Difference fields and algebraic dynamics

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This work is joint with Ehud Hrushovski and is motivated by a theorem of Matthew Baker.

k is an algebraically closed field, K is a function field over k of transcendence degree one and a morphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ defined over K of degree > 1. This defines a function on $\mathbb{P}^1(K) \to \mathbb{P}^1(K)$.

One may fabricate a canonical height $h_f : \mathbb{P}^1(K) \to \mathbb{R}_{\geq 0}$ satisfying $h_f(f(P)) = dh_f(P)$ for $P \in \mathbb{P}^1(K)$. Note that if P is preperiodic, then $h_f(P) = 0$. Is the converse true?

Theorem: (M. Baker) For K and k as above one of the following occurs.

1. $(\exists \epsilon > 0)\{P \in \mathbb{P}^1(K) : h_f(P) < \epsilon\}$ is finite, or
2. $(\exists M \in \mathrm{PGL}_2(K^{\mathrm{alg}}))M^{-1}fM$ is defined over k.

Why should difference fields matter? If there are infinitely many points of small height, then taking an ultrapower we should be able to find a generic one and hence one satisfying a difference equation $\sigma(x) = f(x)$.

Assume for now there are infinitely many points of height zero. The points of height zero are contained in a set which is in bijection with a constructible subset of k^N for some $N \in \mathbb{Z}_+$. The function f on this set then gives a definable function $\widetilde{f}$ defined on a constructible set over k corresponding to f. Thus, if (1) is false, we find some algebraic variety V over k, a positive integer n, and a dominant morphism $h : V \to \mathbb{P}^1$ for which the following diagram commutes.

$$\begin{array}{ccc} V & \xrightarrow{g} & V \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ \mathbb{P}^1 & \xrightarrow{f^{\circ n}} & \mathbb{P}^1 \end{array}$$

Over an algebraically closed field L one may define the category of algebraic dynamical systems over L to have objects (X, f) where X is a variety over L and $f : X \to X$ is dominant rational map with morphisms $h : (X, f) \to (Y, g)$ given by rational maps $h : X \to Y$ for which $g \circ h = h \circ f$.

Dualizing, if $f : V \to V$ is dominant, then given a generic a of K we can define a difference field structure on $L(V)$ via $\sigma(a) = f(a)$ and $\sigma \upharpoonright K = \mathrm{id}_K$.

Theorem: If there exists $h : (W, g) \to (V, f)$ dominant with (W, g) defined over k and (V, f) defined over K^{alg}, then there exists (V_0, f_0) over k and $h_0 : (V, f) \to (V_0, f_0)$ dominant with $\dim V_0 > 0$.

Translating into difference fields, let $b \in W$ be generic over K. Define σ on $K(b)$ via $\sigma(b) = g(b)$ and σ the identity on K. Then $h(b) =: a$ is a generic of V and satisfies $\sigma(x) = f(x)$. We seek a $c \in K(a)$ so that $k(c)$ is a difference field linearly independent from K over k.

The proof is fairly long. The easy case is the one which will yield Baker's theorem. We will speak more about the more difficult case including the case of $\deg(f) = 1$.

One can assume the following "primitivity assumption": $a' \in K(a) \smallsetminus K$, then $a \in K(a')^{\mathrm{alg}}$.

The dichotomy (with some additional work) tells you then that $tp(a/K)$ is of one of the following kinds:

(i) one-based, or

(ii) F-internal, where F is a *fixed field*, i.e., is either

  (a) $\mathrm{Fix}(\sigma)$,

  (b) or possibly $\mathrm{Fix}(\tau)$

  in $(\mathbb{U}, \sigma)$, a large generic difference field containing all the mentioned fields and parameters, where $\tau(x) = \sigma^n(x^{p^m})$ for some $n \neq 0$ and $m \neq 0$, with n, m relatively prime.

[F-internal is a "quantifier-free" version of internal: it means that over some difference field L linearly disjoint from $K(a)$ over K, we have $L(a) \subset FL$.]

Take a finite tuple c in K such that everything is defined over $k(c)$; without loss of generality, we may then assume that $k(b)$ is the field of definition of the difference-algebraic locus of (c, a) over $k(b)$. A crucial observation is then:

$tp(b/K)$ is of the same kind as $tp(a/K)$.

Reversing the rôles and looking at the field of definition of the difference-algebraic locus of b over $K(a)$, we get that if $tp(a/K)$ was of type (i) or (ii)(b), this field is linearly disjoint from K over k. This gives the descent to the constants, and also eventually Baker's theorem (as in case (ii)(a), $deg(f) = 1$).

Definable groups of automorphisms

Let $K = \sigma(K)$ a difference subfield of $\mathbb{U}$. Note: we do not assume that σ is the identity on K. We assume that $\mathrm{tp}(a/K)$ is F-internal for $F = \mathrm{Fix}(\sigma)$, and $K(a) \cap F = K \cap F$.

Consider Q, the set of all tuples of $\mathbb{U}$ satisfying precisely the same K-σ-equations as a. Then there is some $u \in Q^r$ for which $L := K(Q)_\sigma = FK(u)_\sigma$. We consider now $\mathrm{Aut}_\sigma(L/FK)$.

For simplicity, we assume $K = K^{\mathrm{alg}}$. We may then reduce to the case that $Q \subseteq \underline{Q}$, an algebraic variety, and Q is defined by $\phi_Q : \underline{Q} \to \underline{Q}^\sigma$.

One shows then that there is $\underline{G}$, an algebraic group, acting on $\underline{Q}$ and a subgroup G of $\underline{G}(\mathbb{U})$ defined over K by difference equations for which $(G, Q) \cong (\mathrm{Aut}_\sigma(L/FK), Q)$.

This is not quite good enough to prove what we are after. We need some other properties of this group.

Let me first mention a result: If $K \cap F$ is pseudo-finite, then one can take $\underline{G}$ defined over $F \cap K$ and there exists a variety $\underline{Y}$ defined over K on which $\underline{G}$ acts faithfully and $g \in \underline{G}(K)$ for which $(G, Q) \cong (G, \{y \in \underline{Y} : \sigma(y) = gy\})$.

This is the type of result we are after - it gives a nice description of our set Q. In the general case however, the group G will not live over the constants of K.

We wished to extend this result to the case that $K \cap F$ is algebraically closed. However, we only obtained this for a reduct where we replace σ by σ^n for some n.

[Parenthetical note: it may happen that G does not act transitively on Q, though there will only be finitely many orbits.]

What will give us the result is the following: If $K \subseteq F$, then $\sigma \in \mathrm{Aut}_\sigma(L/FK)$. Moreover, $\underline{G}$ is commutative, $L = K(a)$, and Q is a G-torsor. Thus,

Over K, $(G, Q) \cong (G, \{x \in \underline{G}(\mathbb{U}) : \sigma(x) = x + g\})$ for some $g \in G(K)$.

Coming back to our original problem where we have reduced to the case that $K(a)$ does not have proper difference subfields (up to algebraicity), then G is a simple commutative algebraic group: $\mathbb{G}_a$, $\mathbb{G}_m$ or a simple abelian variety.

In the $\mathbb{G}_a$ case we have $\sigma(x) = g + x$ so that $\sigma(x/g) = (x + g)/g = (x/g) + 1$, so that $k(\frac{x}{g})$ is a difference field. In the other cases we use the fact that there are few endomorphisms of $\mathbb{G}_m$ or of a simple abelian variety, and again a canonical base argument.