G}_a . It is simple improperly speaking.

Fact

G is simple improperly speaking if and only if every non-trivial Δ -subgroup is the kernel of a surjective endomorphism of G .

$\mathcal{F}^+[\Delta]$ is the endomorphism ring of \mathbb{G}_a .

Set $\Delta = \{\partial\}$.

$\mathcal{F}^+[\partial]$ is a left and right principal ideal domain. Every Δ -subgroup of \mathbb{G}_a is the kernel of an endomorphism of \mathbb{G}_a .

We have a Jordan-Hölder decomposition:

$$G_0 = \mathbb{G}_a \supset \mathbb{G}_a(C) \supset 0.$$

$\text{Card}(\Delta) \geq 2 \implies \mathcal{F}^+[\Delta]$ is not a left and right principal ideal domain.

$$\Delta = \{\partial_x, \partial_t\}.$$

The homomorphism

$$\begin{aligned} T & : \mathbb{G}_a \longrightarrow \mathbb{G}_a \times \mathbb{G}_a \\ u & \longmapsto (\partial_x u, \partial_t u) \end{aligned}$$

maps \mathbb{G}_a onto the group G' defined by the differential equation

$$\partial_t y_1 - \partial_x y_2 = 0.$$

G' is not isomorphic to \mathbb{G}_a . Its kernel is $\mathbb{G}_a(C)$. \mathbb{G}_a is not simple improperly speaking.

Our first example suggests the possibility of using the *language* and techniques of differential algebraic geometry to study symmetries of partial differential equations (Bluman, Cole, Kumei, Clarkson, Fushchych, Olver, Miller). Morphisms of Δ -varieties are Bäcklund transformations. Endomorphisms of Δ -subgroups of \mathbb{G}_a are *generalized* symmetries. (Miller, 1977, Fushchych, 2002).

Broadening the definition of “symmetry.”

What makes the symbiosis possible? The definition of symmetry has broadened in the last 30 years.

“It is well known that the classical Lie approach does not make it possible to describe completely the symmetry of systems of partial differential equations.... Using the non-Lie approach, in which the symmetries may be differential operators of any order and even integro-differential operators shows that even such well-studied equations as the Dirac and Maxwell equations have more extensive symmetry than the relativistic and conformal invariance.”

(Fushchych, 2002)

We return to Cartan (1909). The group defined by the heat equation resembles the additive group G_a of an ordinary differential field. We ask: Is it also simple improperly speaking?

- If G is a Δ -subgroup of \mathbb{G}_a , every endomorphism of G is the restriction of an endomorphism of \mathbb{G}_a .

Let G be defined by

$$H = \partial_x^2 y - \partial_t y = 0.$$

Let S be a non-trivial endomorphism of G .

$$1 = \Delta\text{-type}(G) = \max(\Delta\text{-type ker } S, \Delta\text{-type im } S).$$

$$\Delta\text{-type ker } S = 0$$

Sit (1974). Thus, S is surjective.

Cartan's 1909 example

Let G be defined by the heat equation

$$H = \partial_x^2 y - \partial_t y = 0.$$

Let $G_1 \subset G$ be the kernel of the operator

$$L = \partial_x - x\partial_t.$$

$$G_1 = \{a(2t + x^2) + b\}, \quad a, b \in \mathbb{C}.$$

$L : G \longrightarrow \mathbb{G}_a$ is a homomorphism with kernel G_1 . The group $G' = L(G)$ is defined by the equation

$$\partial_x^2 y - \partial_t y = \frac{2}{x} \partial_x y.$$

$$L = \partial_x - x\partial_t.$$

L is a *conditional symmetry* Bluman-Cole (1969).

Example

The derivation operator ∂_θ is not a symmetry of the differential equation

$$\frac{1}{r^2}\partial_r (r^2\partial_r y) + \frac{1}{r^2\sin\theta}\partial_\theta (\sin\theta\partial_\theta) - \partial_t^2 y = 0 :$$

a special case of the wave equation in spherical coordinates.
Nevertheless, non-trivial solutions $y(r, t)$ of the wave equation exist.
The solution space contains non-trivial constants of the derivation.

$$L = \partial_x - x\partial_t.$$

Is G' isomorphic to G ? Does there exist a symmetry of the heat equation with kernel G_1 ? Cartan says NO.

A *classical symmetry* is either a point symmetry or the prolongation of a point symmetry.

The *maximal invariance Lie algebra* of the heat equation (the point symmetries) is a 6-dimensional \mathcal{C} -Lie algebra with basis

$$1, \quad \partial_x, \quad \partial_t, \quad t\partial_x + \frac{1}{2}x, \quad t^2\partial_t + tx\partial_x + \frac{t}{2} + \frac{x^2}{4}, \quad x\partial_x + 2t\partial_t + \frac{1}{2},$$

Set

$$S = 2t\partial_x\partial_t + \partial_t^2 - (\partial_x - x\partial_t).$$

The Lie bracket

$$[\partial_x^2 - \partial_t, S] = 0.$$

S is a non-classical symmetry of the heat equation. It is not the prolongation of a symmetry in the maximal invariance Lie algebra.

G_1 has defining Δ -ideal

$$\mathfrak{p} = [\partial_x^2 y - \partial_t y, \partial_x y - x \partial_t y].$$

The linear Δ -polynomials

$$\begin{aligned} H &= \partial_x^2 y - \partial_t y, \\ S &= 2t \partial_x \partial_t y + \partial_t^2 y - (\partial_x - x \partial_t) y \end{aligned}$$

form a characteristic set of generators of \mathfrak{p} .
(Ritt-Kolchin-Rosenfeld reduction algorithm)

G_1 is the kernel of an endomorphism (non-classical symmetry) of the heat equation.

The homomorphism L maps G onto a Δ -subgroup of G_a that is isomorphic to G .

G_1 does not give us a counterexample. Is the Δ -group G defined by the heat equation simple improrement dit? DONT KNOW. If not, there is still hope for our dream.

Conjecture. Every sequence

$$G = G_0 \supset G_1 \supset 0,$$

can be refined to put the kernel of a symmetry between G_1 and G (Cartan 1909).

Equivalently, the defining Δ -ideal of a proper Δ -subgroup of G contains a symmetry of G .

If so, every finite descending sequence of subgroups of the group G defined by the heat equation can be refined to produce a Jordan-Hölder sequence.

The second example realizes a simple Δ -subgroup of $SL(2, \mathcal{F}^+)$ as a symmetry group of a Δ -subvariety of the second Painlevé variety, defined over $\mathcal{C}(x)$. Let \mathcal{F}^+ be an ordinary differentially closed field, with derivation operator ∂_x . We denote $\partial_x y$ by y' , $\partial_x^2 y$ by y'' . In symbols such as $GL(2, \mathcal{F}^+)$ the field \mathcal{F}^+ is understood when the name of the field is suppressed.

Definition

Let k be a field. A *Chevalley* subgroup of $GL(n, k)$ is a simple algebraic subgroup that is defined over the field \mathbb{Q} of rational numbers.

Theorem

Let G be a simple Δ -group. There is a positive integer n , and a Chevalley subgroup H of $GL(n)$ such that one of the following holds:

- 1** G is isomorphic to H .
- 2** $\exists A$ in the Lie algebra of H such that G is the Δ -subgroup of H defined by the equation

$$ZAZ^{-1} + \partial_x Z \cdot Z^{-1} = A.$$

Corollary

Every simple Δ -group is isomorphic to a Zariski dense subgroup G of some Chevalley group H . If $G \neq H$, then, G is conjugate to $H(\mathbb{C})$.

Corollary

G is defined by the matrix equation

$$\partial_x Y = [A, Y],$$

called a Lax equation.

The Chevalley group $SL(2)$ acts on its Lie algebra $sl(2)$ by the *gauge action*

$$A \longmapsto ZAZ^{-1} + \partial_x Z \cdot Z^{-1} -$$

a Δ -group action. It is transitive since \mathcal{F}^+ is differentially closed.

Definition

A Δ -variety V in \mathbb{A}^1 is a Riccati variety if it is the set of solutions of a Riccati equation

$$y' = a_0 + a_1y + a_2y^2.$$

The *affine variety* V is Kolchin closed in \mathbb{P}^1 if and only if $a_2 \neq 0$.

Definition

The matrix

$$A = \begin{pmatrix} (\frac{1}{2})a_1 & a_0 \\ -a_2 & -(\frac{1}{2})a_1 \end{pmatrix}.$$

represents the Riccati equation

$$y' = a_0 + a_1y + a_2y^2.$$

Example

Let

$$A = \begin{pmatrix} 0 & \frac{x}{2} \\ -1 & 0 \end{pmatrix}.$$

A represents the Riccati equation

$$y' = \frac{x}{2} + y^2.$$

The gauge action

$$A \longmapsto ZAZ^{-1} + \partial_x Z \cdot Z^{-1}$$

of $SL(2)$ on $sl(2)$ maps $SL(2)$ into the group of affine transformations of the 3-dimensional \mathcal{F}^+ -vector space.

It induces a transitive action (also called the gauge action) of $SL(2)$ (via linear fractional transformations) on the 3-dimensional \mathcal{F}^+ -vector space of Riccati varieties.

Let

$$Z = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \alpha\delta - \beta\gamma = 1.$$

Let

$$y' = a_0 + a_1y + a_2y^2$$

define a Riccati variety.

Let the Riccati variety

$$y' = b_0 + b_1y + b_2y^2,$$

be its image under the gauge action by Z .

Then,

$$\begin{aligned} & (b_0 \ b_1 \ b_2)^t \\ &= \begin{pmatrix} \alpha^2 & -\alpha\beta & \beta^2 \\ -2\alpha\gamma & \alpha\delta + \beta\gamma & -2\beta\delta \\ \gamma^2 & -\delta\gamma & \delta^2 \end{pmatrix} (a_0 \ a_1 \ a_2)^t \\ &+ (\alpha\beta' - \alpha'\beta \ \delta\alpha' - \delta'\alpha + \beta\gamma' - \beta'\gamma \ \gamma\delta' - \gamma'\delta)^t. \end{aligned}$$

The Riccati variety defined by the equation

$$y' = a_0 + a_1y + a_2y^2$$

is a homogeneous space under the gauge action if and only if G is the isotropy group of the matrix A representing the equation.

The Painlevé equations

The Painlevé equations are 6 ordinary differential equations of *second order* of the form

$$y'' = F(x, y, y', c),$$

where F is a rational function, and c is a constant.

They satisfy

- 1** The Painlevé Property: the absence of movable singularities. The locations of the singularities of the solutions (apart from poles) depend only on the coefficients of F .
- 2** The equation cannot be integrated algebraically, or transformed into a simpler equation or the defining first order differential equation of an elliptic function.

Paul Prudent Painlevé and his student B. Gambier from 1900 to 1906 (about 100 years ago).

The Painlevé equations have Δ -type 0, and typical dimension 2.
The first Painlevé equation is:

$$y'' = 6y^2 + x.$$

Except for the first equation, the Painlevé equations depend on constant parameters.

Since the Painlevé varieties have type 0 and typical dimension 2, we expect them to contain subvarieties of typical dimension 1. This is false for the first Painlevé equation, but true for the others for certain values of the parameters.

We are interested in the second Painlevé equation:

$$P_{II}(c) : y'' = 2y^3 + xy + c.$$

Let $X(c)$ be the variety it defines.

Theorem

The Painlevé variety $X(\frac{1}{2})$ has a Δ -subvariety $V(\frac{1}{2})$ of typical dimension 1. It is a Riccati variety, defined by the equation

$$y' = \left(\frac{1}{2}\right)x + y^2.$$

The Δ -rational map

$$z \longmapsto -\frac{z'}{z}$$

transforms the Airy equation

$$z'' + \frac{x}{2}z = 0$$

into the equation

$$y' = y^2 + \left(\frac{1}{2}\right)x.$$

So, the elements of $V(\frac{1}{2})$ are called *Airy solutions* of $P_{II}(\frac{1}{2})$.

Corollary

The Riccati variety $V\left(\frac{1}{2}\right)$ is a homogeneous space, under the gauge action, of the isotropy group G of the matrix

$$A = \begin{pmatrix} 0 & \frac{x}{2} \\ -1 & 0 \end{pmatrix}.$$

Theorem

(Gromak, 1999)

The Δ -variety $X(c)$ has a Δ -subvariety of typical dimension 1 (and it is unique) if and only if $c = n + \frac{1}{2}$, n an integer.

The family $X(c)$ of second Painlevé varieties are connected by morphisms of Δ -varieties (Bäcklund transformations). They act transitively on the discrete family $X\left(n + \frac{1}{2}\right)$, n an integer. The variety $V\left(\frac{1}{2}\right)$ of Airy solutions of $X\left(\frac{1}{2}\right)$ is transformed into Riccati varieties $V\left(n + \frac{1}{2}\right)$. By composing the action of G on $V\left(\frac{1}{2}\right)$ with the Bäcklund transformations, we get a discrete family of homogeneous spaces for the simple differential algebraic group.

THANK YOU.