QUASI-INVARIANTS OF COMPLEX REFLECTION GROUPS

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Abstract. We introduce quasi-invariant polynomials for an arbitrary finite complex reflection group W. Unlike in the Coxeter case, the space of quasi-invariants of a given multiplicity is not, in general, an algebra but a module Q_k over the coordinate ring of some (singular) affine variety X_k. We extend the main results of [BEG] to this setting: in particular, we show that the variety X_k and the module Q_k are Cohen-Macaulay, and the rings of differential operators on X_k and Q_k are simple rings, Morita equivalent to the Weyl algebra $A_n(\mathbb{C})$, where $n = \dim X_k$. Our approach relies on representation theory of complex Cherednik algebras introduced in [DO] and is parallel to that of [BEG]. As an application, we prove the existence of shift operators for an arbitrary complex reflection group, confirming a conjecture of Dunkl and Opdam [DO]. Another result is a proof of a conjecture of Opdam [O2], concerning certain operations (KZ twists) on the set of irreducible representations of W.

1. Introduction

The notion of a quasi-invariant polynomial for a finite Coxeter group was introduced by A. Veselov and one of the authors in [CV]. Although quasi-invariants were natural generalization of invariants, they first appeared in a slightly disguised form (as symbols of commuting differential operators). More recently, the rings of quasi-invariants and associated varieties have been studied by means of representation theory [FV, EG1, BEG] and found applications in other areas, including noncommutative algebra [BEG], mathematical physics [Be, CFV, FV1] and combinatorics [GW, GW1, BM].

The aim of the present paper is to define quasi-invariants for an arbitrary complex reflection group and give new applications. We begin with a brief overview of our definition, referring the reader to Section 2 for details. Let W be a finite complex reflection group acting in its reflection representation V. Denote by $\mathcal{A} = \{H\}$ the set of reflection hyperplanes of W and write W_H for the (pointwise) stabilizer of $H \in \mathcal{A}$ in W. Each W_H is a cyclic subgroup of W of order $n_H \geq 2$, whose group algebra $\mathbb{C}W_H \subseteq \mathbb{C}W$ is spanned by the idempotents

$$\boldsymbol{e}_{H,\,i} = \frac{1}{n_H} \sum_{w \in W_H} (\det\, w)^i\, w\ , \quad i = 0,\, 1,\, \ldots\, ,\, n_H - 1\ ,$$

where $\det : W \to \mathbb{C}^\times$ is the determinant character of W on V. The group W acts naturally on the polynomial algebra $\mathbb{C}[V]$, and the invariant polynomials $f \in \mathbb{C}[V]^W$

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satisfy the equations

$$\boldsymbol{e}_{H,\,i}(f) = 0\ , \quad i = 1, \ldots, n_H - 1\ . \tag{1.1}$$

More precisely, we have that $f \in \mathbb{C}[V]^W$ if and only if (1.1) hold for all $H \in \mathcal{A}$.

Now, to define quasi-invariants we relax (1.1) in the following way. For each $H \in \mathcal{A}$, we fix a linear form $\alpha_H \in V^*$, such that $H = \operatorname{Ker} \alpha_H$, and choose $n_H - 1$ non-negative multiplicities $k_{H,i} \in \mathbb{Z}$, assuming $k_{H,i} = k_{H',i}$ whenever H and H' are in the same orbit of W in $\mathcal{A}$. Then, we replace equations (1.1) by

$$\boldsymbol{e}_{H,\,i}(f) \equiv 0 \bmod \langle \alpha_H \rangle^{n_H k_{H,i}}\ , \quad i = 1, \ldots, n_H - 1\ , \tag{1.2}$$

where $\langle \alpha_H \rangle$ is the ideal in $\mathbb{C}[V]$ generated by α_H. Letting $k := \{k_{H,i}\}$, we call $f \in \mathbb{C}[V]$ a *k-quasi-invariant* of W if it satisfies (1.2) for all $H \in \mathcal{A}$. It is easy to see that this agrees with the earlier definition of quasi-invariants in the Coxeter case (cf. Example 2.2); however, unlike in that case, the subspace $Q_k(W) \subseteq \mathbb{C}[V]$ of k-quasi-invariants is not necessarily a ring. Still, $Q_k(W)$ contains $\mathbb{C}[V]^W$, and the following remarkable property holds.

Theorem 1.1. *$Q_k(W)$ is a free module over $\mathbb{C}[V]^W$ of rank $|W|$.*

Since $Q_0(W) = \mathbb{C}[V]$, Theorem 1.1 can be viewed as a generalization of a classic result of Chevalley and Serre (see [C]); equivalently, it can be stated by saying that $Q_k(W)$ is a Cohen-Macaulay module. For the Coxeter groups, this was conjectured by Feigin and Veselov in [FV] and proved, by different methods, in [EG1] and [BEG]. It is worth mentioning that the elementary argument of [C] and its refinement in [B] (see *loc. cit*, Ch. V, § 5, Theorem 1) do not work for nonzero k.

We will prove Theorem 1.1 (in fact, the more precise Theorem 8.2) by extending the approach of [BEG], which is based on representation theory of Cherednik algebras. We will also generalize another important result of [BEG] concerning the ring $\mathcal{D}(Q_k)$ of differential operators on quasi-invariants.

Theorem 1.2. *$\mathcal{D}(Q_k)$ is a simple ring, Morita equivalent to $\mathcal{D}(V)$.*

By a general result of Van den Bergh [VdB] (see also [BN]), Theorem 1.2 is actually a strengthening of Theorem 1.1; in this paper, however, we will prove these two theorems by independent arguments, without using [VdB] and [BN].

Although most of the elementary properties of quasi-invariants generalize easily to the complex case, the proofs of Theorem 1.1 and Theorem 1.2 do not. A key observation of [BEG] linking quasi-invariants Q_k to the rational Cherednik algebra H_k is the fact that Q_k is a module over the *spherical* subalgebra $U_k = \boldsymbol{e}H_k\boldsymbol{e}$ of H_k, and U_k is isomorphic to the ring $\mathcal{D}(Q_k)^W$ of invariant differential operators on Q_k. We will see that a similar result holds for an arbitrary complex reflection group; however, unlike in the Coxeter case (cf. [BEG], Lemma 6.4), this can hardly be proved by direct calculation, working with generators of U_k. The problem is that the ring of invariants $\mathbb{C}[V]^W$ of a complex refection group contains no quadratic polynomial, which makes explicit calculations with generators virtually impossible[1]. To remedy this problem, we will work with the Cherednik algebra itself, lifting quasi-invariants at the level of $\mathbb{C}W$-valued polynomials. More precisely, in Section 3, we will define quasi-invariants $\mathbf{Q}_k(\tau)$ with values in an *arbitrary* representation τ of W as a module over the Cherednik algebra H_k. (Checking that $\mathbf{Q}_k(\tau)$ is indeed

[1]In fact, skimming the classification table in [ST] shows that there is an exceptional complex group with minimal fundamental degree as large as 60.

an H_k-module is easy, since H_k is generated by linear forms and first order (Dunkl) operators.) The main observation (Theorem 3.4) is that the usual quasi-invariants Q_k are obtained by symmetrizing the τ-valued ones, $\mathbf{Q}_k(\tau)$, with τ being the regular representation $\mathbb{C}W$. The existence of a natural U_k-module structure on Q_k is a simple consequence of this construction and the fact that H_k and U_k are Morita equivalent algebras for integral k. As we will see in Section 4 (Proposition 4.3), the key isomorphism $U_k \cong \mathcal{D}(Q_k)^W$ also follows easily from this, and Theorem 1.2 (see Section 4.3) can then be proven similarly to [BEG].

In Section 5, we will use quasi-invariants to show the existence of Heckman-Opdam shift operators for an arbitrary complex reflection group. In the Coxeter case, this result was established by an elegant argument by G. Heckman [H], using Dunkl operators. Heckman's proof involves explicit calculations with second order invariant operators, which do not generalize to the complex case (exactly for the reason mentioned above). Still, Dunkl and Opdam [DO] have managed to extend Heckman's construction to the infinite family of complex groups of type $G(m, p, N)$ and conjectured the existence of shift operators in general. Theorem 5.7 proves this conjecture of [DO]. The idea behind the proof is to study symmetries of the family of quasi-invariants $\{\mathbf{Q}_k(\tau)\}$ under certain transformations of multiplicities k, which induce the identity at the level of spherical algebra.

Section 6 reviews the definition and basic properties of the category $\mathcal{O}$ for rational Cherednik algebras. This category was introduced and studied in [DO], [BEG] and [GGOR] as an analogue of the homonymous category of representations of a semisimple complex Lie algebra. In Section 6, we gather together results on the category $\mathcal{O}$ needed for the rest of the paper. Most of these results are either directly borrowed or can be deduced from the above references (in the last case, for reader's convenience, we provide proofs).

In Section 7, we develop some aspects of representation theory of Cherednik algebras, which may be of independent interest. First, in Section 7.1, we introduce a shift functor $\mathcal{T}_{k\to k'} : \mathcal{O}_k \to \mathcal{O}_{k'}$, relating representation categories of Cherednik algebras with different values of multiplicities. This functor is analogous to the Enright completion in Lie theory (see [J]) and closely related to other types of shift functors appeared in the literature. (Some of these relations will be discussed in Section 7.4.)

Next, in Section 7.2, we will study a certain family of permutations $\{\mathrm{kz}_k\}_{k\in\mathbb{Z}}$ on the set $\mathrm{Irr}(W)$ of (isomorphism classes of) irreducible representations of W. These permuations (called KZ twists) were orginally defined by E. Opdam in terms of Knizhnik-Zamolodchikov equations and studied using the finite Hecke algebra $\mathcal{H}_k(W)$ (see [O1, O2, O3]). In [O1], Opdam explicitly described KZ twists for all Coxeter groups; he also discovered the remarkable additivity property:

$$\mathrm{kz}_k \circ \mathrm{kz}_{k'} = \mathrm{kz}_{k+k'}$$

which holds for all integral k and k'. However, the key arguments in [O1] involve continuous deformations in parameter k and work only under the assumption that $\dim \mathcal{H}_k = |W|$, which still remains a conjecture for some exceptional groups in the complex case (see [BMR]). We will derive basic properties of kz_k , including the above additivity, from the properties of the category $\mathcal{O}_k$; thus, we will give a complete case-free proof of Opdam's results (see Theorem 7.11 and Corollary 7.12).

The link to quasi-invariants is explained by Proposition 7.13, which says that, for any $\tau \in \mathrm{Irr}(W)$, the H_k-module $\mathbf{Q}_k(\tau)$ is isomorphic to the so-called standard module $M_k(\tau')$ taken, however, with a twist[2]: $\tau' = \mathrm{kz}_{-k}(\tau)$. We would also like to draw reader's attention to formula (7.2), which gives an intrinsic description of the module $\mathbf{Q}_k(\tau)$ and should be taken, perhaps, as a conceptual definition of quasi-invariants (see Remark 7.14).

In Section 8, we will use the above description of quasi-invariants to prove Theorem 1.1 and find a decomposition of Q_k as a module over the spherical algebra $U_k = \boldsymbol{e}\, H_k \boldsymbol{e}$. In addition, we compute the Poincaré series of Q_k, generalizing the earlier results of [FV1], [EG1] and [BEG] to the complex case. As an application, we give a simple proof of a theorem of Opdam on symmetries of fake degrees of complex reflection groups.

The paper ends with an Appendix, which links our results to the original setting of [CV]. For a general complex reflection group W and W-invariant integral multiplicities $k = \{k_{H,\,i}\}$, we define the *Baker-Akhiezer function* $\psi(\lambda,\, x)$ and establish its basic properties. Although this function is not used in the main body of the paper, it is certainly worth studying.

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2. Definition of Quasi-invariants

2.1. **Complex reflection groups.** Let V be a finite-dimensional vector space over $\mathbb{C}$, and let W be a finite subgroup of $\mathrm{GL}(V)$ generated by complex reflections. We recall that an element $s \in \mathrm{GL}(V)$ is a *complex reflection* if it acts as identity on some hyperplane H_s in V. Since W is finite, there is a positive definite Hermitian form $(\,\cdot\,,\,\cdot\,)$ on V, which is invariant under the action of W. We fix such a form, once and for all, and regard W as a subgroup of the corresponding unitary group $U(V)$. We assume that $(\,\cdot\,,\,\cdot\,)$ is antilinear in its first argument and linear in the second: if $x \in V$, we write $x^* \in V^*$ for the linear form: $V \to \mathbb{C}\,,\ v \mapsto (x,\, v)\,.$ The assignment $x \mapsto x^*$ defines then an antilinear isomorphism $V \xrightarrow{\sim} V^*$, which extends to an antilinear isomorphism of the symmetric algebras $\mathbb{C}[V^*]$ and $\mathbb{C}[V]$.

Let $\mathcal{A}$ denote the set $\{H_s\}$ of reflection hyperplanes of W, corresponding to the reflections $s \in W$. The group W acts on $\mathcal{A}$ by permutations, and we write $\mathcal{A}/W$ for the set of orbits of W in $\mathcal{A}$. If $H \in \mathcal{A}$, the (pointwise) stabilizer of H in W is a cyclic subgroup $W_H \subseteq W$ of order n_H, which depends only on the orbit $C_H \in \mathcal{A}/W$ of H in $\mathcal{A}$. We fix a vector $v_H \in V$, normal to H with respect to $(\,\cdot\,,\,\cdot\,)$, and a

[2]This result corrects an error in [BEG] (cf. Remark 8.3 in Section 8).

covector $\alpha_H \in V^*$, annihilating H in V^*. With above identification, we may (and often will) assume that $\alpha_H = v_H^*$.

Now, we write $\det : W \to \mathbb{C}^\times$ for the character of W obtained by restricting the determinant character of $\mathrm{GL}(V)$. Then, under the natural action of W, the elements

$$\delta := \prod_{H\in\mathcal{A}} \alpha_H \in \mathbb{C}[V] \ , \qquad \delta^* := \prod_{H\in\mathcal{A}} v_H \in \mathbb{C}[V^*] \ , \tag{2.1}$$

transform as relative invariants with characters $\det^{-1}$ and det, respectively. For each $H \in \mathcal{A}$, the characters of W_H form a cyclic group of order n_H generated by $\det|_{W_H}$. We write

$$\boldsymbol{e}_{H,i} := \frac{1}{n_H} \sum_{w\in W_H} (\det\, w)^{-i} w \tag{2.2}$$

for the corresponding idempotents in the group algebra $\mathbb{C}W_H \subseteq \mathbb{C}W$.

More generally, for any orbit $C \in \mathcal{A}/W$, we define

$$\delta_C := \prod_{H\in C} \alpha_H \in \mathbb{C}[V] \ , \qquad \delta_C^* := \prod_{H\in C} v_H \in \mathbb{C}[V^*] \ . \tag{2.3}$$

These are also relative invariants of W, whose characters will be denoted by $\det_C^{-1}$ and $\det_C$. Note that $\det_C(s) = \det(s)$ for any reflection $s \in W$ with $H_s \in C$, while $\det_C(s) = 1$ for all other reflections. The whole group of characters of W is generated by $\det_C$ for various $C \in \mathcal{A}/W$.

Throughout the paper, we will use the following conventions.

1. A W-invariant function on $\mathcal{A}$ and the corresponding function on $\mathcal{A}/W$ will be denoted by the same symbol: for example, if C is the orbit of H in $\mathcal{A}$, we will often write n_C , k_C , ... instead of n_H , k_H , etc.

2. The index set $\{0, 1, 2, \ldots, n_H - 1\}$ will be identified with $\mathbb{Z}/n_H\mathbb{Z}$: thus we will often assume $\{\boldsymbol{e}_{H,i}\}, \{k_{C,i}\}, \ldots$ to be indexed by all integers with understanding that $\boldsymbol{e}_{H,i} = \boldsymbol{e}_{H,\,i+n_H}$, $k_{C,i} = k_{C,\,i+n_C}$, etc.

2.2. **Quasi-invariants.** For each $C \in \mathcal{A}/W$, we fix a sequence of non-negative integers $k_C = \{k_{C,i}\}_{i=0}^{n_C-1}$, with $k_{C,0} = 0$, and let $k := \{k_C\}_{C\in\mathcal{A}/W}$. Following our convention, we will think of k as a collection of *multiplicities* $\{k_{H,i}\}$ assigned to the reflection hyperplanes of W.

Definition 2.1. A *k-quasi-invariant* of W is a polynomial $f \in \mathbb{C}[V]$ satisfying

$$\boldsymbol{e}_{H,-i}(f) \equiv 0 \bmod \langle\alpha_H\rangle^{n_H k_{H,i}} \tag{2.4}$$

for all $H \in \mathcal{A}$ and $i = 0, 1, \ldots, n_H - 1$. Here $\langle\alpha_H\rangle$ stands for the principal ideal of $\mathbb{C}[V]$ generated by α_H. (Note that (2.4) holds automatically for $i = 0$, as we assumed $k_{H,0} = 0$ for all $H \in \mathcal{A}$.)

We write $Q_k(W)$ for the set of all k-quasi-invariants of W: clearly, this is a linear subspace of $\mathbb{C}[V]$.

Example 2.2 ("The Coxeter case")**.** Let W be a finite Coxeter group. Then each W_H is generated by a real reflection s_H of order $n_H = 2$, and the corresponding idempotents (2.2) are given by $\boldsymbol{e}_{H,0} = (1 + s_H)/2$ and $\boldsymbol{e}_{H,1} = (1 - s_H)/2$. As $k_{H,0} = 0$, we have only one (nontrivial) condition (2.4) for each $H \in \mathcal{A}$, defining quasi-invariants: namely, $s_H(f) \equiv f \bmod \langle\alpha_H\rangle^{2k_H}$, with $k_H = k_{H,1}$. This agrees with the original definition of quasi-invariants for the Coxeter groups (cf. [FV]).

Example 2.3 ("The one-dimensional case")**.** Fix an integer $n \geq 2$, and let W be $\mathbb{Z}/n\mathbb{Z}$ acting on $V = \mathbb{C}$ by multiplication by the n-th roots of unity. In this case, we have only one reflection "hyperplane" – the origin – with multiplicities $k = \{k_0 = 0, k_1, \dots, k_{n-1}\}$. Identifying $\mathbb{C}[V] \cong \mathbb{C}[x]$, it is easy to see that

$$Q_k(W) = \bigoplus_{i=0}^{n-1} x^{nk_i+i}\, \mathbb{C}[x^n]\ . \tag{2.5}$$

Observe that the first summand in (2.5) (with $i = 0$) is $\mathbb{C}[x^n] = \mathbb{C}[V]^W$, the ring of invariants of W in $\mathbb{C}[V]$. Observe also that Q_k contains all sufficiently large powers of x and hence the ideal $\langle x\rangle^N \subset \mathbb{C}[V]$ for some $N \gg 0$. In general, Q_k is not a ring: it is not closed under multiplication in $\mathbb{C}[V]$. However, we can define $A_k := \{p \in \mathbb{C}[x] \ :\ p\,Q_k \subseteq Q_k\}$, which is obviously a graded subring of $\mathbb{C}[V]$, Q_k being a graded A_k-module. It is easy to see that A_k also consists of quasi-invariants of W, corresponding to different multiplicities (cf. Lemma 2.4 below). Letting $X_k := \operatorname{Spec}(A_k)$, we note that X_k is a rational cuspidal curve, with a unique singular point "at the origin." The space Q_k can be thought of geometrically, as the space of sections of a rank one torsion-free coherent sheaf on X_k. As a $\mathbb{C}[V]^W$-module, Q_k is freely generated by the monomials $\{x^{nk_i+i}\}$, $i = 0, \dots, n-1$.

2.3. **Elementary properties of quasi-invariants.** We now describe some properties of quasi-invariants, which follow easily from Definition 2.1. First, as in Example 2.3 above, we fix $k = \{k_{H,i}\}$ and set

$$A_k := \{p \in \mathbb{C}[V] \ :\ p\,Q_k \subseteq Q_k\}\ . \tag{2.6}$$

The following lemma is a generalization of [BEG], Lemma 6.3.

Lemma 2.4.

(i) *$A_k = Q_{k'}(W)$ for some $k' = \{k'_{H,i}\}$. In particular, both Q_k and A_k contain $\mathbb{C}[V]^W$ and are stable under the action of W.*

(ii) *A_k is a finitely generated graded subalgebra of $\mathbb{C}[V]$, and Q_k is a finitely generated graded module over A_k of rank 1.*

(iii) *The field of fractions of A_k is $\mathbb{C}(V)$, and the integral closure of A_k in $\mathbb{C}(V)$ is $\mathbb{C}[V]$.*

Proof. For a polynomial $f \in \mathbb{C}[V]$, we define its normal expansion along a hyperplane $H \in \mathcal{A}$ by

$$f(x + tv_H) = \sum_{s\geq 0} c_{H,\,s}(x)\, t^s\ , \quad x \in H\ .$$

It is then easy to see that f satisfies (2.4) if and only if $c_{H,\,s}(x) = 0$ for all $s \in \mathbb{Z}_+ \setminus S$, where

$$S = \bigcup_{i=0}^{n_H-1} \{i + n_H k_{H,i} + n_H\mathbb{Z}_+\}\ .$$

Now, letting $R := \{r \in \mathbb{Z} \ :\ r + S \subseteq S\}$, we observe that $p \in A_k$ if and only if, for each $H \in \mathcal{A}$, the normal expansion of p along H contains no terms t^r with $r \notin R$. To prove (i) it suffices to note that R can be written in the same form as S, maybe with different k's. Indeed, $S \subset \mathbb{Z}$ can be characterized by the property that it is invariant under translation by n_H and contains all integers $s \gg 0$. Clearly, R has the same property and, therefore, a similar description.

To prove (ii) and (iii), we can argue as in [BEG], Lemma 6.3. Since $\mathbb{C}[V]^W \subseteq A_k \subseteq \mathbb{C}[V]$, the Hilbert-Noether Lemma implies that A_k is a finitely generated algebra, and $\mathbb{C}[V]$ is a finite module over A_k. Being a submodule of $\mathbb{C}[V]$, Q_k is then also finite over A_k. Now, both A_k and Q_k contain the ideal of $\mathbb{C}[V]$ generated by a power of $\delta \in \mathbb{C}[V]$. Hence, A_k and $\mathbb{C}[V]$ have the same field of fractions, namely $\mathbb{C}(V)$, and the integral closure of A_k in $\mathbb{C}(V)$ is $\mathbb{C}[V]$. This also implies that $\dim_{\mathbb{C}(V)}[Q_k \otimes_{A_k} \mathbb{C}(V)] = 1$, and thus Q_k is a rank 1 module over A_k. □

It is convenient to state some properties of quasi-invariants in geometric terms. To this end, we write $X_k = \mathrm{Spec}(A_k)$ and let $\mathcal{O}_x = \mathcal{O}_x(X_k)$ denote the local ring of X_k at a point $x \in X_k$. This local ring can be identified with a subring of $\mathbb{C}(V)$ by localizing the algebra embedding $A_k \hookrightarrow \mathbb{C}[V]$. To the module Q_k we can then associate a torsion-free coherent sheaf on X_k, with fibres $(Q_k)_x = Q_k \otimes_{\mathcal{O}} \mathcal{O}_x$. Our definition of quasi-invariants generalizes to this local setting if we require (2.4) to hold for the stabilizer W_x of x under the natural action of W on X_k. This makes sense, since by a theorem of Steinberg [St], W_x is also generated by complex reflections.

Lemma 2.5 (cf. [BEG], Lemma 7.3)**.** *Let* $\mathbb{A}^n := \mathrm{Spec}\,\mathbb{C}[V]$.

(i) *X_k is an irreducible affine variety, with normalization $\tilde{X}_k = \mathbb{A}^n$.*

(ii) *The normalization map $\pi_k :\ \mathbb{A}^n \to X_k$ is bijective.*

(iii) *If we identify the (closed) points of X_k and $\mathbb{A}^n$ via π_k, then for each $x \in \mathbb{A}^n$, $(Q_k)_x$ is the space of k-quasi-invariants in $\mathbb{C}(V)$ with respect to the subgroup $W_x \subseteq W$.*

Proof. The proof given in [BEG] in the case of Coxeter groups (see [BEG], Lemma 7.3) works, *mutatis mutandis*, for all complex reflection groups. We leave this as a (trivial) exercise to the reader. □

3. Quasi-invariants and Cherednik Algebras

3.1. **The rational Cherednik algebra.** We begin by reviewing the definition of Cherednik algebras associated to a complex reflection group. For more details and proofs, we refer the reader to [DO] and [GGOR]. In this section, unless stated otherwise, the multiplicities $k_{C,i}$ are assumed to be arbitrary complex numbers.

We set $V_{\mathrm{reg}} := V \setminus \bigcup_{H\in\mathcal{A}} H$ and let $\mathbb{C}[V_{\mathrm{reg}}]$ and $\mathcal{D}(V_{\mathrm{reg}})$ denote the rings of regular functions and regular differential operators on V_{reg}, respectively. The action of W on V restricts to V_{reg}, so W acts naturally on $\mathbb{C}[V_{\mathrm{reg}}]$ and $\mathcal{D}(V_{\mathrm{reg}})$ by algebra automorphisms. We form the crossed products $\mathbb{C}[V_{\mathrm{reg}}] * W$ and $\mathcal{D}(V_{\mathrm{reg}}) * W$ and denote $\mathcal{D}W := \mathcal{D}(V_{\mathrm{reg}}) * W$. As an algebra, $\mathcal{D}W$ is generated by its two subalgebras $\mathbb{C}W$ and $\mathcal{D}[V_{\mathrm{reg}}]$, and hence, by the elements of W, $\mathbb{C}[V_{\mathrm{reg}}]$ and the derivations ∂_ξ, $\xi \in V$.

Following [DO], we now define the *Dunkl operators* $T_\xi \in \mathcal{D}W$ by

$$T_\xi := \partial_\xi - \sum_{H\in\mathcal{A}} \frac{\alpha_H(\xi)}{\alpha_H} \sum_{i=0}^{n_H-1} n_H k_{H,i} \boldsymbol{e}_{H,i}\ , \quad \xi \in V\ . \tag{3.1}$$

Note that the operators (3.1) depend on $k = \{k_{H,i}\}$, and we sometimes write $T_{\xi,k}$ to emphasize this dependence. The basic properties of Dunkl operators are gathered in the following lemma.

Lemma 3.1 (see [D, DO])**.** *For all* $\xi, \eta \in V$ *and* $w \in W$*, we have*
(i) commutativity: $T_{\xi,k}\, T_{\eta,k} - T_{\eta,k}\, T_{\xi,k} = 0$,
(ii) W-equivariance: $w\, T_\xi = T_{w(\xi)}\, w$,
(iii) homogeneity: T_ξ *is a homogeneous operator of degree* -1 *with respect to the natural (differential) grading on* $\mathcal{D}W$*.*

Properties (ii) and (iii) of Lemma 3.1 follow easily from the definition of Dunkl operators. On the other hand, the commutativity (i) is far from being obvious: it was first proved in [D] in the Coxeter case, and then in [DO] in full generality (*loc. cit.*, Theorem 2.12).

In view of Lemma 3.1, the assignment $\xi \mapsto T_\xi$ extends to an injective algebra homomorphism

$$\mathbb{C}[V^*] \hookrightarrow \mathcal{D}W \ , \quad p \mapsto T_p \ . \tag{3.2}$$

Identifying $\mathbb{C}[V^*]$ with its image in $\mathcal{D}W$ under (3.2), we now define the *rational Cherednik algebra* $H_k = H_k(W)$ as the subalgebra of $\mathcal{D}W$ generated by $\mathbb{C}[V]$, $\mathbb{C}[V^*]$ and $\mathbb{C}W$.

The Cherednik algebras can be also defined directly, in terms of generators and relations, see [EG, BEG, GGOR]. To be precise, H_k is generated by the elements $x \in V^*$, $\xi \in V$ and $w \in W$ subject to the following relations

$$[x,\, x'] = 0 \ , \quad [\xi,\, \xi'] = 0 \ , \quad w\, x\, w^{-1} = w(x) \ , \quad w\, \xi\, w^{-1} = w(\xi) \ ,$$

$$[\xi,\, x] = \langle \xi,\, x\rangle + \sum_{H \in \mathcal{A}} \frac{\langle \alpha_H, \xi\rangle\, \langle x, v_H\rangle}{\langle \alpha_H, v_H\rangle} \sum_{i=0}^{n_H-1} n_H (k_{H,i} - k_{H,i+1})\, \boldsymbol{e}_{H,i} \ .$$

The family $\{H_k\}$ can be viewed as a deformation (in fact, the universal deformation) of the crossed product $H_0 = \mathcal{D}(V) * W$ (see [EG], Theorem 2.16). The embedding of $H_k \hookrightarrow \mathcal{D}W$ is given by $w \mapsto w$, $x \mapsto x$ and $\xi \mapsto T_\xi$ and referred to as the *Dunkl representation* of H_k. The existence of such a representation implies the Poincaré–Birkhoff–Witt (PBW) property for H_k, which says that the multiplication map

$$\mathbb{C}[V] \otimes \mathbb{C}W \otimes \mathbb{C}[V^*] \xrightarrow{\sim} H_k \tag{3.3}$$

is an isomorphism of vector spaces (see [EG])[3].

The algebra $\mathcal{D}W = \mathcal{D}(V_{\rm reg}) * W$ carries two natural filtrations: one is defined by taking $\deg(x) = \deg(\xi) = 1$ and $\deg(w) = 0$, and the other is defined by $\deg(x) = 0$, $\deg(\xi) = 1$ and $\deg(w) = 0$ for all $x \in V^*$, $\xi \in V$ and $w \in W$. We refer to the first filtration as *standard* and to the second as *differential.* Through the Dunkl representation, these two filtrations induce filtrations on H_k for all k. It is easy to see that the associated graded rings $\mathrm{gr}\, H_k$ are isomorphic to $\mathbb{C}[V \times V^*] * W$ in both cases; in particular, they are independent of k.

Note that $\{1,\, \delta,\, \delta^2,\, \ldots\}$, with δ defined in (2.1), is a localizing (Ore) subset in H_k: we write $H_{\rm reg} := H_k[\delta^{-1}]$ for the corresponding localization. Since δ is a unit in $\mathcal{D}W$, the Dunkl embedding $H_k \hookrightarrow \mathcal{D}W$ induces the canonical map $H_{\rm reg} \to \mathcal{D}W$.

Proposition 3.2 (see [EG], Prop. 4.5; [GGOR], Theorem 5.6)**.** *The map* $H_{\rm reg} \to \mathcal{D}W$ *is an isomorphism of algebras.*

---

[3] The PBW property is proven in [EG] for a more general class of symplectic reflection algebras.

Despite its modest appearance, Proposition 3.2 plays an important rôle in representation theory of Cherednik algebras. In particular, it justifies our notation $H_{\rm reg}$ for the localization of H_k (as $H_{\rm reg}$ is indeed independent of k).

Next, we introduce the *spherical subalgebra* U_k of H_k: by definition, $U_k := \boldsymbol{e}\, H_k\, \boldsymbol{e}$, where $\boldsymbol{e} := |W|^{-1} \sum_{w\in W} w$ is the symmetrizing idempotent in $\mathbb{C}W \subset H_k$. For $k = 0$, we have $U_0 = \boldsymbol{e}[\mathcal{D}(V) * W]\boldsymbol{e} \cong \mathcal{D}(V)^W$; thus, the family $\{U_k\}$ is a deformation (in fact, the universal deformation) of the ring of invariant differential operators on V. The standard and differential filtrations on H_k induce filtrations on U_k, and we have $\mathrm{gr}\, U_k \cong \mathbb{C}[V \times V^*]^W$ in both cases.

The relation between H_k and U_k depends drastically on multiplicity values. In the present paper, we will be mostly concerned with integral k's, in which case we have the following result.

Theorem 3.3. *If k is integral, i. e. $k_{C,i} \in \mathbb{Z}$ for all $C \in \mathcal{A}/W$, then H_k and U_k are simple algebras, Morita equivalent to each other.*

Proof. There is a natural functor relating the module categories of H_k and U_k :

$$\mathrm{Mod}(H_k) \to \mathrm{Mod}(U_k)\ , \quad M \mapsto \boldsymbol{e}M\ , \tag{3.4}$$

where $\boldsymbol{e}M := \boldsymbol{e}H_k \otimes_{H_k} M$. By standard Morita theory (see, e. g, [MR], Prop. 3.5.6), this functor is an equivalence if (and only if) $H_k \boldsymbol{e}\, H_k = H_k$. The last condition holds automatically if H_k is simple. So one needs only to prove the simplicity of H_k. In the Coxeter case, this is the result of [BEG], Theorem 3.1. In general, the simplicity of H_k can be deduced from the semi-simplicity of the category $\mathcal{O}_{H_k}$ for integral k's, which, in turn, follows from general results of [GGOR]. We discuss this in detail in Section 6 (see Theorem 6.6 below). □

The restriction of the Dunkl representation $H_k \hookrightarrow \mathcal{D}W$ to $\boldsymbol{e}H_k\boldsymbol{e} \subset H_k$ yields an embedding $U_k \hookrightarrow \boldsymbol{e}\,\mathcal{D}W\boldsymbol{e}$, which is a homomorphism of unital algebras. If we combine this with (the inverse of) the isomorphism $\mathcal{D}(V_{\rm reg})^W \stackrel{\sim}{\to} \boldsymbol{e}\,\mathcal{D}W\boldsymbol{e}$, $D \mapsto \boldsymbol{e}D\boldsymbol{e} = \boldsymbol{e}D = D\boldsymbol{e}$, we get an algebra map

$$\mathrm{Res} : U_k \hookrightarrow \mathcal{D}(V_{\rm reg})^W\ , \tag{3.5}$$

representing U_k by invariant differential operators on $V_{\rm reg}$ (cf. [H]). We will refer to (3.5) as the *Dunkl representation* for the spherical subalgebra U_k.

3.2. **$\mathbb{C}W$-valued quasi-invariants.** The algebra $\mathcal{D}W$ can be viewed as a ring of W-equivariant differential operators on $V_{\rm reg}$, and as such it acts naturally on the space of $\mathbb{C}W$-valued functions. More precisely, using the canonical inclusion $\mathbb{C}[V_{\rm reg}] \otimes \mathbb{C}W \hookrightarrow \mathcal{D}W$, we can identify $\mathbb{C}[V_{\rm reg}] \otimes \mathbb{C}W$ with the cyclic $\mathcal{D}W$-module $\mathcal{D}W/J$, where J is the left ideal of $\mathcal{D}W$ generated by $\partial_\xi \in \mathcal{D}W$, $\xi \in V$. Explicitly, in terms of generators, $\mathcal{D}W$ acts on $\mathbb{C}[V_{\rm reg}] \otimes \mathbb{C}W$ by

$$\begin{aligned} g(f \otimes u) &= gf \otimes u\,, \quad g \in \mathbb{C}[V_{\rm reg}]\ , \\ \partial_\xi(f \otimes u) &= \partial_\xi f \otimes u\,, \quad \xi \in V\ , \\ w(f \otimes u) &= f^w \otimes wu\,, \quad w \in W\ . \end{aligned} \tag{3.6}$$

Now, the restriction of scalars via the Dunkl representation $H_k(W) \hookrightarrow \mathcal{D}W$ makes $\mathbb{C}[V_{\rm reg}] \otimes \mathbb{C}W$ an $H_k(W)$-module. We will call the corresponding action of H_k the *differential action.* It turns out that, in the case of integral k's, the differential action of H_k is intimately related to quasi-invariants $Q_k = Q_k(W)$.

Theorem 3.4. *If k is integral, then $\mathbb{C}[V_{\rm reg}]\otimes\mathbb{C}W$ contains a unique H_k-submodule $\mathbf{Q}_k = \mathbf{Q}_k(W)$, such that $\mathbf{Q}_k$ is finite over $\mathbb{C}[V]\subset H_k$ and*

$$\boldsymbol{e}\,\mathbf{Q}_k = \boldsymbol{e}(\,Q_k\otimes 1) \quad in \quad \mathbb{C}[V_{\rm reg}]\otimes\mathbb{C}W\ . \tag{3.7}$$

We prove Theorem 3.4 in several steps. First, we construct $\mathbf{Q}_k$ as a subspace of $\mathbb{C}[V_{\rm reg}]\otimes\mathbb{C}W$ and verify (3.7). Then we show that $\mathbf{Q}_k$ is stable under the differential action of H_k, and finally we prove its uniqueness.

Besides the diagonal action (3.6), we will use another action of W on $\mathbb{C}[V_{\rm reg}]\otimes\mathbb{C}W$, which is trivial on the first factor : i. e., $f\otimes s\mapsto f\otimes ws$, where $w\in W$ and $f\otimes s\in\mathbb{C}[V_{\rm reg}]\otimes\mathbb{C}W$. We denote this action by $1\otimes w$.

Now, we define $\mathbf{Q}_k$ to be the subspace of $\mathbb{C}[V_{\rm reg}]\otimes\mathbb{C}W$ spanned by the elements φ satisfying

$$(1\otimes\boldsymbol{e}_{H,i})\,\varphi\equiv 0 \bmod \langle\alpha_H\rangle^{n_H k_{H,i}}\otimes\mathbb{C}W\ , \tag{3.8}$$

for all $H\in\mathcal{A}$ and $i=0,1,\dots,n_H-1$. Here, as in Definition 2.4, $\langle\alpha_H\rangle$ stands for the ideal of $\mathbb{C}[V]$ generated by α_H.

It is immediate from (3.8) that $\mathbf{Q}_k\subseteq\mathbb{C}[V]\otimes\mathbb{C}W$, and $\mathbf{Q}_k$ is closed in $\mathbb{C}[V]\otimes\mathbb{C}W$ under the natural action of $\mathbb{C}[V]$. Hence, as W is finite and $\mathbb{C}[V]$ is Noetherian, $\mathbf{Q}_k$ is a finitely generated $\mathbb{C}[V]$-module.

Lemma 3.5. $\mathbf{Q}_k$ *satisfies* (3.7).

Proof. We need to show that $\boldsymbol{e}(f\otimes 1)\in\boldsymbol{e}\,\mathbf{Q}_k$ if and only if $f\in Q_k$. First, for any $f\in\mathbb{C}[V]$ and $s\in W$, we compute

$$(1\otimes s)[\boldsymbol{e}(f\otimes 1)] = \frac{1}{|W|}\sum_{w\in W} f^w\otimes sw = \frac{1}{|W|}\sum_{w\in W} f^{s^{-1}w}\otimes w\,.$$

Now, multiplying this by appropriate characters and summing up over all $s\in W_H$, we get

$$(1\otimes\boldsymbol{e}_{H,i})[\boldsymbol{e}(f\otimes 1)] = \frac{1}{|W|}\sum_{w\in W}\boldsymbol{e}_{H,-i}(f^w)\otimes w\ .$$

It follows from (3.8) that $\boldsymbol{e}(f\otimes 1)\in\boldsymbol{e}\mathbf{Q}_k$ if and only if $f^w\in Q_k$ for all $w\in W$. The latter is equivalent to $f\in Q_k$, since Q_k is W-stable. □

Lemma 3.6. $\mathbf{Q}_k$ *is stable under the differential action of* H_k*.*

Proof. As already mentioned above, $\mathbf{Q}_k$ is closed under the action of $\mathbb{C}[V]\subset H_k$. To see that $\mathbf{Q}_k$ is stable under the diagonal action of W, we observe that

$$w\,(1\otimes\boldsymbol{e}_{H,i}) = w\otimes w\boldsymbol{e}_{H,i} = (1\otimes\boldsymbol{e}_{wH,i})\,w$$

as endomorphisms of $\mathbb{C}[V_{\rm reg}]\otimes\mathbb{C}W$. Since (3.8) hold for each $H\in\mathcal{A}$ and $k_{H,i}$'s depend only on the orbit of H in $\mathcal{A}$, we have $w\,\mathbf{Q}_k\subseteq\mathbf{Q}_k$ for all $w\in W$.

Thus, we need only to check that $\mathbf{Q}_k$ is preserved by the Dunkl operators (3.1). For each $H\in\mathcal{A}$, let $\mathbf{Q}_k^H$ denote the subspace of $\mathbb{C}[V_{\rm reg}]\otimes\mathbb{C}W$ spanned by all φ's satisfying (3.8) only for the given H. Clearly $\mathbf{Q}_k=\bigcap_{H\in\mathcal{A}}\mathbf{Q}_k^H$, so it suffices to show that

$$T_\xi(\mathbf{Q}_k)\subseteq\mathbf{Q}_k^H \quad \text{for all } H\in\mathcal{A}\ . \tag{3.9}$$

Writing $T_\xi = T_0 + T_1$ with

$$T_0 := \partial_\xi - \frac{\alpha_H(\xi)}{\alpha_H} \sum_{i=0}^{n_H-1} n_H k_{H,i} \boldsymbol{e}_{H,i}\,,$$

$$T_1 := \sum_{H' \neq H} \frac{\alpha_{H'}(\xi)}{\alpha_{H'}} \sum_{i=0}^{n_{H'}-1} n_{H'} k_{H',i} \boldsymbol{e}_{H',i}\,,$$

we will verify (3.9) separately for T_0 and T_1.

Since $\mathbf{Q}_k$ is W-stable, $\boldsymbol{e}_{H',i}(\mathbf{Q}_k) \subseteq \mathbf{Q}_k \subseteq \mathbf{Q}_k^H$. Next, $\alpha_{H'}^{-1} \in \mathbb{C}[V_{\rm reg}]$ is regular along H, therefore, $\alpha_{H'}^{-1}\,\mathbf{Q}_k^H \subseteq \mathbf{Q}_k^H$. Combining these two facts together, we get $\alpha_{H'}^{-1}\boldsymbol{e}_{H',i}(\mathbf{Q}_k) \subseteq \mathbf{Q}_k^H$, and hence $T_1(\mathbf{Q}_k) \subseteq \mathbf{Q}_k^H$.

It remains to show that $T_0(\mathbf{Q}_k) \subseteq \mathbf{Q}_k^H$. In fact, we have $\mathbf{Q}_k \subseteq \mathbf{Q}_k^H$, so it suffices to show that $T_0(\mathbf{Q}_k^H) \subseteq \mathbf{Q}_k^H$. Note that the definition of both $\mathbf{Q}_k^H$ and T_0 involve only one hyperplane H and the group W_H, so the statement can be checked in dimension one, in which case it is straightforward, see Example 3.9 below. ☐

Lemma 3.7. *If k is integral, there exists at most one H_k-submodule $\mathbf{Q}_k \subset \mathbb{C}[V_{\rm reg}] \otimes \mathbb{C}W$, satisfying* (3.7).

Proof. Suppose that $\mathbf{Q}_k$ and $\mathbf{Q}'_k$ are two such submodules. Replacing one of them by their sum, we may assume that $\mathbf{Q}_k \subset \mathbf{Q}'_k$, with $e\mathbf{Q}_k = e\mathbf{Q}'_k$. Setting $M := \mathbf{Q}'_k/\mathbf{Q}_k$, we get $eM = 0$. This forces $M = 0$, since (3.4) is a fully faithful functor by Theorem 3.3. Thus $\mathbf{Q}'_k = \mathbf{Q}_k$, as required. ☐

Lemmas 3.5, 3.6 and 3.7 combined together imply Theorem 3.4. As a simple consequence of this theorem, we get

Corollary 3.8. *Q_k is stable under the action of U_k on $\mathbb{C}[V_{\rm reg}]$ via the Dunkl representation* (3.5). *Thus Q_k is a U_k-module, with U_k acting on Q_k by invariant differential operators.*

Proof. Theorem 3.4 implies that $\boldsymbol{e}H_k\boldsymbol{e}(\boldsymbol{e}\mathbf{Q}_k) \subseteq \boldsymbol{e}\mathbf{Q}_k$. Recall that for every element $\boldsymbol{e}L\boldsymbol{e} \in \boldsymbol{e}H_k\boldsymbol{e}$ we have $\boldsymbol{e}L\boldsymbol{e} = \boldsymbol{e}\,{\rm Res}\,L$, by the definition of the map (3.5). As a result,

$$\boldsymbol{e}\,({\rm Res}\,L[Q_k] \otimes 1) = \boldsymbol{e}\,{\rm Res}\,L\,[Q_k \otimes 1] = (\boldsymbol{e}\,L\,\boldsymbol{e})[\mathbf{Q}_k] \subseteq \boldsymbol{e}\mathbf{Q}_k = \boldsymbol{e}(Q_k \otimes 1)\,.$$

It follows that $({\rm Res}\,L)[Q_k] \subseteq Q_k$, since $\boldsymbol{e}\,(f \otimes 1) = 0$ in $\mathbb{C}[V_{\rm reg}] \otimes \mathbb{C}W$ forces $f = 0$. ☐

Example 3.9. We illustrate Theorem 3.4 in the one-dimensional case. Let $W = \mathbb{Z}/n\mathbb{Z}$ and $k = (k_0, \ldots, k_{n-1})$ be as in Example 2.3. Then

$$\mathbf{Q}_k = \bigoplus_{i=0}^{n-1} x^{nk_i}\mathbb{C}[x] \otimes \boldsymbol{e}_i\,, \quad \boldsymbol{e}_i = \frac{1}{n}\sum_{w \in W} (\det w)^{-i} w\,. \tag{3.10}$$

Clearly, $\mathbf{Q}_k$ is stable under the action of W and $\mathbb{C}[x]$. On the other hand, if $k_i \in \mathbb{Z}$, a trivial calculation shows that the Dunkl operator $T := \partial_x - x^{-1}\sum_{i=0}^{n-1} nk_i\boldsymbol{e}_i$ annihilates the elements $x^{nk_i} \otimes \boldsymbol{e}_i$, and hence preserves $\mathbf{Q}_k$ as well. Now, acting on $\mathbf{Q}_k$ by $\boldsymbol{e} = \boldsymbol{e}_0$ and using (2.5), we get

$$\boldsymbol{e}\mathbf{Q}_k = \bigoplus_{i=0}^{n-1} x^{nk_i+i}\mathbb{C}[x^n] \otimes \boldsymbol{e}_i = \bigoplus_{i=0}^{n-1} \boldsymbol{e}\,(x^{nk_i+i}\mathbb{C}[x^n] \otimes 1) = \boldsymbol{e}\,(Q_k \otimes 1)\,, \tag{3.11}$$

which agrees with Theorem 3.4.

3.3. **Generalized quasi-invariants.** In our construction of quasi-invariants, the regular representation $\mathbb{C}W$ played a distinguished rôle. We now outline a generalization, in which $\mathbb{C}W$ is replaced by an arbitrary W-module τ. For a more conceptual definition of quasi-invariants in terms of shift functors, we refer the reader to Section 7 (see Remark 7.14).

First, we observe that the left ideal J of $\mathcal{D}W$ generated by the derivations ∂_ξ, $\xi \in V$, is stable under *right* multiplication by the elements of $\mathbb{C}W \subset \mathcal{D}W$. Hence $\mathcal{D}W/J$ is naturally a $\mathcal{D}W$-$\mathbb{C}W$-bimodule. For any W-module τ, we can form then the left $\mathcal{D}W$-module $\mathcal{D}W/J \otimes_{\mathbb{C}W} \tau \cong \mathbb{C}[V_{\rm reg}] \otimes \tau$. The action of $\mathcal{D}W$ on $\mathbb{C}[V_{\rm reg}] \otimes \tau$ is given by the same formulas (3.6), with $w \in W$ acting now in representation τ, and H_k operates via its Dunkl representation. Now, generalizing (3.8), we define the module $\mathbf{Q}_k(\tau)$ of *τ-valued quasi-invariants* as the span of all $\varphi \in \mathbb{C}[V_{\rm reg}] \otimes \tau$ satisfying

$$(1 \otimes \boldsymbol{e}_{H,i})\, \varphi \equiv 0 \bmod \langle \alpha_H \rangle^{n_H k_{H,i}} \otimes \tau \tag{3.12}$$

for all $H \in \mathcal{A}$ and $i = 0, 1, \ldots, n_H - 1$. It is convenient to write $\mathbf{Q}_k(\tau)$ as the intersection of subspaces corresponding to the reflection hyperplanes $H \in \mathcal{A}$:

$$\mathbf{Q}_k(\tau) = \bigcap_{H \in \mathcal{A}} \mathbf{Q}_k^H(\tau)\,, \quad \mathbf{Q}_k^H(\tau) := \bigoplus_{i=0}^{n_H-1} \langle \alpha_H \rangle^{n_H k_{H,i}} \otimes \boldsymbol{e}_{H,i}\tau\,. \tag{3.13}$$

The same argument as in Lemma 3.6 above proves the following

Proposition 3.10. *The space $\mathbf{Q}_k(\tau) \subset \mathbb{C}[V_{\rm reg}] \otimes \tau$ is stable under the action of H_k. The subspace $\boldsymbol{e}\mathbf{Q}_k(\tau)$ of W-invariant elements in $\mathbf{Q}_k(\tau)$ is then a module over the spherical subalgebra $\boldsymbol{e}H_k\boldsymbol{e}$.*

In addition, we have

Lemma 3.11. *Let $\mathbf{Q}_k^H(\tau)$ be as in (3.13), and let $\boldsymbol{e}_{H,0} := \frac{1}{n_H}\sum_{w \in W_H} w$. Then*

$$\boldsymbol{e}\mathbf{Q}_k(\tau) = \bigcap_{H \in \mathcal{A}} \boldsymbol{e}_{H,0}\, \mathbf{Q}_k^H(\tau)\,. \tag{3.14}$$

Proof. First, it is clear that the right-hand side of (3.14) lies in the intersection (3.13) and thus belongs to $\mathbf{Q}_k(\tau)$. Furthermore, it is contained in $\boldsymbol{e}_{H,0}\mathbf{Q}_k^H(\tau)$ and therefore stable under the action of W_H. Since H is arbitrary, this proves that the right-hand side of (3.14) is stable under the whole of W and hence contained in the left-hand side. The opposite inclusion follows from $\boldsymbol{e}\mathbf{Q}_k(\tau) \subseteq \boldsymbol{e}_{H,0}\mathbf{Q}_k(\tau) \subseteq \boldsymbol{e}_{H,0}\mathbf{Q}_k^H(\tau)$. ∎

We can decompose each subspace $\boldsymbol{e}_{H,0}\mathbf{Q}_k^H(\tau)$ in (3.14) as in the one-dimensional case (see Example 3.9, (3.11)). To be precise, let $\mathbb{C}[V_{\rm reg}^H]$ denote the subring of functions in $\mathbb{C}[V_{\rm reg}]$ that are regular along H. This ring carries a natural action of W_H, so we write $\mathbb{C}[V_{\rm reg}^H]^{W_H}$ for its subring of invariants. With this notation, we have

$$\boldsymbol{e}_{H,0}\mathbf{Q}_k^H(\tau) = \bigoplus_{i=0}^{n_H-1} \alpha_H^{n_H k_{H,i}+i}\, \mathbb{C}[V_{\rm reg}^H]^{W_H} \otimes \boldsymbol{e}_{H,i}\tau\,. \tag{3.15}$$

We close this section with a few remarks.

1. As an immediate consequence of the definition (3.12), we have

$$\delta^r\mathbb{C}[V]\otimes\tau\subset\mathbf{Q}_k(\tau)\subset\delta^{-r}\mathbb{C}[V]\otimes\tau\,, \tag{3.16}$$

where $r>0$ is sufficiently large (precisely, $r>\max\{n_H k_{H,i}\}$). More generally, for integral k,k', it is easy to show that

$$\delta^r\mathbf{Q}_k(\tau)\subseteq\mathbf{Q}_{k'}(\tau)\subseteq\delta^{-r}\mathbf{Q}_k(\tau)\ , \tag{3.17}$$

where $r\gg 0$ depends only on the difference $k'-k$.

2. If τ is a direct sum of W-modules, say τ_i, then $\mathbb{C}[V_{\rm reg}]\otimes\tau$ and $\mathbf{Q}_k(\tau)$ are also direct sums of $\mathbb{C}[V_{\rm reg}]\otimes\tau_i$ and $\mathbf{Q}_k(\tau_i)$, respectively. In particular, replacing τ by $\mathbb{C}W=\sum_{\tau\in{\rm Irr}(W)}\tau\otimes\tau^*$, we get

$$\mathbf{Q}_k=\sum_{\tau\in{\rm Irr}(W)}\mathbf{Q}_k(\tau)\otimes\tau^*\,. \tag{3.18}$$

Thus, the structure of $\mathbf{Q}_k$ is determined by the modules $\mathbf{Q}_k(\tau)$ associated to irreducible representations of W. We will study these modules in detail in Section 8.

3. As was mentioned already, on the space $\mathbb{C}[V_{\rm reg}]\otimes\mathbb{C}W$ one has yet another (left) W-action sending $f\otimes u$ to $f\otimes uw^{-1}$. It is clear from the definitions, that it commutes with the action of $\mathcal{D}W$ and preserves both $\mathbf{Q}_k$ and $\boldsymbol{e}\mathbf{Q}_k$. Note that this action preserves each summand in (3.18), acting on τ^*. Under (3.7), it translates into the standard action of W on $Q_k\subset\mathbb{C}[V]$.

4. Differential Operators on Quasi-invariants

4.1. **Rings of differential operators.** We briefly recall the definition of differential operators in the algebro-geometric setting (see [Gr] or [MR], Chap. 15).

Let A be a commutative algebra over $\mathbb{C}$, and let M be an A-module. The filtered ring of (linear) differential operators on M is defined by

$$\mathcal{D}_A(M):=\bigcup_{n\ge 0}\mathcal{D}^n_A(M)\ \subseteq\ {\rm End}_{\mathbb{C}}(M)\ ,$$

where $\mathcal{D}^0_A(M):={\rm End}_A(M)$ and $\mathcal{D}^n_A(M)$, with $n\ge 1$, are given inductively:

$$\mathcal{D}^n_A(M):=\{D\in{\rm End}_{\mathbb{C}}(M)\mid[\,D,\,a\,]\in\mathcal{D}^{n-1}_A(M)\text{ for all }a\in A\}\ .$$

The elements of $\mathcal{D}^n_A(M)\setminus\mathcal{D}^{n-1}_A(M)$ are called *differential operators of order* n on M. Note that the commutator of two operators in $\mathcal{D}^n_A(M)$ of orders n and m has order at most $n+m-1$. Hence the associated graded ring ${\rm gr}\,\mathcal{D}_A(M):=\bigoplus_{n\ge 0}\mathcal{D}^n_A(M)/\mathcal{D}^{n-1}_A(M)$ is a commutative algebra.

If X is an affine variety with coordinate ring $A=\mathcal{O}(X)$, we denote $\mathcal{D}_A(A)$ by $\mathcal{D}(X)$ and call it the *ring of differential operators on* X. If X is irreducible, then each differential operator on X has a unique extension to a differential operator on $\mathbb{K}:=\mathbb{C}(X)$, the field of rational functions of X, and thus we can identify (see [MR], Theorem 15.5.5):

$$\mathcal{D}(X)=\{\,D\in\mathcal{D}(\mathbb{K})\mid D(f)\in\mathcal{O}(X)\text{ for all }f\in\mathcal{O}(X)\,\}\ .$$

Slightly more generally, we have

Lemma 4.1 (cf. [BW], Prop. 2.6)**.** *Suppose that* $M\subseteq\mathbb{K}$ *is a (nonzero)* A-*submodule of* $\mathbb{K}$. *Then*

$$\mathcal{D}_A(M)=\{\,D\in\mathcal{D}(\mathbb{K})\mid D(f)\in M\text{ for all }f\in M\,\}\ .$$

We apply these concepts in case when $A = A_k$ and $M = Q_k$, denoting $\mathcal{D}_A(M)$ in this case by $\mathcal{D}(Q_k)$. By Lemma 2.4(iii), $X_k = \mathrm{Spec}(A_k)$ is an irreducible variety with $\mathbb{K} = \mathbb{C}(V)$, so, by Lemma 4.1, we have

$$\mathcal{D}(Q_k) = \{D \in \mathcal{D}(\mathbb{K}) \,|\, D(f) \subseteq Q_k \text{ for all } f \in Q_k\} . \tag{4.1}$$

Note that the differential filtration on $\mathcal{D}(Q_k)$ is induced from the differential filtration on $\mathcal{D}(\mathbb{K})$. Thus (4.1) yields a canonical inclusion $\mathrm{gr}\,\mathcal{D}(Q_k) \subseteq \mathrm{gr}\,\mathcal{D}(\mathbb{K})$, with $\mathcal{D}^0(Q_k) = A_k$, see (2.6). In particular, if $k = \{0\}$, then $Q_k = \mathbb{C}[V]$ and (4.1) becomes the standard realization of $\mathcal{D}(V)$ as a subring of $\mathcal{D}(\mathbb{K})$.

Apart from Q_k, we may also apply Lemma 4.1 to $\mathbb{C}[V_{\mathrm{reg}}]$, which is naturally a subalgebra of $\mathbb{K} = \mathbb{C}(V)$. This gives the identification

$$\mathcal{D}(V_{\mathrm{reg}}) = \{D \in \mathcal{D}(\mathbb{K}) \,|\, D(f) \subseteq \mathbb{C}[V_{\mathrm{reg}}] \text{ for all } f \in \mathbb{C}[V_{\mathrm{reg}}]\} . \tag{4.2}$$

Lemma 4.2. *With identifications* (4.1) *and* (4.2)*, we have*

$$\text{(i) } \mathcal{D}(Q_k) \subseteq \mathcal{D}(V_{\mathrm{reg}}) \quad \textit{and} \quad \text{(ii) } \mathrm{gr}\,\mathcal{D}(Q_k) \subseteq \mathrm{gr}\,\mathcal{D}(V) .$$

Proof. This can be deduced from general results of [SS] or [BEG] (see, e.g., [BEG], Lemma A.1). However, for reader's convenience, we give a shorter argument here. First, recall that $\delta^N \mathbb{C}[V] \subseteq Q_k \subseteq \mathbb{C}[V]$ for some $N \geq 1$. Hence, for any $D \in \mathcal{D}(Q_k)$, we have

$$D\,\delta^N(\mathbb{C}[V]) \subseteq D(Q_k) \subseteq Q_k \subseteq \mathbb{C}[V] .$$

It follows that $D\delta^N \in \mathcal{D}(V)$ for all $D \in \mathcal{D}(Q_k)$ and $\mathcal{D}(Q_k) \subseteq \mathcal{D}(V)\,\delta^{-N}$ proving the first claim of the lemma. The last inclusion also implies that $\mathrm{gr}\,\mathcal{D}(Q_k) \subseteq \delta^{-N}\mathrm{gr}\,\mathcal{D}(V)$. Since $\mathrm{gr}\,\mathcal{D}(Q_k)$ is closed under multiplication, this is possible only if $\mathrm{gr}\,\mathcal{D}(Q_k) \subseteq \mathrm{gr}\,\mathcal{D}(V)$, which is the second claim of the lemma. □

4.2. **Invariant differential operators.** Recall that, by Lemma 2.4, Q_k is stable under the action of W on $\mathbb{C}[V_{\mathrm{reg}}]$. Hence W acts naturally on $\mathcal{D}(Q_k)$, and this action is compatible with the inclusion of Lemma 4.2(i). It follows that $\mathcal{D}(Q_k)^W \subseteq \mathcal{D}(V_{\mathrm{reg}})^W$. Now, we recall the algebra embedding (3.5), which defines the Dunkl representation for the spherical subalgebra of H_k.

Proposition 4.3. *The image of* $\mathrm{Res} : U_k \hookrightarrow \mathcal{D}(V_{\mathrm{reg}})^W$ *coincides with* $\mathcal{D}(Q_k)^W$. *Thus the Dunkl representation of* U_k *yields an algebra isomorphism* $U_k \cong \mathcal{D}(Q_k)^W$.

Proof. In the Coxeter case, this is the result of [BEG], Proposition 7.22. In general, the proof is similar, *provided* the results of the previous section are available. Indeed, by Corollary 3.8, the image of Res is contained in $\mathcal{D}(Q_k)^W$. So we need only to see that the map $\mathrm{Res} : U_k \to \mathcal{D}(Q_k)^W$ is surjective. Passing to the associated graded algebras, we first note that $\mathrm{gr}\,\mathcal{D}(Q_k)^W \subseteq \mathrm{gr}\,\mathcal{D}(V)^W$ by Lemma 4.2 (ii). On the other hand, by the PBW property (3.3) of H_k, the Dunkl representation induces an isomorphism $\mathrm{gr}\,U_k \cong \mathrm{gr}\,\mathcal{D}(V)^W$. Hence, the associated graded map $\mathrm{gr}\,U_k \to \mathrm{gr}\,\mathcal{D}(Q_k)^W$ is surjective, and so is the map $U_k \to \mathcal{D}(Q_k)^W$. □

Corollary 4.4. $\mathrm{gr}\,\mathcal{D}(V)$ *is a finite module over* $\mathrm{gr}\,\mathcal{D}(Q_k)$. *Consequently* $\mathrm{gr}\,\mathcal{D}(Q_k)$ *is a finitely generated (and hence, Noetherian) commutative* $\mathbb{C}$*-algebra.*

Proof. We have already seen that $\mathrm{gr}\,\mathcal{D}(Q_k)^W \subseteq \mathrm{gr}\,\mathcal{D}(Q_k) \subseteq \mathrm{gr}\,\mathcal{D}(V)$. On the other hand, by Proposition 4.3, $\mathrm{gr}\,\mathcal{D}(Q_k)^W = \mathrm{gr}\,U_k = \mathrm{gr}\,U_0 = \mathrm{gr}\,\mathcal{D}(V)^W = [\mathrm{gr}\,\mathcal{D}(V)]^W$. Since W is finite, $\mathrm{gr}\,\mathcal{D}(V)$ is a finite module over $[\mathrm{gr}\,\mathcal{D}(V)]^W$, and hence *a fortiori* over $\mathrm{gr}\,\mathcal{D}(Q_k)$. This proves the first claim of the corollary. The second claim follows from the first by the Hilbert-Noether Lemma. □

Remark 4.5. Following [Kn], let us say that an algebra $A \subseteq \mathcal{D}(\mathbb{K})$ is *graded cofinite* in $\mathcal{D}(V)$ if $\operatorname{gr} A \subseteq \operatorname{gr} \mathcal{D}(V)$ and $\operatorname{gr} \mathcal{D}(V)$ is a finite module over $\operatorname{gr} A$. Under the assumption that $A \subseteq \mathcal{D}(V)$, such algebras are described in [Kn]. Corollary 4.4 shows that $\mathcal{D}(Q_k)$ is graded cofinite in $\mathcal{D}(V)$, although it is actually *not* a subalgebra of $\mathcal{D}(V)$. It might be interesting to see whether the geometric description of graded cofinite algebras given in [Kn] extends to our more general situation. Another interesting problem is to study the structure of $\operatorname{gr} \mathcal{D}(Q_k)$ as a module over $\operatorname{gr} \mathcal{D}(Q_k)^W$. This is a natural "double" of the $\mathbb{C}[V]^W$-module Q_k. In contrast to Theorem 1.1, the module $\operatorname{gr} \mathcal{D}(Q_k)$ is not free over $\operatorname{gr} \mathcal{D}(Q_k)^W$ and consequently $\mathcal{D}(Q_k)$ is not free over $\mathcal{D}(Q_k)^W$, although the latter module is projective (see Corollary 4.6 below).

4.3. **Simplicity and Morita equivalence.** We now prove Theorem 1.2 from the Introduction, which is a generalization of [BEG], Theorem 9.7. Our proof is similar to that of [BEG], except for the fact that Q_k may not be a ring in general. We give some details for completeness.

Proof of Theorem 1.2. First, by Theorem 3.3, U_k is a simple ring, and hence so is $\mathcal{D}(Q_k)^W$, by Proposition 4.3. An easy argument (see [BEG], p. 319) shows that $\mathbb{C}[V]^W \cap I \neq \{0\}$ for any nonzero two-sided ideal I of $\mathcal{D}(Q_k)$. Since $\mathbb{C}[V]^W = Q_k^W \subset \mathcal{D}(Q_k)^W$, we have $\mathcal{D}(Q_k)^W \cap I \neq \{0\}$ and therefore (by simplicity of $\mathcal{D}(Q_k)^W$) $1 \in I$. This proves the simplicity of $\mathcal{D}(Q_k)$.

Now, letting $\mathcal{P} := \{\, D \in \mathcal{D}(\mathbb{K}) \mid D(f) \in Q_k \text{ for all } f \in \mathbb{C}[V] \,\}$, we note that $\mathcal{P} \subset \mathcal{D}(V)$ is a right ideal of $\mathcal{D}(V)$, with $\operatorname{End}_{\mathcal{D}(V)} \mathcal{P} \cong \mathcal{D}(Q_k)$. To see the latter, we can argue as in [SS], Proposition 3.3. First, it is clear that $\mathcal{P}$ is closed under the left multiplication by the elements of $\mathcal{D}(Q_k)$ in $\mathcal{D}(\mathbb{K})$: this gives an embedding $\mathcal{D}(Q_k) \subseteq \operatorname{End}_{\mathcal{D}(V)} \mathcal{P}$. On the other hand, $\mathcal{P}(\mathbb{C}[V]) = Q_k$, since the $\mathcal{D}(Q_k)$-module $Q_k/\mathcal{P}(\mathbb{C}[V])$ has a nonzero annihilator (containing $\mathcal{P}$), and hence, must be 0, by simplicity of $\mathcal{D}(Q_k)$. Identifying now $\operatorname{End}_{\mathcal{D}(V)} \mathcal{P} \cong \{\, D \in \mathcal{D}(\mathbb{K}) \mid D.\mathcal{P} \subseteq \mathcal{P} \,\}$, we have $D(Q_k) = D\,\mathcal{P}(\mathbb{C}[V]) \subseteq \mathcal{P}(\mathbb{C}[V]) = Q_k$ for any $D \in \operatorname{End}_{\mathcal{D}(V)} \mathcal{P}$, whence $\operatorname{End}_{\mathcal{D}(V)} \mathcal{P} \subseteq \mathcal{D}(Q_k)$.

Finally, since $\mathcal{D}(V)$ and $\mathcal{D}(Q_k)$ are both simple rings, $\operatorname{End}_{\mathcal{D}(V)} \mathcal{P} \cong \mathcal{D}(Q_k)$ implies that $\mathcal{P}$ is a progenerator in the category of right $\mathcal{D}(V)$-modules, and $\mathcal{D}(V)$ and $\mathcal{D}(Q_k)$ are Morita equivalent rings. □

As a simple consequence of Proposition 4.3 and Theorem 1.2, we get

Corollary 4.6. *$\mathcal{D}(Q_k)$ is a (right) projective module over $\mathcal{D}(Q_k)^W$.*

Proof. Since $\mathcal{D}(Q_k)^W$ and $\mathcal{D}(Q_k)$ are simple rings, $\operatorname{End}_{\mathcal{D}(Q_k)^W} \mathcal{D}(Q_k) \cong \mathcal{D}(Q_k) * W$ is a simple ring, Morita equivalent to $\mathcal{D}(Q_k)^W$ (see [M], Theorem 2.4). It follows that $\mathcal{D}(Q_k)$ is a progenerator in the category of right $\mathcal{D}(Q_k)^W$-modules; in particular, $\mathcal{D}(Q_k)$ is f. g. projective over $\mathcal{D}(Q_k)^W$. □

5. Shift Operators

5.1. **Automorphisms of $\mathcal{D}W$.** We start by describing certain automorphisms of the algebra $\mathcal{D}W$ and their action on the subalgebras H_k and $U_k = \boldsymbol{e}H_k\boldsymbol{e}$. Recall that $\mathcal{D}W$ is generated by the elements $w \in W$, $x \in V^*$ and $\xi \in V$, so any automorphism of $\mathcal{D}W$ is determined by its action on these elements.

Given a one-dimensional character χ of W, we define our first automorphism by

$$w \mapsto \chi(w)w\,, \quad x \mapsto x\,, \quad \partial_\xi \mapsto \partial_\xi\,. \tag{5.1}$$

Under (5.1), the subalgebras H_k and U_k transform to $H_{k'}$ and $\boldsymbol{e}_\chi H_{k'} \boldsymbol{e}_\chi$, where $\boldsymbol{e}_\chi \in \mathbb{C}W$ is the idempotent corresponding to χ, and $k'_{H,i} := k_{H,i+a_H}$ with $a_H \in \mathbb{Z}$ determined by $\chi|_{W_H} = (\det)^{a_H}$.

To define the second automorphism we fix a W-orbit $C \subseteq \mathcal{A}$ and a W-invariant closed 1-form ω on $V_{\rm reg}$:

$$\omega = \lambda\, d\, \log \delta_C = \lambda \sum_{H \in C} \frac{d\,\alpha_H}{\alpha_H}\ , \quad \lambda \in \mathbb{C}\ . \tag{5.2}$$

Then, regarding $\xi \in V$ as a constant vector field on $V_{\rm reg}$, we define

$$w \mapsto w\,, \quad x \mapsto x\,, \quad \partial_\xi \mapsto \partial_\xi + \omega(\xi)\,, \tag{5.3}$$

This automorphism maps the algebras H_k and U_k to $H_{k'}$ and $U_{k'}$, where k' is given by $k'_{C,i} = k_{C,i} - \lambda/n_C$ and $k'_{C',i} = k_{C',i}$ for $C' \neq C$.

Finally, for a fixed $C \in \mathcal{A}/W$, we consider the automorphism $u \mapsto \delta_C u\, \delta_C^{-1}$ given by conjugation by the element (2.3). It is easy to see that this automorphism is the composition of the automorphism (5.1), with $\chi = \det_C$, and the automorphism (5.3), with $\lambda = -1$. Therefore, it maps H_k, U_k to $H_{k'}$ and $\epsilon_C\, H_{k'} \epsilon_C$, where

$$\epsilon_C = \delta_C\, \boldsymbol{e}\, \delta_C^{-1} = |W|^{-1} \sum_{w \in W} (\det{}_C w) w\,, \tag{5.4}$$

and k' is related to k by

$$k'_{C,i} = k_{C,i+1} + 1/n_C \quad \text{and} \quad k'_{C',i} = k_{C',i} \ \text{ for } C' \neq C\,. \tag{5.5}$$

5.2. **Twisted quasi-invariants.** For the purposes of this section, we redefine quasi-invariants in a slightly greater generality to allow fractional multiplicities. Precisely, we fix a W-invariant function $a : \mathcal{A} \to \mathbb{Z}$ and choose $k_{C,i} \in \mathbb{Q}$ so that

$$k_{C,i} \equiv a_C/n_C \mod \mathbb{Z}\ . \tag{5.6}$$

(In particular, $a = 0$ corresponds to the case of integral k's.) For such k, we take Q_k to be the subspace of all $f \in \mathbb{C}[V_{\rm reg}]$ satisfying

$$\boldsymbol{e}_{H,-i-a_H}(f) \equiv 0 \mod \langle \alpha_H^{n_H k_{H,i}} \rangle \tag{5.7}$$

for all $H \in \mathcal{A}$ and $i = 0, 1, \ldots, n_H - 1$. In the case of negative multiplicities, $\langle \alpha_H^{n_H k_{H,i}} \rangle$ should be understood as the span of rational functions $f \in \mathbb{C}[V_{\rm reg}]$ for which $f \cdot \alpha_H^{-n_H k_{H,i}}$ is regular along H (although it may still have poles along other hyperplanes).

The proof of Theorem 3.4 will work in this more general situation, if we modify the definition of $\mathbf{Q}_k \subset \mathbb{C}[V_{\rm reg}] \otimes \mathbb{C}W$ in the following way, cf. (3.8):

$$\varphi \in \mathbf{Q}_k \quad \Longleftrightarrow \quad (1 \otimes \boldsymbol{e}_{H,i+a_H})\varphi \equiv 0 \mod \langle \alpha_H^{n_H k_{H,i}} \rangle \otimes \mathbb{C}W \tag{5.8}$$

for all $H \in \mathcal{A}$ and $i = 0, 1, \ldots, n_H - 1$.

Example 5.1. Let $W = \mathbb{Z}/n\mathbb{Z}$ and suppose that $k_i \equiv a/n \ (\mathrm{mod}\, \mathbb{Z})$. In that case, we have

$$\mathbf{Q}_k = \bigoplus_{i=0}^{n-1} x^{n k_i} \mathbb{C}[x] \boldsymbol{e}_{i+a}\ , \quad \boldsymbol{e}_i = \frac{1}{n} \sum_{w \in W} (\det w)^{-i} w\,. \tag{5.9}$$

On the other hand, it is easy to see that the subspace $Q_k \subseteq \mathbb{C}[V]$ is still described by formula (2.5), which is actually independent of a. As a consequence, for different values of k, we may get the same Q_k. For example, if we take k' to be

$$k'_i = k_{i-1} - \frac{1}{n} \text{ for } i = 1, \dots, n-1\,,\ k'_0 = k_{n-1} - \frac{1}{n} + 1\,, \tag{5.10}$$

then the formula (2.5) gives that $Q_{k'} = Q_k$. More generally, this holds for all iterations of (5.10), which form a cyclic group of order n. In the next section, we extend this observation to an arbitrary group W.

5.3. **Symmetries of the Dunkl representation.** The Dunkl representation defines a flat family of subalgebras $\{U_k\}$ of $\mathcal{D}(V_{\mathrm{reg}})^W$, with $\mathrm{gr}(U_k) = \mathbb{C}[V \times V^*]^W$ for any k. It turns out that this family is invariant under a certain subgroup G of affine transformations of k, so that $U_k = U_{k'}$ whenever $k' = g \cdot k$ with $g \in G$. This kind of invariance is not obvious from definitions: we will deduce it by studying the action of G on modules Q_k of quasi-invariants.

First, as in Example 5.1, for $C \in \mathcal{A}/W$ we define the transformation $g_C : k \mapsto k'$ by

$$k'_{C,i} = k_{C,i-1} - \frac{1}{n_C} + \delta_{i,0} \text{ and } \quad k'_{C',i} = k_{C',i} \quad \text{for } C' \neq C\,. \tag{5.11}$$

Note that $(g_C)^{n_C} = \mathrm{Id}$. Note also that if k satisfies the conditions (5.6), then k' satisfies the same conditions, with a replaced by $a' := a - 1_C$, where $1_C : \mathcal{A} \to \mathbb{Z}$ is the characteristic function of the orbit C.

Proposition 5.2. *Let G denote the (abelian) group generated by all g_C with $C \in \mathcal{A}/W$. Then $Q_{k'} = Q_k$ for any $k' \in G \cdot k$, provided k satisfies* (5.6).

Proof. A straightforward calculation shows that the two systems of congruences (5.7) for k and $k' = g_C \cdot k$ are equivalent. As in Example 5.1 above, this implies the equality $Q_{k'} = Q_k$. □

For the purposes of Section 8, we will need an analogue of the above result for the modules of τ-valued quasi-invariants $\mathbf{Q}_k(\tau)$. First, we need to modify their definition similarly to (5.8):

$$\varphi \in \mathbf{Q}_k(\tau) \quad \Longleftrightarrow \quad (1 \otimes \boldsymbol{e}_{H,i+a_H})\varphi \equiv 0 \bmod \langle \alpha_H^{n_H k_{H,i}} \rangle \otimes \tau \tag{5.12}$$

for all $H \in \mathcal{A}$ and $i = 0, 1, \dots, n_H - 1$. Then it is easy to see that $\mathbf{Q}_k(\tau)$ can be described similarly to (3.13):

$$\mathbf{Q}_k(\tau) = \bigcap_{H \in \mathcal{A}} \mathbf{Q}_k^H(\tau)\,, \quad \mathbf{Q}_k^H(\tau) = \bigoplus_{i=0}^{n_H - 1} \langle \alpha_H \rangle^{n_H k_{H,i}} \otimes \boldsymbol{e}_{H,i+a_H} \tau\,. \tag{5.13}$$

As before, the space $\mathbf{Q}_k(\tau) \subset \mathbb{C}[V_{\mathrm{reg}}] \otimes \tau$ is invariant under the differential action of H_k. As a result, the subspace $\boldsymbol{e}\mathbf{Q}_k(\tau)$ of W-invariant elements in $\mathbf{Q}_k(\tau)$ becomes a module over the spherical subalgebra $\boldsymbol{e}H_k\boldsymbol{e}$. Furthermore, the proof of Lemma 3.11 applies verbatim, so we have the formula

$$\boldsymbol{e}\mathbf{Q}_k(\tau) = \bigcap_{H \in \mathcal{A}} \boldsymbol{e}_{H,0} \mathbf{Q}_k^H(\tau)\,, \tag{5.14}$$

with each of the subspaces $\boldsymbol{e}_{H,0}\mathbf{Q}_k^H(\tau)$ described similarly to (3.15):

$$\boldsymbol{e}_{H,0}\mathbf{Q}_k^H(\tau) = \bigoplus_{i=0}^{n_H-1} \alpha_H^{n_H k_{H,i}+i}\,\mathbb{C}[V_{\mathrm{reg}}^H]^{W_H} \otimes \boldsymbol{e}_{H,i+a_H}\tau\,. \tag{5.15}$$

Finally, using (5.14) and (5.15), we obtain similarly to Proposition 5.2 the following result.

Proposition 5.3. *Let G denote the abelian group generated by all transformations* (5.11)*. Then for any k satisfying* (5.6) *and any $k' \in G\cdot k$, we have $\boldsymbol{e}\mathbf{Q}_{k'}(\tau) = \boldsymbol{e}\mathbf{Q}_k(\tau)$ as subspaces in $\mathbb{C}[V_{\mathrm{reg}}]\otimes\tau$.*

Proposition 5.2 has the following important consequence.

Proposition 5.4. *Let k be arbitrary and $k' \in G\cdot k$. Then the spherical subalgebras $U_k = \boldsymbol{e}H_k\boldsymbol{e}$ and $U_{k'} = \boldsymbol{e}H_{k'}\boldsymbol{e}$ coincide as subsets in $\mathcal{D}W$ and hence are isomorphic. Furthermore, we have $\boldsymbol{e}T_{p,k}\boldsymbol{e} = \boldsymbol{e}T_{p,k'}\boldsymbol{e}$ for any $p \in \mathbb{C}[V^*]^W$, or equivalently, $L_{p,k} = L_{p,k'}$, where $L_{p,k} := \mathrm{Res}\,(\boldsymbol{e}T_{p,k}\boldsymbol{e})$.*

Proof. First, we prove the claim under the integrality assumption (5.6). By Proposition 5.2, we have $Q_k = Q_{k'}$, so that $\mathcal{D}(Q_k)^W = \mathcal{D}(Q_{k'})^W$. On the other hand, Proposition 4.3 says that $U_k = \boldsymbol{e}\mathcal{D}(Q_k)^W$ and $U_{k'} = \boldsymbol{e}\mathcal{D}(Q_{k'})^W$. Whence $U_k = U_{k'}$.

To prove the second claim, let L, L' denote $L_{p,k}$ and $L_{p,k'}$, respectively. From the defintion of the Dunkl operators it easily follows that L and L' have the same principal symbol $p(\partial)$, and their lower order coefficients are rational functions of negative homogeneous degrees. Hence $L - L'$ is a differential operator whose all coefficients have negative homogeneous degree. But, by Proposition 5.2 and Theorem 3.8, both L and L' are in $\mathcal{D}(Q_k)$, so, by Lemma 4.2(ii), the principal symbol of $L - L'$ must be regular. This proves that $L = L'$.

To extend the above results to arbitrary k, take $k' = g\cdot k$, with *fixed* $g \in G$. For the standard filtration, we have $\mathrm{gr}\,U_k = \mathrm{gr}\,U_{k'} \cong \mathbb{C}[V\times V^*]^W$. Thus, we may view $k \mapsto U_k$ and $k \mapsto U_{g\cdot k}$ as two flat families of filtered subspaces in $\mathcal{D}W$. We know that these subspaces coincide when k takes rational values satisfying (5.6). Since the set of such values of k is Zariski dense in the space of all complex multiplicities, we conclude that $U_k = U_{g\cdot k}$ holds for all k. In the same spirit, we have $L_{p,k} = L_{p,k'}$ for rational k, and both sides of this equality depend polynomially in k, hence the same must be true for all k. □

5.4. **Isomorphisms of spherical algebras.** In this section, we will regard $k = \{k_{C,i}\}$ as a vector in $\mathbb{C}^N$, with $N = \sum_{C\in\mathcal{A}/W} n_C$. Let $\{\ell_{C,i}\}$ denote the standard basis in this vector space, so that $k = \sum_{C\in\mathcal{A}/W}\sum_{i=0}^{n_C-1} k_{C,i}\ell_{C,i}$. (As usual, we assume $\ell_{C,i}$ to be periodic in i, so that $\ell_{C,n_C} = \ell_{C,0}$.)

The next proposition describes the transformation of U_k under translations $k \mapsto k + \ell_{C,n_C-1}$. In the Coxeter case, this result was first established in [BEG] for generic ('regular') multiplicities and later extended in [Go] to arbitrary k's when W is crystallographic. We now prove it in full generality: for an arbitrary complex reflection group and arbitrary multiplicities.

Proposition 5.5. *For a fixed $C \in \mathcal{A}/W$, we have the following isomorphisms*

(1) $\boldsymbol{e}H_k\boldsymbol{e} \cong \epsilon_C H_{k'}\epsilon_C$, $k' = k + \ell_{C,n_C-1}$;

(2) $\boldsymbol{e}H_k\boldsymbol{e} \cong \epsilon H_{k'}\epsilon$, $k' = k + \sum_{C\in\mathcal{A}/W}\ell_{C,n_C-1}$,

where ϵ is the sign idempotent on W and ϵ_C is given by (5.4).

Proof. Let $f = f_C$ and $g = g_C$ be the transformations $k \mapsto k'$ defined by (5.5) and (5.11), respectively. Recall that f describes the effect of the conjugation by δ_C, so that

$$\delta_C\, T_{\xi,k}\, \delta_C^{-1} = T_{\xi,f(k)} \quad \text{and} \quad \delta_C\, H_k\, \delta_C^{-1} = H_{f(k)}\,. \tag{5.16}$$

On the other hand, by Proposition 5.4, $\boldsymbol{e}H_k\boldsymbol{e} = \boldsymbol{e}H_{g(k)}\boldsymbol{e}$. Now, a simple calculation shows that $k' := fg(k) = k + \ell_{C,n_C-1}$. Combining all these together, we get

$$\boldsymbol{e}H_k\boldsymbol{e} = \boldsymbol{e}H_{g(k)}\boldsymbol{e} = \boldsymbol{e}\,\delta_C^{-1}H_{fg(k)}\delta_C\,\boldsymbol{e} = \delta_C^{-1}\,\epsilon_C H_{k'}\epsilon_C\,\delta_C \cong \epsilon_C H_{k'}\epsilon_C\ ,$$

which is our first isomorphism. The second isomorphism is proved in a similar way, using $f = \prod_{C\in\mathcal{C}} f_C$ and $g = \prod_{C\in\mathcal{C}} g_C$ instead of f_C, g_C. ☐

Note that the above proof gives a bit more than stated in the proposition: it shows that $\boldsymbol{e}H_k\boldsymbol{e} = \boldsymbol{e}\delta_C^{-1}H_{k'}\delta_C\boldsymbol{e}$ as *subsets* in $\mathcal{D}W$. Now, arguing as in (the proof of) Proposition 5.4, we conclude that $\boldsymbol{e}\,T_{p,k}\,\boldsymbol{e} = \boldsymbol{e}\,\delta_C^{-1}T_{p,k'}\delta_C\,\boldsymbol{e}$ for any W-invariant polynomial p. More generally, we have the following result, which answers a question of Dunkl and Opdam (see [DO], Question 3.22).

Proposition 5.6. *For fixed $C \in \mathcal{A}/W$ and $a = 1, \ldots, n_C - 1$, let*

$$k' = k + \sum_{i=1}^{a} \ell_{C,n_C-i}\ . \tag{5.17}$$

Then $\boldsymbol{e}H_k\boldsymbol{e} = \boldsymbol{e}\delta_C^{-a}H_{k'}\delta_C^{a}\boldsymbol{e}$ in $\mathcal{D}W$, and $\boldsymbol{e}\,T_{p,k}\,\boldsymbol{e} = \boldsymbol{e}\,\delta_C^{-a}T_{p,k'}\delta_C^{a}\,\boldsymbol{e}$ for all $p \in \mathbb{C}[V^*]^W$.

This is proved by replacing the transformations $f = f_C$, $g = g_C$ in the proof of Proposition 5.5 by their iterates, f^a and g^a. ☐

5.5. **Shift operators.** We are now in position to construct the Heckman-Opdam shift operators for the group W, extending an idea of G. Heckman [H]. Fix $C \in \mathcal{A}/W$ and $a \in \{0, \ldots, n_C - 1\}$ as above, and recall the elements δ_C, δ_C^*, see (2.3). For an arbitrary k, define k' by (5.17) (with $k' := k$ in the case $a = 0$) and introduce the following differential operators

$$S_k := \mathrm{Res}\left(\delta_C^{1-a}T_{\delta_C^*,k'}\delta_C^{a}\right)\ , \quad S_k^- := \mathrm{Res}\left(\delta_C^{-a}(T_{\delta_C^*,k'})^{n_C-1}\delta_C^{a-1}\right)\ . \tag{5.18}$$

Note that both expressions under Res are W-invariant.

Theorem 5.7. *For all $p \in \mathbb{C}[V^*]^W$, the operators S_k and S_k^- satisfy the following intertwining relations*

$$L_{p,\widetilde{k}} \circ S_k = S_k \circ L_{p,k}\ , \quad L_{p,k} \circ S_k^- = S_k^- \circ L_{p,\widetilde{k}}\ ,$$

where $\widetilde{k} = k + \ell_{C,n_C-a}$.

Proof. Let $f = f_C$ and $g = g_C$ be the same as in the proof of Proposition 5.5, and let k' be as in (5.17). A direct calculation shows that $k' = f^a g^a(k)$ and $g^{1-a}fg^a(k) = k + \ell_{C,n_C-a}$. As a result, if we let $k_1 := f^{1-a}(k')$ and $k_2 := f^{-a}(k')$, then $k = g^{1-a}(k_1)$ and $k_2 = g^a(k)$. By Proposition 5.4, this implies

$$L_{p,\widetilde{k}} = L_{p,k_1} \quad \text{and} \quad L_{p,k} = L_{p,k_2}\,.$$

To prove the first identity it thus suffices to show that S_k intertwines L_{p,k_1} and L_{p,k_2}. Writing δ , δ^* for δ_C, δ^*_C, we have

$$\begin{aligned}
\boldsymbol{e}L_{p,k_1}S_k &= \boldsymbol{e}T_{p,k_1}\delta^{1-a}T_{\delta^*,k'}\delta^a\boldsymbol{e} \\
&= \boldsymbol{e}\delta^{1-a}T_{p,f^{a-1}(k_1)}T_{\delta^*,k'}\delta^a\boldsymbol{e} && \text{(by (5.16))} \\
&= \boldsymbol{e}\delta^{1-a}T_{p,k'}T_{\delta^*,k'}\delta^a\boldsymbol{e} \\
&= \boldsymbol{e}\delta^{1-a}T_{\delta^*,k'}T_{p,k'}\delta^a\boldsymbol{e} && \text{(by Lemma 3.1(i))} \\
&= \boldsymbol{e}\delta^{1-a}T_{\delta^*,k'}\delta^aT_{p,f^{-a}(k')}\boldsymbol{e} \\
&= \boldsymbol{e}\delta^{1-a}T_{\delta^*,k'}\delta^a\boldsymbol{e}\cdot\boldsymbol{e}T_{p,k_2}\boldsymbol{e} && \text{(by Lemma 3.1(ii))} \\
&= \boldsymbol{e}S_kL_{p,k_2}\ .
\end{aligned}$$

The second identity involving S_k^- is proved in a similar fashion. □

6. Category $\mathcal{O}$

Throughout this section, we will use the following notation: if A is an algebra, we write $\mathrm{Mod}(A)$ for the category of all left modules over A, and $\mathrm{mod}(A)$ for its subcategory consisting of finitely generated modules. In particular, when A is a finite-dimensional algebra over $\mathbb{C}$ (e.g., $A = \mathbb{C}W$), $\mathrm{mod}(A)$ is the category of finite-dimensional modules over A.

6.1. **Standard modules.** Recall that the Cherednik algebra $H_k = H_k(W)$ admits a decomposition $H_k \cong \mathbb{C}[V]\otimes\mathbb{C}W\otimes\mathbb{C}[V^*]$, which is similar to the PBW decomposition $U(\mathfrak{g}) \cong U(\mathfrak{n}_-)\otimes U(\mathfrak{h})\otimes U(\mathfrak{n}_+)$ for the universal enveloping algebra of a complex semisimple Lie algebra $\mathfrak{g}$. This suggests to view the subalgebras $\mathbb{C}[V]$, $\mathbb{C}[V^*]$ and $\mathbb{C}W$ of H_k as analogues of $U(\mathfrak{n}_-)$, $U(\mathfrak{n}_+)$ and $U(\mathfrak{h})$ respectively, and introduce a category of 'highest weight modules' over H_k by analogy with the Bernstein-Gelfand-Gelfand category $\mathcal{O}_{\mathfrak{g}}$ in Lie theory.

Precisely, the category $\mathcal{O}_k := \mathcal{O}_{H_k}$ is defined as the full subcategory of $\mathrm{mod}(H_k)$, consisting of modules on which the elements of $V\subset\mathbb{C}[V^*]$ act locally nilpotently:

$$\mathcal{O}_k := \{M\in\mathrm{mod}(H_k)\ :\ \xi^d m = 0\ ,\ \forall\, m\in M\ ,\ \forall\,\xi\in V\ ,\ \forall\, d\gg 0\}\ .$$

It is easy to see that $\mathcal{O}_k$ is closed under taking subobjects, quotients and extensions in $\mathrm{mod}(H_k)$: in other words, $\mathcal{O}_k$ is a Serre subcategory of $\mathrm{mod}(H_k)$.

The structure of $\mathcal{O}_k$ is determined by so-called standard modules, which play a rôle similar to Verma modules in Lie theory. To define such modules we fix an irreducible representation τ of W and extend the W-structure on τ to a $\mathbb{C}[V^*]*W$-module structure by letting $\xi\in V$ act trivially. The *standard H_k-module of type τ* is then given by

$$M(\tau) := \mathrm{Ind}_{\mathbb{C}[V^*]*W}^{H_k}\,\tau\ =\ H_k \underset{\mathbb{C}[V^*]*W}{\otimes}\tau\ . \tag{6.1}$$

It is easy to see from the relations of H_k that $M(\tau)\in\mathcal{O}_k$. Moreover, the PBW theorem (3.3) implies that $M(\tau)\cong\mathbb{C}[V]\otimes\tau$ as a $\mathbb{C}[V]$-module.

The basic properties of standard modules are summarized in the following

Proposition 6.1. *Let* $\mathrm{Irr}(W)$ *be the set of irreducible representations of* W*.*

(1) $\{M(\tau)\}_{\tau\in\mathrm{Irr}(W)}$ *are pairwise non-isomorphic indecomposable objects of* $\mathcal{O}_k$*.*

(2) *Each* $M(\tau)$ *has a unique simple quotient* $L(\tau)$*, and* $\{L(\tau)\}_{\tau\in\mathrm{Irr}(W)}$ *is a complete set of simple objects of* $\mathcal{O}_k$*.*

(3) *Every module* $M \in \mathcal{O}_k$ *admits a finite filtration*

$$\{0\} = F_0 \subset F_1 \subset \ldots \subset F_N = M \ , \tag{6.2}$$

with $F_i \in \mathcal{O}_k$ *and* $F_i/F_{i-1} \cong L(\tau_i)$ *for some* $\tau_i \in \mathrm{Irr}(W)$.

Proof. The first claim follows from [DO], Proposition 2.27 and Corollary 2.28. The second and the third are [GGOR], Proposition 2.11 and Corollary 2.16, respectively. □

6.2. **The Knizhnik-Zamolodchikov (KZ) functor.** Introduced by Opdam and Rouquier, this functor is one of the main tools for studying the category $\mathcal{O}$. We briefly review its construction referring the reader to [GGOR] for details and proofs.

First, using Proposition 3.2, we introduce the localization functor

$$\mathrm{Mod}(H_k) \to \mathrm{Mod}(\mathcal{D}W) \ , \quad M \mapsto M_{\mathrm{reg}} := \mathcal{D}W \underset{H_k}{\otimes} M \ . \tag{6.3}$$

By definition, $\mathrm{Mod}(\mathcal{D}W)$ is the category of W-equivariant $\mathcal{D}$-modules on V_{reg}. Since W acts freely on V_{reg}, this category is equivalent to the category $\mathrm{Mod}\,\mathcal{D}(V_{\mathrm{reg}}/W)$ of $\mathcal{D}$-modules on the quotient variety V_{reg}/W. The full subcategory of $\mathrm{Mod}\,\mathcal{D}(V_{\mathrm{reg}}/W)$ consisting of $\mathcal{O}$-coherent $\mathcal{D}$-modules is equivalent to the category of vector bundles on V_{reg}/W equipped with a regular flat connection, which is, in turn, equivalent to the category of finite-dimensional representation of the Artin braid group $B_W := \pi_1(V_{\mathrm{reg}}/W, *)$ (the Riemann-Hilbert correspondence).

Now, in view of Proposition 6.1, localizing an object in the category $\mathcal{O}_k \subset \mathrm{Mod}(H_k)$ yields a $\mathcal{D}W$-module, which is finite over $\mathbb{C}[V_{\mathrm{reg}}]$. Hence, combined with above equivalences, the restriction of (6.3) to $\mathcal{O}_k$ gives an exact additive functor

$$\mathtt{KZ}_k \, : \, \mathcal{O}_k \, \to \, \mathrm{mod}(\mathbb{C}B_W) \ . \tag{6.4}$$

We illustrate this construction by applying (6.4) to a standard module $M = M(\tau)$ (cf. [BEG], Prop. 2.9). Since $M(\tau) \cong \mathbb{C}[V] \otimes \tau$ as a $\mathbb{C}[V]$-module, M_{reg} can be identified with $\mathbb{C}[V_{\mathrm{reg}}] \otimes \tau$ as a $\mathbb{C}[V_{\mathrm{reg}}]$-module and thus can be thought of as (the space of sections of) a trivial vector bundle on V_{reg} of rank $\dim \tau$. With this identification, the $\mathcal{D}$-module structure on M_{reg} is described by

$$\partial_\xi(f \otimes v) = \partial_\xi(f) \otimes v + f \otimes \partial_\xi(v) \ , \quad \forall\, \xi \in V \ , \tag{6.5}$$

where $f \in \mathbb{C}[V_{\mathrm{reg}}]$ and $v \in \tau$. Since $\xi\, v = 0$ in M and ξ corresponds under localization to the Dunkl operator T_ξ, we have $T_\xi(v) = 0$, or equivalently

$$\partial_\xi v - \sum_{H \in \mathcal{A}} \frac{\alpha_H(\xi)}{\alpha_H} \sum_{i=0}^{n_H-1} n_H k_{H,i} \boldsymbol{e}_{H,i}(v) = 0 \ , \quad \forall\, \xi \in V \ . \tag{6.6}$$

The relations (6.5) can thus be rewritten as

$$\partial_\xi(f \otimes v) = \partial_\xi(f) \otimes v + \sum_{H \in \mathcal{A}} \frac{\alpha_H(\xi)}{\alpha_H} \sum_{i=0}^{n_H-1} n_H k_{H,i} f \otimes \boldsymbol{e}_{H,i} v \ , \tag{6.7}$$

which gives an explicit formula for a regular flat connection on $M_{\mathrm{reg}} = \mathbb{C}[V_{\mathrm{reg}}] \otimes \tau$. This connection is called a *KZ connection* with values in τ : its horizontal sections $y : V_{\mathrm{reg}} \to \tau$ satisfy the following KZ equations

$$\partial_\xi y + \sum_{H \in \mathcal{A}} \frac{\alpha_H(\xi)}{\alpha_H} \sum_{i=0}^{n_H-1} n_H k_{H,i} e_{H,i}(y) = 0 \ , \quad \forall\, \xi \in V \ . \tag{6.8}$$

Remark 6.2. Notice a formal similarity between the systems (6.6) and (6.8). Apart from inessential change of sign, there is, however, an important difference: in (6.8), the group elements $w \in W$ act on the *values* of the functions involved, while in (6.6) on their arguments.

It is easy to check that if y is a local solution of (6.8) near a point $x_0 \in V_{\rm reg}$, then ${}^w y := wyw^{-1}$ is a local solution near wx_0. Thus, the system (6.8) is W-equivariant and descends to a regular holonomic system on $V_{\rm reg}/W$. The space of local solutions of this holonomic system has dimension $\dim\tau$, and its monodromy gives a linear representation of the braid group B_W in this space. The corresponding $\dim(\tau)$-dimensional $\mathbb{C}B_W$-module is the value of the functor (6.4) on $M(\tau)$. We remark that for complex reflection groups, the system (6.8) and its monodromy have been studied in detail in [K], [BMR] and [O2].

6.3. **The Hecke algebra.** It is crucial for applications that the KZ functor (6.4) factors through representations of the Hecke algebra of W. To define this algebra, we recall that, for every $H \in \mathcal{A}$, there is a unique reflection $s_H \in W_H$ with $\det s_H = \exp 2\pi i/n_H$. It is known that the braid group B_W is generated by the elements σ_H which correspond to s_H as generators of monodromy around $H \in \mathcal{A}$ (see [BMR]). Given now complex parameters $k = \{k_{H,i}\}$, with $k_{H,0} = 0$, the Hecke algebra $\mathcal{H}_k(W)$ is defined as the quotient of $\mathbb{C}B_W$ by the following relations

$$\prod_{j=0}^{n_H-1}\left(\sigma_H - (\det s_H)^{-j}e^{2\pi i k_{H,j}}\right) = 0\ , \quad \forall\, H \in \mathcal{A}\ .$$

Notice that, for $k_{H,j} \in \mathbb{Z}$, these relations become $(\sigma_H)^{n_H} = 1$, so in that case $\mathcal{H}_k(W)$ is canonically isomorphic to the group algebra of W. In general, $\mathcal{H}_k$ should be viewed as a deformation of $\mathbb{C}W$.

Restricting scalars via the natural projection $\mathbb{C}B_W \twoheadrightarrow \mathcal{H}_k(W)$, we can regard $\mathrm{mod}(\mathcal{H}_k)$ as a full subcategory of $\mathrm{mod}(\mathbb{C}B_W)$. It turns out that

Theorem 6.3 ([GGOR], Theorem 5.13)**.** *For each k, the KZ functor* (6.4) *has its image in* $\mathrm{mod}(\mathcal{H}_k)$*, i. e.* $\mathtt{KZ}_k : \mathcal{O}_k \to \mathrm{mod}(\mathcal{H}_k)$.

The next two results require the assumption that $\dim\mathcal{H}_k = |W|$. It will be crucial for us that this assumption holds automatically for *all* W whenever $k_{H,j} \in \mathbb{Z}$, since $\mathcal{H}_k \cong \mathbb{C}W$ in this case[4].

Let $\mathcal{O}_{\rm tor}$ denote the full subcategory of $\mathcal{O}_k$ consisting of modules M such that $M_{\rm reg} = 0$. Clearly, $\mathcal{O}_{\rm tor}$ is a Serre subcategory of $\mathcal{O}_k$, so that the quotient $\mathcal{O}_k/\mathcal{O}_{\rm tor}$ is defined as an abelian category.

Proposition 6.4 ([GGOR], Theorem 5.14)**.** *Assume that* $\dim\mathcal{H}_k = |W|$. *Then the KZ functor induces an equivalence*

$$\mathtt{KZ}_k\ :\ \mathcal{O}_k/\mathcal{O}_{\rm tor} \xrightarrow{\sim} \mathrm{mod}(\mathcal{H}_k)\ .$$

In addition, one can prove

Theorem 6.5 ([GGOR], Theorems 5.15, 5.16)**.** *Assume that* $\dim\mathcal{H}_k = |W|$. *Then there exist projective objects* $P \in \mathcal{O}_k$ *and* $Q \in \mathrm{mod}(\mathcal{H}_k)$ *such that*

$$\mathcal{H}_k \cong (\mathrm{End}_{\mathcal{O}_k}P)^{\rm opp} \quad \textit{and} \quad \mathcal{O}_k \simeq \mathrm{mod}\ (\mathrm{End}_{\mathcal{H}_k}Q)^{\rm opp}\ .$$

[4]In general, the equlaity $\dim\mathcal{H}_k = |W|$ is known to be true for almost all complex reflection groups, except for a few exceptional ones, in which case it still remains a conjecture (see [BMR]).

6.4. **Regularity.** The structure of the category $\mathcal{O}_k$ depends on the values of the parameters k. For generic k's, $\mathcal{O}_k$ is a semisimple category, while for special values of k it has a more complicated structure (in particular, it has homological dimension > 0). Likewise, the Hecke algebra $\mathcal{H}_k$ is semisimple for generic k's, but becomes more complicated for certain special values. Using the KZ functor, we can show that the special values in both cases actually coincide. Precisely, we have the following

Theorem 6.6. *Assume that* $\dim \mathcal{H}_k = |W|$. *Then the following are equivalent.*
1. $\mathcal{H}_k$ *is a semisimple algebra.*
2. $\mathcal{O}_k$ *is a semisimple category.*
3. H_k *is a simple ring.*

Proof. We give a detailed proof of this result following R.Vale's dissertation [V] (cf. *loc. cit.*, Theorem 2.1).

1 $\Rightarrow$ *2*. If $\mathcal{H}_k$ is semisimple and $Q \in \mathrm{mod}(\mathcal{H}_k)$, then $\mathrm{End}_{\mathcal{H}_k} Q$ is a semisimple algebra. Thus, by Theorem 6.5, $\mathcal{O}_k$ is a semisimple category.

2 $\Rightarrow$ *1*. By Proposition 6.1, the standard modules are indecomposable. Hence, if $\mathcal{O}_k$ is semisimple, all $M(\tau)$ are simple and we have $L(\tau) = M(\tau)$. Each $M(\tau)$ is torsion free over $\mathbb{C}[V]$, so $L(\tau)_{\mathrm{reg}} = M(\tau)_{\mathrm{reg}} \neq 0$. Since any $M \in \mathcal{O}_k$ can be filtered as in Proposition 6.1, we conclude that $M_{\mathrm{reg}} \neq 0$ and $\mathcal{O}_{\mathrm{tor}} = 0$. As a result, by Proposition 6.4, the category $\mathrm{mod}(\mathcal{H}_k)$ is equivalent to a semisimple category $\mathcal{O}_k$. Thus, $\mathcal{H}_k$ is semisimple.

2 $\Rightarrow$ *3*. If $\mathcal{O}_k$ is semisimple, then as above $M(\tau) = L(\tau)$ for all $\tau \in \mathrm{Irr}(W)$. Suppose now that $0 \neq I \subset H_k$ is a proper two-sided ideal. H_k and I are torsion free over $\mathbb{C}[V]$. Therefore $0 \neq I_{\mathrm{reg}} \subset H_{\mathrm{reg}}$ is a two-sided ideal of $H_{\mathrm{reg}} = \mathcal{D}W$, which is a simple algebra. Hence, $I_{\mathrm{reg}} = H_{\mathrm{reg}}$.

Now, we can always find a primitive ideal $J \subset H_k$, containing I. By [G], Theorem 2.3, every primitive ideal is the annihilator of some simple module in $\mathcal{O}$. Therefore, $I \subset \mathrm{Ann}_{H_k} L(\tau)$ for some τ. But $I_{\mathrm{reg}} = H_{\mathrm{reg}}$ implies that $I \cap \mathbb{C}[V] \neq 0$, while $\mathrm{Ann}_{H_k} L(\tau) \cap \mathbb{C}[V] = 0$ because $L(\tau) = M(\tau)$ is torsion free over $\mathbb{C}[V]$. Contradiction.

3 $\Rightarrow$ *2*. Assuming H_k is simple, we get that $\mathrm{Ann}_{H_k} L(\tau) = 0$ for all $\tau \in \mathrm{Irr}(W)$. Then $L(\tau)_{\mathrm{reg}}$ must be nonzero. Indeed, otherwise $L(\tau)$ would be annihilated by some power of δ, which contradicts $\mathrm{Ann}_{H_k} L(\tau) = 0$.

Thus, $L(\tau)_{\mathrm{reg}} \neq 0$ for all τ. In that case, each $L(\tau)$ is a submodule of some standard module, by [GGOR], Proposition 5.21. By [DO], 2.5, we have $[M(\tau) : L(\tau)] = 1$ and it follows that $L(\tau) \subset M(\tau)$ only if both are the same. Hence, if $L(\tau) \neq M(\tau)$ then it must be a submodule of some $M(\sigma)$ with $\sigma \neq \tau$.

By *loc.cit.*, we can order the elements $\tau_1 < \cdots < \tau_d$ of $\mathrm{Irr}(W)$ in such a way that the matrix with the entries $[M(\tau_i) : L(\tau_j)]$ is upper-triangular. From the previous paragraph it follows that if $L(\tau_i) \neq M(\tau_i)$ then the i-th column of this (upper-triangular) matrix has at least one nonzero off-diagonal entry. This gives us immediately that $M(\tau_1) = L(\tau_1)$ is simple. Therefore, $[M(\tau_1) : L(\tau_2)] = 0$, which implies that $L(\tau_2) = M(\tau_2)$ is simple, and so on. As a result, we conclude that $L(\tau_i) = M(\tau_i)$ for all i, i.e. all standard modules are simple. Now, the BGG reciprocity (see [GGOR], Section 2.6.2 and Proposition 3.3) implies that each $L(\tau) = M(\tau)$ is projective and $\mathcal{O}$ is semisimple (cf. the concluding remark of [BEG], Section 2). $\square$

Remark 6.7. The implication "*2 ⇒ 3*" holds without the assumption $\dim \mathcal{H}_k = |W|$, since the KZ functor is not used in the proof. Note also that this implication is equivalent to

$$L(\tau) = M(\tau)\ , \quad \forall \tau \in \mathrm{Irr}(W) \quad \Rightarrow \quad H_k \text{ is a simple ring },$$

which was one of the key observations of [BEG] (see Section 3 of *loc.cit*).

We now call a multiplicity vector $k = \{k_{C,i}\} \in \mathbb{C}^{\sum_{C\in\mathcal{A}/W} n_C}$ *regular* if the category $\mathcal{O}_k(W)$ is semisimple. Write $\mathrm{Reg}(W)$ for the subset of all regular vectors in $\mathbb{C}^{\sum_{C\in\mathcal{A}/W} n_C}$. In view of Theorem 6.6, for those groups W, where it is known that $\dim \mathcal{H}_k = |W|$, $\mathrm{Reg}(W)$ coincides with the set of all k's, for which the Hecke algebra $\mathcal{H}_k(W)$ is semisimple and the Cherednik algebra $H_k(W)$ is simple. In general, we will need the following fact.

Lemma 6.8. *For any group W, $\mathrm{Reg}(W)$ is a connected subset in $\mathbb{C}^{\sum_{C\in\mathcal{A}/W} n_C}$.*

Proof. Put

$$z(k) = \sum_{H\in\mathcal{A}} \sum_{i=0}^{n_H-1} n_H k_{H,i} e_{H,i} \in \mathbb{C}W\,. \tag{6.9}$$

The element $z(k)$ is central in $\mathbb{C}W$, hence it acts on each $\tau \in \mathrm{Irr}(W)$ as a scalar, which we denote by $c_\tau(k)$. Obviously, $c_\tau(k)$ is a linear function of k. Moreover, according to [DO], Lemma 2.5, $c_\tau(k)$ is a linear function with *nonnegative integer* coefficients. By *loc.cit.*, Proposition 2.31, $M(\tau)$ is simple if $c_\sigma(k) - c_\tau(k) \notin \mathbb{N}$ for all $\sigma \in \mathrm{Irr}(W)$. Hence, if k is generic, namely,

$$c_\sigma(k) - c_\tau(k) \notin \mathbb{N}\ , \quad \forall \sigma, \tau \in \mathrm{Irr}(W)\,, \tag{6.10}$$

then all standard modules are simple and, as in the proof of Theorem 6.6, the category $\mathcal{O}_k$ is semisimple.

It follows that the complement to $\mathrm{Reg}(W)$ is contained in a locally finite union of hyperplanes, thus $\mathrm{Reg}(W)$ itself is connected. □

7. Shift Functors and KZ Twists

7.1. **Shift functors.** Recall that $\mathcal{O}_k$ is the full subcategory of $\mathrm{Mod}(H_k)$ consisting of finitely generated modules on which the elements $\xi \in V$ act locally nilpotently. It is convenient to enlarge $\mathcal{O}_k$ by dropping the finiteness assumption: following [GGOR], we denote the corresponding category by $\mathcal{O}_k^{\mathrm{ln}}$. The inclusion functor $\mathcal{O}_k^{\mathrm{ln}} \hookrightarrow \mathrm{Mod}(H_k)$ has then a right adjoint $\mathfrak{r}_k : \mathrm{Mod}(H_k) \to \mathcal{O}_k^{\mathrm{ln}}$, which assigns to $M \in \mathrm{Mod}(H_k)$ its submodule

$$\mathfrak{r}_k(M) := \{m \in M \ :\ \xi^d m = 0\ ,\ \forall\, \xi \in V\ ,\ d \gg 0\}\ .$$

Thus, $\mathfrak{r}_k(M)$ is the largest submodule (i.e., the sum of all submodules) of M belonging to $\mathcal{O}_k^{\mathrm{ln}}$. When restricted to finitely generated modules, $\mathfrak{r}_k$ defines a functor $\mathrm{mod}(H_k) \to \mathcal{O}_k$; however, $\mathfrak{r}_k(M) \notin \mathcal{O}_k$ for an arbitrary $M \in \mathrm{Mod}(H_k)$.

We will combine $\mathfrak{r}_k$ with localization to define functors between module categories of H_k, with different values of k. To this end, for each k, we identify $H_k[\delta^{-1}] = \mathcal{D}W$ using the Dunkl representation (see Proposition 3.2) and write $\theta_k : H_k \to \mathcal{D}W$ for the corresponding localization map. Associated to θ_k is a pair of natural functors: the localization $(\theta_k)^* : \mathrm{Mod}(H_k) \to \mathrm{Mod}(\mathcal{D}W), M \mapsto \mathcal{D}W \otimes_{H_k} M$, and its right adjoint – the restriction of scalars $(\theta_k)_*: \mathrm{Mod}(\mathcal{D}W) \to \mathrm{Mod}(H_k)$ via θ_k. Given a pair of multiplicities, k and k' say, we now define

$$\mathcal{T}_{k\to k'} := \mathfrak{r}_{k'}\,(\theta_{k'})_*\,(\theta_k)^* : \ \mathrm{Mod}(H_k) \to \mathrm{Mod}(H_{k'})\ .$$

Proposition 7.1. *The functor $\mathcal{T}_{k\to k'}$ restricts to a functor: $\mathcal{O}_k \to \mathcal{O}_{k'}$.*

Proof. Given $M \in \mathcal{O}_k$, let $N := (\theta_{k'})_*(\theta_k)^* M \in \mathrm{Mod}(H_{k'})$. To prove the claim we need only to show that $\mathfrak{r}_{k'}(N)$ is a finitely generated module over $H_{k'}$. Assuming the contrary, we may construct an infinite *strictly* increasing chain of submodules $N_0 \subset N_1 \subset N_2 \subset \ldots \subset \mathfrak{r}_{k'}(N) \subset M_{\mathrm{reg}}$, with $N_i \in \mathcal{O}_{k'}$. Localizing this chain, we get an infinite chain of H_{reg}-submodules of M_{reg}. Since M_{reg} is finite over $\mathbb{C}[V_{\mathrm{reg}}]$ and $\mathbb{C}[V_{\mathrm{reg}}]$ is Noetherian, this localized chain stabilizes at some i. Thus, omitting finitely many terms, we may assume that $(N_i)_{\mathrm{reg}} = (N_0)_{\mathrm{reg}}$ for all i. In that case all the inclusions $N_i \subset N_{i+1}$ are essential extensions, and since each $N_i \in \mathcal{O}_{k'}$, the above chain of submodules can be embedded into an injective hull of N_0 in $\mathcal{O}_{k'}$ and hence stabilizes for $i \gg 0$. (The injective hulls in $\mathcal{O}_{k'}$ exist and have finite length, since $\mathcal{O}_{k'}$ is a highest weight category, see [GGOR], Theorem 2.19.) This contradicts the assumption that the inclusions are strict. Thus, we conclude that $\mathfrak{r}_{k'}(N)$ is finitely generated. □

Definition 7.2. We call $\mathcal{T}_{k\to k'}: \mathcal{O}_k \to \mathcal{O}_{k'}$ the *shift functor* from $\mathcal{O}_k$ to $\mathcal{O}_{k'}$.

The following lemma establishes basic properties of the functors $\mathcal{T}_{k\to k'}$.

Lemma 7.3. *Let k, k', k'' be arbitrary complex multiplicities, and let $M \in \mathcal{O}_k$.*
(i) *If $k \in \mathrm{Reg}$, then $\mathcal{T}_{k\to k}(M) \cong M$.*
(ii) *If $k, k' \in \mathrm{Reg}$ and M is simple, then $\mathcal{T}_{k\to k'}(M)$ is either simple or zero.*
(iii) *If $k \in \mathrm{Reg}$ and M is simple with $\mathcal{T}_{k\to k'}(M) \neq 0$, then*

$$[\mathcal{T}_{k'\to k''} \circ \mathcal{T}_{k\to k'}](M) \cong \mathcal{T}_{k\to k''}(M)\ .$$

Proof. To simplify the notation, we will write M_{reg} for both $(\theta_k)^* M \in \mathrm{Mod}(\mathcal{D}W)$ and $(\theta_{k'})_*(\theta_k)^* M \in \mathrm{Mod}(H_{k'})$ whenever this does not lead to confusion.

(i) For regular k, $\mathcal{O}_{\mathrm{tor}} = 0$, hence $M_{\mathrm{reg}} \neq 0$ whenever $M \neq 0$, and M is naturally an H_k-submodule of M_{reg}. We need to show that M is the maximal submodule of M_{reg} belonging to $\mathcal{O}_k$. If $M \subseteq N \subset M_{\mathrm{reg}}$, with $N \in \mathcal{O}_k$, then $N_{\mathrm{reg}} = M_{\mathrm{reg}}$. Since $\mathcal{O}_{\mathrm{tor}} = 0$, this forces $N = M$, proving (i).

(ii) For regular k, the simple objects in $\mathcal{O}_k$ are the standard modules $M(\tau)$. If $M = M(\tau)$ is such a module, then M_{reg} is a simple $\mathcal{D}W$-module. Hence, if $0 \neq N \subseteq (\theta_{k'})_*\,(\theta_k)^*(M)$, then $N_{\mathrm{reg}} = M_{\mathrm{reg}}$. As a result, if $0 \neq N \subsetneq N' \subset (\theta_{k'})_*\,(\theta_k)^*(M)$ are two submodules N, $N' \in \mathcal{O}_{k'}$, then $N_{\mathrm{reg}} = N'_{\mathrm{reg}}$ and $(N'/N)_{\mathrm{reg}} = 0$. But this contradicts the fact that $(\mathcal{O}_{k'})_{\mathrm{tor}} = 0$. Thus $(\theta_{k'})_*\,(\theta_k)^*(M)$ may have at most one nontrivial submodule $N \in \mathcal{O}_{k'}$ which, therefore, must be simple.

(iii) If $M \in \mathcal{O}_k$ is simple, then M_{reg} is simple. Hence, if $N = \mathcal{T}_{k\to k'}(M) \neq 0$, then $N_{\mathrm{reg}} = M_{\mathrm{reg}}$, and therefore $\mathfrak{r}_{k''}(M_{\mathrm{reg}}) = \mathfrak{r}_{k''}(N_{\mathrm{reg}})$. □

Remark 7.4. Part (ii) of Lemma 7.3 can be restated as follows: if k, $k' \in \mathrm{Reg}$, then $\mathcal{T}_{k\to k'}: \mathcal{O}_k \to \mathcal{O}_{k'}$ transforms standard modules either to standard modules or zero.

Corollary 7.5. *Assume that k, $k' \in \mathrm{Reg}$. Then the following are equivalent:*
(1) $\mathcal{T}_{k\to k'}[M_k(\tau)] \cong M_{k'}(\tau')$,
(2) $M_k(\tau)_{\mathrm{reg}} \cong M_{k'}(\tau')_{\mathrm{reg}}$ *as H_{reg}-modules.*

Proof. (1) $\Rightarrow$ (2). Let $M = M_k(\tau)$. Since $k \in \mathrm{Reg}$, M is a simple H_k-module and M_{reg} is a simple H_{reg}-module. Then, if (1) holds, $M_{k'}(\tau')_{\mathrm{reg}}$ is a submodule of a simple module M_{reg}, and hence $M_{k'}(\tau')_{\mathrm{reg}} = M_{\mathrm{reg}}$, as needed.

(2) $\Rightarrow$ (1). If (2) holds, M_{reg} contains a copy of $M_{k'}(\tau')$. Lemma 7.3(ii) then implies that $M_{k'}(\tau') \cong \mathcal{T}_{k\to k'}(M)$. □

7.2. **KZ twists.** Throughout this section we assume that $k_{H,i} \in \mathbb{Z}$. In that case the Hecke algebra $\mathcal{H}_k$ is isomorphic to the group algebra $\mathbb{C}W$, so that $\dim \mathcal{H}_k = |W|$. We can use the results of the previous section, which we summarize in the following

Proposition 7.6. *If k is integral, then the algebra H_k is simple, the category $\mathcal{O}_k$ is semisimple, all standard modules $M_k(\tau) \in \mathcal{O}_k$ are irreducible, and the functor* $\mathtt{KZ}$ *is an equivalence: $\mathcal{O}_k \xrightarrow{\sim} \mathrm{mod}(\mathbb{C}W)$.*

Proof. The first two claims follow from Theorem 6.6. The irreducibility of $M(\tau)$ then follows from the fact that these modules are indecomposable. Finally, $L_k(\tau) = M_k(\tau)$ implies that $\mathcal{O}_{\mathrm{tor}} = 0$, so the last claim is a consequence of Proposition 6.4. □

Now, applying the KZ functor to $M_k(\tau)$, we see that, for integral k's, any local solution to the KZ system (6.8) is a global single-valued function $y : V_{\mathrm{reg}} \to \tau$. Thus we have the following result, due to Opdam.

Proposition 7.7 (see [O1, O2])**.** *If k is integral, every local solution of the system* (6.8) *extends to a rational function on V, with possible poles along $H \in \mathcal{A}$. The monodromy of this system on V_{reg}/W is given by the W-action ${}^w y := wyw^{-1}$ on the space of global solutions.*

Remark 7.8. If $\{e_i\}$ is a basis of τ, then any global solution of (6.8) can be written in the form $y_i = \sum f_{ij} \otimes e_j$, with $f_{ij} \in \mathbb{C}[V_{\mathrm{reg}}]$. Since $\{y_i\}$ are linearly independent at each point $x \in V_{\mathrm{reg}}$, the matrix $F := (f_{ij})$ is invertible, with $F^{-1} \in \mathbb{C}[V_{\mathrm{reg}}] \otimes \mathrm{End}_{\mathbb{C}}\tau$.

Next, the last statement of Proposition 7.6 implies that the functor $\mathtt{KZ}$ induces a bijection between the simple objects of $\mathcal{O}_k$ and $\mathrm{mod}(\mathbb{C}W)$, i.e. between the sets $\{M_k(\tau)\}_{\tau \in \mathrm{Irr}(W)}$ and $\mathrm{Irr}(W)$. For any integral k, this defines a permutation

$$\mathrm{kz}_k : \mathrm{Irr}(W) \to \mathrm{Irr}(W) , \quad \mathrm{kz}_k(\tau) := \mathtt{KZ}\,[M_k(\tau)] ,$$

which we call a *KZ twist*. It is obvious from the definition that $\mathrm{kz}_0(\tau) = \tau$ for all τ. It is also clear that kz_k preserves dimension.

As mentioned in the Introduction, our aim is to establish the following *additivity property* of KZ twists:

$$\mathrm{kz}_k \circ \mathrm{kz}_{k'} = \mathrm{kz}_{k+k'} , \quad \forall\, k,\, k' \in \mathbb{Z}^{\sum_{C \in \mathcal{A}/W} n_C} , \tag{7.1}$$

which was first proved (under the assumption that $\dim \mathcal{H}_k = |W|$) in [O1, O2]. We begin by relating kz_k to localization in the category $\mathcal{O}$.

Proposition 7.9. *If k is integral, there is an isomorphism of H_{reg}-modules*

$$M_k(\tau)_{\mathrm{reg}} \cong M_0(\sigma)_{\mathrm{reg}} ,$$

where $\sigma = \mathrm{kz}_k(\tau)$ and $M_0(\sigma)$ is the standard module over $H_0 = \mathcal{D}(V) * W$ corresponding to σ.

Proof. Choose a basis $\{e_i\}$ of τ, and let $M = M_k(\tau)$. By 6.2 and 7.2, we have a flat connection ∂ on $M_{\rm reg} \cong \mathbb{C}[V_{\rm reg}] \otimes \tau$, and a space σ of the horizontal sections of this connection, with a basis $y_i = \sum f_{ij} \otimes e_j$. The action of W on σ is given by

$$wy_i = \sum f_{ij} \circ w^{-1} \otimes we_j \;=\; {}^w y_i\,,$$

that is, it coincides with the monodromy of the connection, cf. Proposition 7.7. Thus there is a subspace $\sigma \subset M_{\rm reg}$ which is isomorphic to ${\rm kz}_k(\tau)$ as a W-module and such that $\partial_\xi \sigma = 0$ for all $\xi \in V$. Also, by Remark 7.8, we have

$$\mathbb{C}[V_{\rm reg}] \cdot \sigma = \mathbb{C}[V_{\rm reg}] \cdot \tau = M_{\rm reg}\,.$$

It follows that $M_{\rm reg} \cong M_0(\sigma)_{\rm reg}$, with $\sigma = {\rm kz}_k(\tau)$, as required. □

Taking $\tau' = {\rm kz}_{k'}^{-1} \circ {\rm kz}_k(\tau)$ in Proposition 7.9, we get

Corollary 7.10. *For any integral k, k' there is a permutation $\tau \mapsto \tau'$ on ${\rm Irr}(W)$, such that $M_k(\tau)_{\rm reg} \cong M_{k'}(\tau')_{\rm reg}$ for all $\tau \in {\rm Irr}(W)$.*

Now, we are in position to state the main result of this section.

Theorem 7.11. *Let k and k' be complex multiplicities such that $k'_{H,i} - k_{H,i} \in \mathbb{Z}$ for all H and i. Then*
(1) $\mathcal{T}_{k\to k'}(M) \neq 0$ *for any standard module* $M \in \mathcal{O}_k$.
(2) *If* $k, k' \in {\rm Reg}$, *then* $\mathcal{T}_{k\to k'}[M_k(\tau)] \cong M_{k'}(\tau')$, *or equivalently,*

$$M_k(\tau)_{\rm reg} \cong M_{k'}(\tau')_{\rm reg}\ , \quad \tau = {\rm kz}_{k'-k}(\tau')\ .$$

Before proving Theorem 7.11 (see Section 7.3 below), we deduce some of its implications. First, Theorem 7.11 implies the additivity property (7.1) of KZ twists.

Corollary 7.12 (Conjecture in [O2, O3])**.** *The map $k \mapsto {\rm kz}_k$ is a homomorphism from the additive group of integral multiplicities to the group of permutations on* ${\rm Irr}(W)$.

Indeed, all integral values of k are regular, so by Theorem 7.11 and Lemma 7.3(iii),

$$\mathcal{T}_{0\to k+k'}[M_0(\tau)] \cong \mathcal{T}_{k\to k+k'} \circ \mathcal{T}_{0\to k}[M_0(\tau)]\ .$$

Hence ${\rm kz}_{k+k'}(\tau) = {\rm kz}_{k'} \circ {\rm kz}_k(\tau))$, as required. □

Next, we will prove one of the key results for describing the structure of quasi-invariants in Section 8. For this, recall the module $\mathbf{Q}_k(\tau)$ defined in Section 3.3: by construction, this is a submodule of $\mathbb{C}[V_{\rm reg}] \otimes \tau$ under the differential action of H_k. Using notation of Section 7.1, we now identify $\mathbb{C}[V_{\rm reg}] \otimes \tau$ with $(\theta_k)_*(\theta_0)^*(M)$, where $M = M_0(\tau)$. The Dunkl operators $T_{\xi,k}$ act on $\mathbb{C}[V_{\rm reg}] \otimes \tau$ by lowering the degree. Together with property (3.16), this implies that $\mathbf{Q}_k(\tau) \in \mathcal{O}_k$. Lemma 7.3 shows then

$$\mathbf{Q}_k(\tau) \cong \mathcal{T}_{0\to k}[M_0(\tau)]\ . \tag{7.2}$$

On the other hand, by Proposition 7.9, we have

$$\mathcal{T}_{0\to k}[M_0(\tau)] \cong M_k(\tau')\ , \quad \tau = {\rm kz}_k(\tau')\ . \tag{7.3}$$

Combining (7.2) and (7.3), we arrive at the following conclusion.

Proposition 7.13. *There is an isomorphism of H_k-modules $\mathbf{Q}_k(\tau) \cong M_k(\tau')$, where $\tau = {\rm kz}_k(\tau')$.*

Remark 7.14. Formula (7.2) suggests a conceptual way to define quasi-invariants with values in an arbitrary W-module τ (cf. Section 3.3). Specifically, for any $k = \{k_{H,i}\}$, with $k_{H,i} \in \mathbb{Z}$, the module $\mathbf{Q}_k(\tau)$ can be described by

$$\mathbf{Q}_k(\tau) = \{\varphi \in \mathbb{C}[V_{\mathrm{reg}}] \otimes \tau \ : \ \theta_k(\xi)^d \varphi = 0 \ , \ \forall \xi \in V \ , \ d \gg 0\} \ ,$$

where $\theta_k \ : \ H_k \hookrightarrow \mathcal{D}W$, and $\mathcal{D}W$ operates on $\mathbb{C}[V_{\mathrm{reg}}] \otimes \tau$ via the identification $\mathbb{C}[V_{\mathrm{reg}}] \otimes \tau \cong (\mathcal{D}W/J) \otimes_{\mathbb{C}W} \tau$, by formulas (3.6).

7.3. **Proof of Theorem 7.11.** We first prove the result for integral k, k' and then use a deformation argument in k. We begin with some preparations. Given $M = M_k(\tau) \in \mathcal{O}_k$, we identify $M_{\mathrm{reg}} \cong \mathbb{C}[V_{\mathrm{reg}}] \otimes \tau$ as a $\mathbb{C}[V_{\mathrm{reg}}] * W$-module. The action of ∂_ξ gives then a flat connection on $\mathbb{C}[V_{\mathrm{reg}}] \otimes \tau$, depending on k, which is the KZ connection (6.7). The algebra $H_{k'}$ also acts on M_{reg}, with $\xi \in V$ acting as the Dunkl operator

$$T_{\xi,k'} = \partial_\xi - \sum_{H \in \mathcal{A}} \frac{\alpha_H(\xi)}{\alpha_H} \sum_{i=0}^{n_H-1} n_H k'_{H,i} \boldsymbol{e}_{H,i} \ , \tag{7.4}$$

where ∂_ξ acts by formula (6.7). Clearly, for $k' = k + b$ with b fixed, the action of both $T_{\xi,k}$ and $T_{\xi,k'}$ on $M_{\mathrm{reg}} = \mathbb{C}[V_{\mathrm{reg}}] \otimes \tau$ depends polynomially on k.

Recall that $\mathbb{C}[V_{\mathrm{reg}}]$ is obtained from $\mathbb{C}[V]$ by inverting the homogeneous polynomial δ, so the standard grading on $\mathbb{C}[V]$ extends naturally to a $\mathbb{Z}$-grading on $\mathbb{C}[V_{\mathrm{reg}}]$ and M_{reg}.

Now, we choose dual bases $\{\xi_i\}$ and $\{x_i\}$ in V and V^*, and, following [DO], consider the (deformed) Euler operator

$$E(k) := \sum_i x_i T_{\xi_i,k} \in \mathcal{D}W \ . \tag{7.5}$$

It is easy to see that $E(k) = E(0) - z(k)$, with $E(0) = \sum_i x_i \partial_{\xi_i}$ and $z(k)$ given by (6.9). Using formula (6.7) for the action of ∂_ξ on M_{reg}, we get

$$E(0)(f \otimes v) = E(0)(f) \otimes v + f \otimes z(k)(v) \, .$$

Being a central element in $\mathbb{C}W$, $z(k)$ acts on $\tau \in \mathrm{Irr}(W)$ as a scalar $c_\tau(k)$, so that

$$\mathrm{tr}\, z(k)|_\tau = c_\tau(k) \, \dim \tau \ . \tag{7.6}$$

For any homogeneous $f \otimes v \in M_k(\tau)_{\mathrm{reg}}$, we have then

$$E(k')(f \otimes v) = (m + c_\tau(k) - z(k'))(f \otimes v) \, , \quad m = \deg f \, .$$

This gives the following result (cf. [DO], Lemma 2.26).

Lemma 7.15. *Let $\sigma \in \mathrm{Irr}(W)$ and $m \in \mathbb{Z}$. Let $M_{\sigma,m}$ be a homogeneous subspace of $M_k(\tau)_{\mathrm{reg}}$ of degree m, which is isomorphic to $\sigma \in \mathrm{Irr}(W)$ as a W-module. Then $E(k')$ acts on $M_{\sigma,m}$ as multiplication by $m + c_\tau(k) - c_\sigma(k')$.*

Arguing as in [DO], Proposition 2.27, from Lemma 7.15 we deduce

Lemma 7.16. *Every $H_{k'}$-submodule of $M_{\mathrm{reg}} = M_k(\tau)_{\mathrm{reg}}$ is graded. With respect to this grading, the actions of $T_{\xi,k'}$, W and V^* have degrees -1, 0 and 1, respectively.*

Now, let us summarize what we have so far in the case of integral k, k'. By Corollary 7.10 and Corollary 7.5,

$$M_k(\tau)_{\mathrm{reg}} \cong M_{k'}(\tau')_{\mathrm{reg}} \quad \text{and} \quad \mathcal{T}_{k \to k'}[M_k(\tau)] \cong M_{k'}(\tau') \tag{7.7}$$

for some $\tau' \in \mathrm{Irr}(W)$. Thus, viewed as a $H_{k'}$-module, $M_k(\tau)_{\mathrm{reg}}$ contains a (unique) submodule $N \in \mathcal{O}_{k'}$, which is isomorphic to $M_{k'}(\tau')$. Note that both $M_k(\tau)_{\mathrm{reg}}$ and $M_{k'}(\tau')_{\mathrm{reg}}$ are free over $\mathbb{C}[V_{\mathrm{reg}}]$, so the first isomorphism in (7.7) implies that $\dim\tau = \dim\tau'$. Further, we claim that $N \subseteq M_k(\tau)_{\mathrm{reg}}$ satisfies

$$\delta^r M_k(\tau) \subset N \subset \delta^{-r} M_k(\tau)\,, \tag{7.8}$$

where $r \gg 0$ depends on the difference $k' - k$ but not on k. To see this, we can use Proposition 7.13 to identify $M = M_k(\tau)$ with one of the modules $\mathbf{Q}_k(\sigma)$. Under such an identification, $N = \mathcal{T}_{k',\,k}(M)$ gets identified with $\mathbf{Q}_{k'}(\sigma)$, and then (7.8) follows from (3.17). Now, (7.8) and Lemma 7.16 show that the subspace τ' generating N sits in $\delta^{-r}M_k(\tau)$, and its homogeneity degree $\deg\tau' \le r\deg\delta$. Thus, summing up, we have

Lemma 7.17. *Assume that k and k' are integral, and let $M := M_k(\tau)$, with $\tau \in \mathrm{Irr}(W)$. Then M_{reg} contains a subspace τ', such that $\dim\tau' = \dim\tau$, $T_{\xi,k'}(\tau') = 0$ for all $\xi \in V$, and*

$$\tau' \subset \delta^{-r}M\,, \quad \deg\tau' \le r\deg\delta\ , \tag{7.9}$$

where r depends only on $k' - k$.

Proof of Theorem 7.11. Let k be arbitrary complex-valued and let $k' = k + b$, where b is integral. Throughout the proof we will keep b fixed, while regarding k as a parameter. As above, we identify $M = M_k(\tau)$ with one and the same vector space $\mathbb{C}[V] \otimes \tau$ for all k. The localized modules M_{reg} are then identified with $\mathbb{C}[V_{\mathrm{reg}}] \otimes \tau$, and the information about k is encoded in the connection (6.7).

Let $(M_{\mathrm{reg}})^0$ denote the subspace of all elements in M_{reg} that are annihilated by $T_{\xi,k'}$ for all ξ. Obviously, $(M_{\mathrm{reg}})^0$ is preserved by the action of W. If $\mathcal{W} \subset (M_{\mathrm{reg}})^0$ is a W-invariant subspace isomorphic to some $\sigma \in \mathrm{Irr}(W)$, then we have a nonzero homomorphism from $M_{k'}(\sigma)$ to M_{reg} (by the universality of the standard modules). Therefore, to see that $\mathcal{T}_{k\to k'}M \neq 0$ it suffices to see that $(M_{\mathrm{reg}})^0 \neq 0$.

We put on M_{reg} a positive increasing filtration $\{F_j\}$, with

$$F_j = \{m \in M_{\mathrm{reg}} \,|\, m = \delta^{-j}u\,,\ \text{where } u \in M \text{ and } \deg u \le 2j\deg\delta\}\,.$$

Each F_j is finite-dimensional, and it is easy to see that $T_{\xi,k'}F_j \subseteq F_{j+1}$ for all $\xi \in V$.

Set $(F_j)^0 := F_j \cap (M_{\mathrm{reg}})^0$, so that $(F_j)^0 = \{m \in F_j \,|\, T_{\xi,k'}(m) = 0\,,\ \forall\xi \in V\}$. For each $j \ge 0$, the operators $T_{\xi,k'}$ induce linear maps between the finite-dimensional spaces F_j and F_{j+1}. All these maps depend polynomially on k, and the subspace $(F_j)^0$ is their common kernel. It follows that $(F_j)^0$ has constant dimension, independent of k, over some dense Zariski open subset in the parameter space. Now, for integral k, we have Lemma 7.17, which says that $(F_j)^0 \neq 0$ for some $j = r$, which depends only on $b = k' - k$. Therefore, for this particular j, $(F_j)^0 \neq 0$ for all integral k, and hence for all k. As a result, $(M_{\mathrm{reg}})^0 \neq 0$ for all k, which proves the first claim of the theorem. Moreover, it follows that $\dim(F_j)^0 \ge \dim\tau$ for all k.

Recall the set Reg of regular values of k. For a fixed integral b, put $\mathrm{Reg}_b := \mathrm{Reg} \cap (b + \mathrm{Reg})$; this is the set of all k such that both k and $k' = k + b$ are regular. It follows from Lemma 6.8 and Theorem 7.6 that the set Reg_b is connected and contains all integral points. Since we already know that $\mathcal{T}_{k\to k'}(M) \neq 0$, Lemma 7.3(ii) implies that, for $k \in \mathrm{Reg}_b$, there is a (unique) submodule $N \in \mathcal{O}_{k'}$ inside M_{reg} (considered as a $H_{k'}$-module). Moreover, we know that $N \cong M_{k'}(\tau')$ for some

$\tau' \in \mathrm{Irr}(W)$. It remains to show that τ' satisfies $\tau = \mathrm{kz}_{k'-k}(\tau')$. Note that this is certainly true when $k = 0$, see (7.3).

If we regard the generating space τ' of N as a subspace in $M_{\mathrm{reg}} \cong \mathbb{C}[V_{\mathrm{reg}}] \otimes \tau$, then we know that (1) $\dim \tau' = \dim \tau$, (2) $\tau' = (M_{\mathrm{reg}})^0$ for $k \in \mathrm{Reg}_b$, and (3) $\dim(F_j)^0 \geq \dim \tau$ for all k. Since $(F_j)^0 \subset (M_{\mathrm{reg}})^0$, this immediately implies that $(F_j)^0 = \tau'$ for all $k \in \mathrm{Reg}_b$, in particular, it has the same dimension. Thus, the dimension of $(F_j)^0$ does not jump at any of the regular values $k \in \mathrm{Reg}_b$, therefore the subspace $(F_j)^0 \subset \mathbb{C}[V_{\mathrm{reg}}] \otimes \tau$ varies continuously with k varying inside Reg_b. As a result, $\tau' = (F_j)^0$ does not deform as a W-module, so it is the same as for $k = 0$, in which case we know already that $\tau = \mathrm{kz}_{k'-k}(\tau')$. This finishes the proof. □

The above arguments allow us to prove the following property of KZ twists, which is obtained by a different method in [O3], Corollary 3.8(*vi*).

Corollary 7.18. *If* $\tau \in \mathrm{Irr}(W)$ *and* $\tau' = \mathrm{kz}_b(\tau)$*, then* $c_\tau(k) = c_{\tau'}(k)$*.*

Proof. The proof of Theorem 7.11 shows that, for all regular k and $k' = k+b$ (b here is fixed and integral), there is a homogeneous subspace $\tau' \subset M_k(\tau)_{\mathrm{reg}} = \mathbb{C}[V_{\mathrm{reg}}] \otimes \tau$ annihilated by all $T_{\xi,k'}$ and therefore by the Euler operator $E(k')$. This subspace varies continuously with k, hence its homogeneity degree remains constant. By Lemma 7.15, this degree is given by $m = c_{\tau'}(k') - c_\tau(k) = c_{\tau'}(b) + c_{\tau'}(k) - c_\tau(k)$. Therefore $c_{\tau'}(k) - c_\tau(k)$ is constant in k, hence zero. □

7.4. **Heckman-Opdam shift functors.** We briefly explain the relation between our functors $\mathcal{T}$ and the Heckman-Opdam shift functors introduced in [BEG] and studied in [GS].

Assume that k' is related to k by (5.17), for some $C \in \mathcal{A}/W$ and $a = 1, \ldots, n_C - 1$. Then, by Proposition 5.6, we have

$$\boldsymbol{e} H_k \boldsymbol{e} = \boldsymbol{e}\, \delta_C^{-a}\, H_{k'}\, \delta_C^{a}\, \boldsymbol{e}\,.$$

It follows that $\boldsymbol{e}\, H_{k'}\, \delta_C^a\, \boldsymbol{e}$ is a $\boldsymbol{e}H_{k'}\boldsymbol{e}$-$\boldsymbol{e}H_k\boldsymbol{e}$-bimodule. Thus, one can define a functor $\mathcal{S}_{k\to k'} : \mathrm{Mod}(H_k) \to \mathrm{Mod}(H_{k'})$ by

$$M \mapsto H_{k'}\boldsymbol{e} \otimes_{\boldsymbol{e}H_{k'}\boldsymbol{e}} \boldsymbol{e}H_{k'}\delta_C^a\boldsymbol{e} \otimes_{\boldsymbol{e}H_k\boldsymbol{e}} \boldsymbol{e}M\,.$$

It is easy to check that $\mathcal{S}_{k'\to k}$ restricts to a functor from $\mathcal{O}_k$ to $\mathcal{O}_{k'}$. Similarly, one defines $\mathcal{S}_{k'\to k} : \mathcal{O}_{k'} \to \mathcal{O}_k$ by

$$M \mapsto H_k\boldsymbol{e} \otimes_{\boldsymbol{e}H_k\boldsymbol{e}} \boldsymbol{e}\delta_C^{-a}H_{k'}\boldsymbol{e} \otimes_{\boldsymbol{e}H_{k'}\boldsymbol{e}} \boldsymbol{e}M\,.$$

Now, checking on standard modules, it is easy to prove

Proposition 7.19. *If* $k,\, k' \in \mathrm{Reg}$*, then* $\mathcal{T}_{k\to k'} \cong \mathcal{S}_{k\to k'}$ *.*

In general, however, the functors $\mathcal{T}$ and $\mathcal{S}$ are not isomorphic: for example, since $\mathcal{T}$ factors through localization, it always kills torsion (in particular, finite-dimensional modules), while $\mathcal{S}$ does not. On the other hand, if $k' = k$, then $\mathcal{S}$ is, by definition, isomorphic to the identity functor, while $\mathcal{T}$ is not (for projective objects $P \in \mathcal{O}_k$, we still have $\mathcal{T}(P) \cong P$, by [GGOR], Theorem 5.3).

We can also define shift functors using shift operators constructed in Section 5.5. Briefly, if $k' - k$ is integral, then there is a W-invariant differential operator S satisfying

$$T_{p,k'}\, \boldsymbol{e}S = \boldsymbol{e}S\, T_{p,k}\ , \qquad \forall\, p \in \mathbb{C}[V^*]^W\,. \tag{7.10}$$

(Such S is a composition of elementary shift operators of Theorem 5.7.) Regard $H_{\rm reg}$ as a $H_{k'}$-H_k-bimodule via the Dunkl representation, and consider its sub-bimodule P generated by $\boldsymbol{e}S$, i.e. $P := H_{k'}\,(\boldsymbol{e}S)\,H_k \subset H_{\rm reg}$. Now, given $M \in \mathcal{O}_k$, define $M' := P \otimes_{H_k} M$. Clearly, if $M_{\rm reg} \neq 0$, then M' is a nonzero $H_{k'}$-module embedded in $H_{\rm reg} \otimes_{H_k} M = M_{\rm reg}$. To prove that $M' \in \mathcal{O}_{k'}$, it is suffices to check that $T_{p,k'}$, with $p \in \mathbb{C}[V^*]^W$, act locally nilpotently on M'. But this follows immediately from (7.10) and the well-known fact that the adjoint action of $\mathbb{C}[V^*]^W$ on H_k is locally nilpotent. Thus, $P \otimes_{H_k} —$ defines a functor $\mathcal{O}_k \to \mathcal{O}_{k'}$. Again, when $k, k' \in \mathrm{Reg}$, it is easy to show that this functor is isomorphic to $\mathcal{T}_{k\to k'}$ (and hence, $\mathcal{S}_{k\to k'}$, by Proposition 7.19). We can use this to prove the following useful observation.

Proposition 7.20. *Let k, k' be integral, and let S be a composition of shift operators of Theorem 5.7, such that*

$$L_{p,k'} \circ S = S \circ L_{p,k}\ , \qquad \forall\, p \in \mathbb{C}[V^*]^W\ .$$

Then $S[Q_k] \subseteq Q_{k'}$, where Q_k and $Q_{k'}$ are the corresponding modules of quasi-invariants.

Proof. Using the fact that $\mathcal{T}_{k\to k'}(\mathbf{Q}_k) = \mathbf{Q}_{k'}$ and the above relation between $\mathcal{T}$ and S, we have

$$\boldsymbol{e}(S[Q_k] \otimes 1) = \boldsymbol{e}S\,[Q_k \otimes 1] = (\boldsymbol{e}S)[\mathbf{Q}_k] \subseteq H_{k'}\boldsymbol{e}SH_k \otimes_{H_k} \mathbf{Q}_k \subset \mathbf{Q}_{k'} = \boldsymbol{e}(Q_{k'} \otimes 1)\,.$$

Thus $S[Q_k] \subseteq Q_{k'}$, as required. □

8. The Structure of Quasi-invariants

8.1. **Cohen-Macaulayness.** First, we consider the module of W-valued quasi-invariants $\mathbf{Q}_k$ introduced in Section 3.2. By (3.18), this is a $H_k \otimes \mathbb{C}W$-module, which can be decomposed as

$$\mathbf{Q}_k = \bigoplus_{\tau\in \mathrm{Irr}(W)} \mathbf{Q}_k(\tau) \otimes \tau^*\ ,$$

with $\mathbf{Q}_k(\tau) \subset \mathbb{C}[V] \otimes \tau$ defined by (3.12). By Proposition 7.13, $\mathbf{Q}_k(\tau) \cong M_k(\tau')$, where $\tau' = \mathrm{kz}_{-k}(\tau)$. Hence we have

Proposition 8.1. *The $H_k \otimes \mathbb{C}W$-module $\mathbf{Q}_k$ has the direct sum decomposition*

$$\mathbf{Q}_k \cong \bigoplus_{\tau\in \mathrm{Irr}(W)} M_k(\tau') \otimes \tau^*\ , \tag{8.1}$$

where $\tau' = \mathrm{kz}_{-k}(\tau)$. In particular, $\mathbf{Q}_k$ is a free module over $\mathbb{C}[V]$.

Now, by Theorem 3.4, the module Q_k of the usual quasi-invariants is isomorphic to $\boldsymbol{e}\mathbf{Q}_k$ as a $\boldsymbol{e}H_k\boldsymbol{e}\otimes\mathbb{C}W$-module. This gives the following result generalizing [BEG], Proposition 6.6.

Theorem 8.2. *The $\boldsymbol{e}H_k\boldsymbol{e} \otimes \mathbb{C}W$-module Q_k has the direct sum decomposition*

$$Q_k \cong \bigoplus_{\tau\in \mathrm{Irr}(W)} \boldsymbol{e}M_k(\tau') \otimes \tau^*\ , \tag{8.2}$$

where $\tau' = \mathrm{kz}_{-k}(\tau)$. In particular, Q_k is free over $\mathbb{C}[V]^W$ and, hence, Cohen-Macaulay.

Proof. The decomposition (8.2) follows directly from (8.1). Each $\boldsymbol{e}M_k(\tau)$ is isomorphic to $(\mathbb{C}[V]\otimes\tau)^W$ as a $\mathbb{C}[V]^W$-module and, hence, free over $\mathbb{C}[V]^W$. With (8.2), this implies the last claim of the theorem. □

Remark 8.3. Our proof of Theorem 8.2 is similar to [BEG], however the result is slightly different, because of a KZ twist. In [BEG], it was erroneously claimed that $M_k(\tau)_{\rm reg}\cong M_0(\tau)_{\rm reg}$. By Theorem 7.11, this is true only for those groups W and values of k, for which kz_k is the identity on $\mathrm{Irr}(W)$. In general, even in the Coxeter case, there are examples when kz_k is non-trivial (see [O1]).

8.2. **Poincaré series.** Given a graded module $M=\bigoplus_{i=0}^{\infty}M^{(i)}$, with finite-dimensional components $M^{(i)}$, we write $P(M,t):=\sum_{i=0}^{\infty}t^i\dim M^{(i)}$ for the Poincaré series of M. Using Theorem 8.2, we will compute this series for Q_k. Our computation is slightly different from [BEG] as we begin with $\mathbf{Q}_k$.

We equip $\mathbb{C}[V_{\rm reg}]\otimes\tau$ with a natural grading, so that $\deg V^*=1$ and $\deg\tau=0$. Each $\mathbf{Q}_k(\tau)$ is then a graded submodule of $\mathbb{C}[V_{\rm reg}]\otimes\tau$, and by Proposition 7.13, we know that $\mathbf{Q}_k(\tau)\cong M_k(\tau')$, with $\tau=\mathrm{kz}_k(\tau')$. Now, by Lemma 7.15 and Corollary 7.18, the degree of the generating subspace τ' of $\mathbf{Q}_k(\tau)$ is equal to $\deg\tau'=c_{\tau'}(k)=c_\tau(k)$. Hence

$$P(\mathbf{Q}_k(\tau),\,t)=(\dim\tau)\,t^{c_\tau(k)}\,(1-t)^{-\dim V}\,.$$

As a result, by Proposition 8.1, the Poincaré series for $\mathbf{Q}_k$ is given by

$$P(\mathbf{Q}_k,\,t)=\sum_{\tau\in\mathrm{Irr}(W)}(\dim\tau)^2\,t^{c_\tau(k)}\,(1-t)^{-\dim V}\,.$$

Now, to compute $P(Q_k,\,t)=P(\boldsymbol{e}\mathbf{Q}_k,\,t)$ we simply take the W-invariant part of $\mathbf{Q}_k$. This can be done separately for each summand in (8.1). The Poincaré series of $\boldsymbol{e}M_k(\tau)$ is obtained by multiplying the Poincaré series of $(\mathbb{C}[V]\otimes\tau)^W$ by $t^{c_\tau(k)}$. Hence, writing

$$\chi_\tau(t):=P((\mathbb{C}[V]\otimes\tau)^W,\,t) \tag{8.3}$$

for the Poincaré series of $(\mathbb{C}[V]\otimes\tau)^W$, we get

$$P(\boldsymbol{e}\mathbf{Q}_k(\tau),\,t)=t^{c_{\tau'}(k)}\chi_{\tau'}(t)\;, \tag{8.4}$$

where $\tau'=\mathrm{kz}_{-k}(\tau)$. Finally, summing up over all $\tau\in\mathrm{Irr}(W)$ as in (8.2), we find (cf. [BEG])

$$P(Q_k,\,t)=\sum_{\tau\in\mathrm{Irr}(W)}(\dim\tau)\,t^{c_\tau(k)}\,\chi_\tau(t)\;. \tag{8.5}$$

8.3. **Symmetries of fake degrees.** It was pointed out to us by E. Opdam that the above results could be used to give another proof of an interesting symmetry of fake degrees of complex reflection groups (see [O2], Theorem 4.2). Below, we will show that property for the series (8.3); for the relation of (8.3) to fake degrees we refer the reader to Opdam's paper [O2].

Fix a collection of integers $a=\{a_C\}_{C\in\mathcal{A}/W}$, with $a_C\in\{0,1,\ldots,n_C-1\}$. Put $\delta_a:=\prod_{C\in\mathcal{A}/W}(\delta_C)^{a_C}$ and write ϵ_a for the corresponding one-dimensional representation of W, with character $\prod_{C\in\mathcal{A}/W}(\det_C)^{-a_C}$. Now, define $k=\{k_{C,i}\}$ by $k_{C,i}:=a_C/n_C$ for all C,i. Then, for every $\tau\in\mathrm{Irr}(W)$, the space $\mathbf{Q}_k(\tau)$ has a simple description:

$$\mathbf{Q}_k(\tau)=\delta_a\,\mathbb{C}[V]\otimes\tau\;, \tag{8.6}$$

which is easily seen from the definition (3.12).

On the other hand, consider $k' = g \cdot k$, with $g := \prod_{C\in\mathcal{A}/W}(g_C)^{a_C}$ and g_C defined by (5.11). A straightforward calculation shows that

$$k' = \sum_{C\in\mathcal{A}/W}\ \sum_{1\le i\le a_C} \ell_{C,n_C-i}\,,$$

where we use the same notation as in Proposition 5.5. Now, from Proposition 5.3, it follows that $\boldsymbol{e}\mathbf{Q}_k(\tau) = \boldsymbol{e}\mathbf{Q}_{k'}(\tau)$; hence, these two modules have the same Poincaré series. For $\boldsymbol{e}\mathbf{Q}_k(\tau)$, we can compute its Poincaré series directly from (8.6): with notation (8.3), the result reads $t^{\deg\delta_a}\chi_{\epsilon_a\otimes\tau}$. On the other hand, for $\boldsymbol{e}\mathbf{Q}_{k'}(\tau)$, we apply (8.4). Equating the resulting Poincaré series, we get

$$t^{\deg\delta_a}\chi_{\epsilon_a\otimes\tau}(t) = t^{c_{\tau'}(k')}\chi_{\tau'}(t)\,,\qquad \tau = \mathrm{kz}_{k'}(\tau')\ ,$$

which is equivalent to [O2], Theorem 4.2.

9. Appendix: The Baker-Akhiezer Function

In the case of integral multiplicities k, we can construct a common eigenfunction for the ring of commuting differential operators $L_{p,k}$, $p\in\mathbb{C}[V^*]^W$, by successively applying the shift operators (5.18) to the exponential function $e^{\langle\lambda,x\rangle}$. We will call such a function the *Baker-Akhiezer function*; our goal is to establish some properties of this function, generalizing the results of [VSC] and [CFV] in the Coxeter case. Perhaps, the most curious property (the 'bispectral' symmetry) is given by Proposition 9.1. This proposition has been proven in [SV] for the complex groups of type $G(m, p, N)$, and although our proof is somewhat different, the key idea to use the pairing (9.10) is borrowed from [SV].

We restrict ourselves to the case when $k_{C,i}\in\mathbb{Z}_{\ge 0}$, with $k_{H,0}=0$. In that case, the successive application of the elementary shift operators S_k, see (5.18), produces a function $\psi(\lambda,x)$ on $V^*\times V$ of the form

$$\psi(\lambda,x) = P(\lambda,x)\,e^{\langle\lambda,x\rangle}\,, \tag{9.1}$$

where $\langle\lambda,x\rangle$ is the natural pairing, and $P\in\mathbb{C}[V^*\times V]$ is a polynomial with leading term

$$P_0 = \prod_{C\in\mathcal{A}/W}(\delta_C^*(\lambda)\,\delta_C(x))^{N_C}\quad,\quad N_C := \sum_{i=0}^{n_C-1}k_{C,i}\ . \tag{9.2}$$

Since the elementary shift operators (5.18) are all homogeneous of degree zero, so is their composition, and hence P has degree zero with respect to the grading defined by $\deg V^* = 1$ and $\deg V = -1$.

By construction, ψ is a common eigenfunction of the generalized Calogero-Moser operators $L_{p,k} = \mathrm{Res}\,T_{p,k}$:

$$L_{p,k}[\psi] = p(\lambda)\psi\ ,\quad \forall\, p\in\mathbb{C}[V^*]^W\ . \tag{9.3}$$

It is analytic in both variables, and by Proposition 7.20,

$$\psi(\lambda,x)\in\mathcal{Q}_k\quad\text{as a function of } x\,, \tag{9.4}$$

where $\mathcal{Q}_k$ denotes the analytic completion of the module of quasi-invariants Q_k. Note also that the shift operators in Theorem 5.7 are W-invariant, thus,

$$\psi(w\lambda,x) = \psi(\lambda,wx)\qquad\forall\, w\in W\ . \tag{9.5}$$

Recall the antilinear isomorphism $* : V \to V^*$ determined by the W-invariant Hermitian form on V, see Section 2.1. It is easy to check that $*$ respects the canonical pairing between V and V^* and is W-equivariant, see [DO], Proposition 2.17(i). It extends to an anti-linear map $\mathbb{C}[V^* \times V] \to \mathbb{C}[V \times V^*]$, which we denote by the same symbol. Note that $*$ induces a natural antilinear map $* : \mathrm{End}_{\mathbb{C}}(\mathbb{C}[V]) \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}[V^*])$, and it is easy to check that

$$(T_{p,k})^* = T_{p^*,\overline{k}}\,, \quad p \in \mathbb{C}[V^*]\,, \tag{9.6}$$

where $\overline{k}$ denotes the complex conjugate (in our case, $\overline{k} = k$). Here, the Dunkl operators on the right are defined in the same way as $T_{p,k}$ but with respect to the *dual* representation V^* of W.

Applying $*$ to ψ, we get

$$\psi^*(x, \lambda) = P^*(x, \lambda)\, e^{\langle x, \lambda \rangle}\,. \tag{9.7}$$

Let us write $\psi = \psi_V(\lambda, x)$ to indicate the dependence of ψ on the reflection representation V of W. It follows then from (9.6) that $\psi^* = \psi_{V^*}(x, \lambda)$. In particular, ψ^* is a common eigenfunction of the 'dual' family of operators with respect to the λ-variable:

$$L_{q,k}[\psi^*] = q(x)\psi^*\,, \quad \forall\, q \in \mathbb{C}[V]^W\,. \tag{9.8}$$

Now, by 'bispectral symmetry' of the Baker-Akhiezer function we mean the following property.

Proposition 9.1. $\psi_V(\lambda, x) = \psi_{V^*}(x, \lambda)$. *In particular,* $\psi = \psi_V$ *is a common solution to the eigenvalue problems* (9.3) *and* (9.8).

For the proof, we consider

$$\Phi(\lambda, x) := \sum_{w \in W} \psi(w\lambda, x) = \sum_{w \in W} \psi(\lambda, wx)\,. \tag{9.9}$$

Lemma 9.2. *The function* (9.9) *has the following properties:*

(1) Φ *is global analytic in* x *and* λ*;*

(2) Φ *is* W*-invariant in each of the variables,* x *and* λ*;*

(3) $T_{p,k}\Phi = p(\lambda)\Phi$ *for all* $p \in \mathbb{C}[V^*]^W$*;*

(4) *In a neighborhood of* $\lambda = 0$, Φ *admits an expansion* $\Phi = \sum_i \Phi_i$*, where* $\Phi_i \in \mathbb{C}[V]^W \otimes \mathbb{C}[V^*]^W$ *is homogeneous of degree* i *in both* λ *and* x*;*

(5) $\Phi(0, x) = \Phi(\lambda, 0) = \Phi(0, 0) \neq 0$.

Proof. The first four properties are immediate from the definition; only (5) needs a proof. Let us define a bilinear map $\mathbb{C}[V^*] \times \mathbb{C}[V] \to \mathbb{C}$ by

$$(p, q)_k := T_{p,k}(q)(0)\,, \qquad p \in \mathbb{C}[V^*]\,,\ q \in \mathbb{C}[V]\,. \tag{9.10}$$

This is closely related to the pairing on $\mathbb{C}[V] \times \mathbb{C}[V]$ defined in [DO], which equals $(p^*, q)_k$ in our notation.

It follows from [DO], Proposition 2.20 and Theorem 2.18, that (9.10) is a nondegenerate pairing for any $k \in \mathrm{Reg}$ satisfying

$$(p, q)_k = \overline{(q^*, p^*)_{\overline{k}}}\,, \qquad \forall\, (p, q) \in \mathbb{C}[V] \times \mathbb{C}[V^*]\,. \tag{9.11}$$

(For integral k, we have $\overline{k} = k$.) Moreover, by Proposition 2.17(iii) of *loc.cit*, the restriction of $(-,-)_k$ to W-invariants is also nondegenerate.

We need to prove that $\Phi_0 = \Phi(0,0) \neq 0$. Assuming the contrary, let us take the first nonzero term Φ_i. Then, substituting the expansion $\Phi = \sum_i \Phi_i$ into the equations (3), we see that $T_{p,k}\Phi_i = 0$ for all $p \in \mathbb{C}[V^*]^W$. This implies that

$$(p, \Phi_i)_k = 0\ , \quad \forall\, p \in \mathbb{C}[V^*]^W\ . \tag{9.12}$$

Note that Φ_i is W-invariant as a function of x. Thus, (9.12) contradicts the non-degeneracy of $(-,-)_k$ and proves that $\Phi_0 = \Phi(0,0) \neq 0$. □

Proof of Proposition 9.1. We can normalize Φ so that $\Phi(0,0) = 1$. Taking a homogeneous basis $\{p_i\}$ of $\mathbb{C}[V^*]^W$, with $0 = \deg p_0 \le \deg p_1 \le \deg p_2 \le \dots$, we can expand Φ (as a function of λ) into a series in p_i :

$$\Phi(\lambda, x) = \sum_{i\ge 0} p_i(\lambda) q_i(x)\ , \quad \text{with some } q_i \in \mathbb{C}[V]^W\ .$$

Evaluating both sides of $T_{p,k}\Phi = p(\lambda)\Phi$ at $x = 0$, we conclude that the elements q_i form the basis dual to $\{p_i\}$ with respect to the pairing (9.10).

If $\{p_i\}$ and $\{q_i\}$ are dual bases, then so are $\{q_i^*\}$ and $\{p_i^*\}$, by (9.11). Therefore, we also have

$$\Phi(\lambda, x) = \sum_{i\ge 0} p_i(\lambda) q_i(x) = \sum_{i \ge 0} q_i^*(\lambda) p_i^*(x) = \overline{\Phi(x^*, \lambda^*)}\,.$$

Using the definition (9.9) of Φ, and the fact that $\langle \mu, x\rangle = \overline{\langle x^*, \mu^*\rangle}$, we easily conclude that $\psi(\lambda, x) = \overline{\psi(x^*, \lambda^*)} = \psi^*(x, \lambda)$, which finishes the proof. □

Thus, the properties of ψ in x (say) mirror those in λ, but with V replaced by V^*. For instance, letting $Q_k^* := Q_k(W, V^*)$, we have a counterpart of (9.4):

$$\psi(\lambda, x) \in \mathcal{Q}_k^* \quad \text{as a function of } \lambda\ . \tag{9.13}$$

Having this, we can now characterize, similarly to [VSC], the Baker-Akhiezer function $\psi(\lambda, x)$ as a *unique* function satisfying (9.1), (9.2) and (9.13). Furthermore, we get the following result, which for a Coxeter group W was first established in [VSC] (see also [CFV]). Recall the subalgebra $A_k \subset \mathbb{C}[V]$, see (2.6), and denote by $A_k^* \subset \mathbb{C}[V^*]$ its 'dual' counterpart related to Q_k^*.

Proposition 9.3. *For any $p \in A_k^*$, there exists a differential operator $L_p \in \mathcal{D}(V_{\rm reg})$ in the x-variable, with a constant principal symbol p, such that $L_p\psi = p(\lambda)\psi$. The operators $\{L_p\}_{p\in A_k^*}$ pairwise commute and generate a subalgebra of $\mathcal{D}(V_{\rm reg})$, isomorphic to A_k^*.*

Note that, by bispectral symmetry, we also have a similar commutative subalgebra of differential operators in the 'spectral' variable λ.

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