

MACDONALD POLYNOMIALS AND ALGEBRAIC INTEGRABILITY

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ABSTRACT. We construct explicitly (non-polynomial) eigenfunctions of the difference operators by Macdonald in case $t = q^k$, $k \in \mathbb{Z}$. This leads to a new, more elementary proof of several Macdonald conjectures, first proved by Cherednik. We also establish the algebraic integrability of Macdonald operators at $t = q^k$ ($k \in \mathbb{Z}$), generalizing the result of Etingof and Styrkas. Our approach works uniformly for all root systems including BC_n case and related Koornwinder polynomials. Moreover, we apply it for a certain deformation of A_n root system where the previously known methods do not work.

1. INTRODUCTION

One of the goals of this paper is to present a new (essentially, "non-polynomial") approach to Macdonald polynomials. These polynomials were introduced in late 80-s by I.G. Macdonald in [M1] as, informally speaking, a "discrete spectrum" of certain remarkable symmetric difference operators. Since these (Macdonald) operators are self-adjoint with respect to a specific scalar product, Macdonald's theory leads directly to the families of multivariable orthogonal polynomials. From that point of view, they generalize various classical orthogonal polynomials of one variable. In fact, there are as many families of Macdonald polynomials, as simple complex Lie algebras, or root systems. Each family depends, apart from a root system, on (at least) two parameters q, t and specializes to several remarkable families of symmetric functions. Among them are Schur functions and characters of corresponding Lie groups, Hall–Littlewood functions, Jack polynomials or, more generally, multivariable Jacobi polynomials due to Heckman and Opdam [HO]. All this makes Macdonald polynomials very interesting from the point of view of the representation theory, combinatorics, special function theory and mathematical physics. This also makes clear that Macdonald polynomials are highly non-trivial. Thus, it is not surprising that their various properties formulated by Macdonald as conjectures, remained unproven for quite a long time. A remarkable progress has been achieved by Cherednik, who proved the so-called norm and evaluation conjectures and the symmetry identity for all root systems [C1, C2]. Cherednik's approach is based on his theory of double affine Hecke algebras and it remains one of the major achievements in this area. As an introduction into Cherednik's theory we recommend to the reader a nicely written survey by Kirillov [Ki1].

One of the results of the present paper is an independent proof of these three Macdonald's conjectures. Our approach uses some remarkable properties of the Macdonald operators in case $t = q^k$ with *integer* k . Ideologically, it goes back to the

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paper by Veselov and the author [CV], where the quantum Calogero–Sutherland–Moser problem was considered for some special values of the coupling constant. Recall that the Calogero–Sutherland–Moser problem [Ca, Su] describes N particles x_1, \dots, x_N on the line whose pairwise interaction is given by the potential $u(x) = m(m+1)\sin^{-2}x$. In the quantum case its Hamiltonian is the following Schrödinger operator in \mathbb{R}^N :

$$H = -\Delta + \sum_{i < j} m(m+1)\sin^{-2}(x_i - x_j).$$

It is a celebrated example of a completely integrable N -body problem, and there are quite a few exact results about it. In particular, it is completely integrable in Liouville sense, i.e. H is a member of a family of N commuting partial differential operators (quantum integrals) $H_1 = H, H_2, \dots, H_N$. Moreover, as it was demonstrated in [CV, CSV], for special values of the coupling constant m this problem becomes "much more" integrable. Namely, it turns out that the Calogero–Sutherland–Moser problem for $m \in \mathbf{Z}$ is *algebraically integrable*, i.e. its quantum integrals H_1, \dots, H_N are a part of some bigger commutative ring \mathfrak{R} of partial differential operators (see [CSV] for precise formulations and results). Moreover, $\text{Spec}(\mathfrak{R})$ is an affine algebraic variety whose points parameterize Bloch eigenfunctions of H . This is a multidimensional analogue of a phenomenon, well known from the finite-gap theory in dimension one [DMN, Kr1]. The main difference with the one-dimensional case is that the corresponding algebro-geometrical data are very "rigid" and really exceptional, which makes the existence problem for related multidimensional Baker–Akhiezer functions extremely difficult. This difficulty was overcome in [CSV] with the help of the theory of multivariable hypergeometric functions due to Heckman and Opdam [HO]. Recently, a direct independent proof was obtained in [Ch1].

Continuing [Ch1], here we demonstrate that a similar phenomenon appears for H being replaced by any of the Macdonald difference operators, namely, they all are *algebraically integrable* for special integer values of the parameter(s). Note that in case of R being of A_n type the Macdonald operators coincide (up to a certain gauge) with the trigonometric version of the elliptic Ruijsenaars operators from [R], introduced as a generalization of the elliptic Calogero–Moser problem. We should mention at this point the paper by Etingof and Styrkas [ES] where the algebraic integrability has been established for the Macdonald–Ruijsenaars operators with $t = q^k, k \in \mathbf{Z}$. Their approach was based on earlier results by Etingof and Kirillov [EK1, EK2] who gave an interpretation of Macdonald polynomials for $R = A_n$ in terms of the representation theory for quantum groups. This delivers independent proofs of several results in case $R = A_n$, see [EK3, EK4, Ki2]. It is worth noticing that the symmetry identity in this case has been proved first by Koornwinder [Ko1] (see chapter VI of Macdonald's book [M4]). However, his method, as well as the methods of [EK2, ES], does not extend to other root systems.

One of the advantages of our approach is that it works equally well for all root systems and (we believe) is simpler comparing to [C1, C2]. The main object is what is natural to call a *Baker–Akhiezer function* ψ for Macdonald operators. In case $R = A_n$ this coincides with the ψ -function from [ES]. In that part which goes back to the papers [CV, CSV], our considerations have very much in common with [ES]. The main new ingredient is a fairly elementary construction of the ψ -function. Our main observation (Proposition 2.1) is that the Macdonald operators in case $t = q^k$,

$k \in \mathbb{Z}_-$, act naturally in the coordinate ring of a certain very specific affine algebraic variety. This implies the existence of ψ which is our central result. From this we derive the duality, which reflects a certain symmetry between the two arguments x, z of ψ -function. Basically, it means that, as a function of x , ψ is an eigenfunction of the Macdonald operators related to the root system R , while, as a function of z , it is an eigenfunction of the Macdonald operators related to the *dual* root system. Thus, we observe on this level the *bispectrality* of Macdonald operators, if one uses the terminology of the fundamental work by Duistermaat and Grünbaum on bispectral problem [DG]. It is worth mentioning that in contrast with the one-dimensional case, bispectral problem in higher dimensions is much less investigated. For an interesting example related to Knizhnik–Zamolodchikov equation see recent paper [TV].

Notice that our proof of the existence of ψ is an effective one and gives a closed expression for it. The resulting formula generalizes the main result of [Ch1] and it is a discrete version of one remarkable formula by Berest, who found in [Be] an elegant "universal" expression for the ψ -function for the quantum (rational) Calogero–Moser problem. His derivation, however, was based on a crucial assumption that such a ψ *does exist*. As we mentioned above, that type of existence problems is highly nontrivial in dimension > 1 . Remarkably enough, our approach being inspired by the Berest's result, allows us to do these two things simultaneously: we prove the existence of ψ by a direct derivation of a discrete analogue of the Berest's formula.

Everything extends without much difficulties to BC_n case. A proper generalization of Macdonald's theory for this case was suggested by Koornwinder [Ko1]. The resulting orthogonal polynomials (Koornwinder polynomials) can be viewed as a multivariable analog of the celebrated Askey–Wilson polynomials [AW]. Van Diejen [vD1] showed that Koornwinder polynomials are joint eigenfunctions of n commuting difference operators for which he gave an explicit expression (initially, Koornwinder constructed one operator only). Further, in [vD2] the symmetry identity (or *self-duality*) was established for a certain subfamily of Koornwinder polynomials. Then, finally, Sahi [S] proved duality in general case, using a proper generalization of Cherednik's double affine Hecke algebra. Together with earlier van Diejen's results [vD2] this implied also the evaluation identity and the norm identity, conjectured by Macdonald. Our approach leads to an independent proof of these results. As above, the key ingredient is the algebraic integrability of the Koornwinder operator for special integer values of the parameters.

One of our primary motivations for this work was, in fact, our attempt to find a difference version of the deformed Calogero–Moser problem from [CFV1]. It is related to what can be viewed as a one-parameter deformation of A_n root system. In [Ch2], guided by duality, we were able to find a rational difference version of that quantum problem. Here we consider its trigonometric version, proving its (algebraic) integrability. Similar to the usual A_n case, the constructed difference operator is self-dual: corresponding Baker–Akhiezer function $\psi(x, z)$ is invariant under permuting x, z . A natural elliptic version seems to be integrable, too. We would like to stress that while for the root systems the approach based on affine Hecke algebras seems to be the most adequate and powerful, in the deformed case none of the previously known methods can be applied (at least, straightforwardly).

Thus, it would be very interesting to find an algebraic structure which underlies that "deformed" root system.

The paper is organized as follows. In Section 2 we recall the definitions of difference operators and polynomials due to Macdonald. Then we make our central observation about Macdonald operators in case $t = q^{-m}$ ($m \in \mathbb{Z}_+$). The following two sections form the core of the paper.

In Section 3, we define a Baker–Akhiezer function $\psi(x, z)$ associated to a datum which consists of a root system R and some additional integer parameter(s). ψ is determined by prescribing its analytic properties in the z -variable, which is a proper modification of the approach from [CV, ES]. We prove first its uniqueness up to a normalization and then the existence, obtaining as a by-product a discrete version of the Berest's formula.

In Section 4, we explain how one should normalize ψ to achieve a remarkable symmetry between x and z . This leads us directly to the duality theorem, which is the central result of this section.

In Section 5 we derive various corollaries from achieved results. First, we obtain algebraic and complete integrability of the Macdonald operators and prove the existence of the so-called shift operators. Then we explain how our ψ relates to Macdonald polynomials, this generalizes Weyl's character formula and is similar to a relation between symmetric and nonsymmetric Macdonald polynomials (see [M3]). As a corollary, we observe a nice "localization" property for Macdonald polynomials in case $t = q^{-m}$ ($m \in \mathbb{Z}_+$). We conclude the section by explaining how our results lead to a proof of the norm identity, evaluation formula and duality for Macdonald polynomials.

In Section 6 we sketch how to extend our approach to BC_n case. This is parallel to the previous sections, so we skip most of the proofs. The main difference comes from $n = 1$ case which is technically more difficult compared to A_1 .

Finally, in Section 7 we discuss a generalized Macdonald–Ruijsenaars model, related to the deformed A_n system.

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2. DIFFERENCE OPERATORS AND POLYNOMIALS BY MACDONALD

2.1. Notations. Let $V = \mathbb{R}^n$ be a Euclidean space with the scalar product denoted as (u, v) . Consider an arbitrary root system $R \in V$ which is, by definition, a finite set of vectors (roots) $\alpha \in V$ with the following two properties:

- (1) $\forall \alpha \in R$ the orthogonal reflection s_α in V

$$s_\alpha : x \longmapsto x - 2 \frac{(\alpha, x)}{(\alpha, \alpha)} \alpha$$

leaves R invariant, $s_\alpha(R) = R$;

- (2) $\forall \alpha, \beta \in R \quad 2 \frac{(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z}$

(see [B] for the details).

The second property implies that the so-called root lattice Q generated over \mathbb{Z} by the roots $\alpha \in R$ is invariant under all the reflections s_α and, therefore, under the whole Weyl group W generated by s_α , $\alpha \in R$. The vectors $\alpha^\vee = 2\alpha/(\alpha, \alpha)$ form the dual root system R^\vee and we denote by Q^\vee the lattice generated by all

$\alpha^\vee \in R^\vee$. Introduce also weight and coweight lattices P, P^\vee as

$$\begin{aligned} P &= \{\pi \in V \mid (\pi, \alpha^\vee) \in \mathbb{Z} \quad \forall \alpha^\vee \in R^\vee\} && \text{(weights)}, \\ P^\vee &= \{\pi \in V \mid (\pi, \alpha) \in \mathbb{Z} \quad \forall \alpha \in R\} && \text{(coweights)}. \end{aligned}$$

From the definitions one has inclusions $Q \subset P$, $Q^\vee \subset P^\vee$. Taking $\mathbb{Z}_{\geq 0}$ instead of \mathbb{Z} in the last two definitions leads to dominant weights (coweights) P_+ and P_+^\vee , respectively.

Let us fix some basis of simple roots $\alpha_1, \dots, \alpha_n$ in R , this determines a decomposition of R and R^\vee into positive and negative parts:

$$R = R_+ \cup (-R_+), \quad R^\vee = R_+^\vee \cup (-R_+^\vee).$$

The elements ω_i of the basis, dual to $\alpha_1^\vee, \dots, \alpha_n^\vee$, $(\omega_i, \alpha_j^\vee) = \delta_{ij}$, are called the *fundamental weights* for the system R . Similarly, one defines the fundamental coweights b_i : $(b_i, \alpha_j) = \delta_{ij}$.

In these terms the root lattice Q is simply

$$Q = \bigoplus_{i=1}^n \mathbb{Z}\alpha_i.$$

Its *positive part* Q_+ is obtained by replacing \mathbb{Z} by \mathbb{Z}_+ . Similarly, the cone of the dominant weights P_+ is

$$P_+ = \bigoplus_{i=1}^n \mathbb{Z}_+\omega_i.$$

Below we will assume that $R = A_n, \dots, G_2$ is reduced and irreducible. The case $R = BC_n$ is considered in section 6.

2.2. Macdonald operators. Let R be a (reduced, irreducible) root system in Euclidean space V . Let us fix $q \in \mathbb{C}^\times$ and a set k of W -invariant parameters $k_\alpha = k_{\alpha^\vee}$, i.e. $k_\alpha = k_{w\alpha}$ for any $\alpha \in R$ and $w \in W$. Below we will sometimes use the related parameters $t_\alpha = t_{\alpha^\vee} := q^{k_\alpha}$, denoting by $t = q^k$ the whole set $\{t_\alpha\}$ ¹. Throughout the paper we will suppose that q is not a root of unity. Something still can be done in case of roots of unity, though we will not touch these issues here (see [Ki2, C2]).

For $v \in V$ we denote by T^v the operator acting on a function $f(x)$ as a shift by v in $x \in V$:

$$(T^v f)(x) = f(x + v).$$

Later, we will deal with the functions depending on two variables $x, z \in V$, in that case we will use subscripts to distinguish between shifts T_x^v and T_z^v , acting in x and z , respectively. To introduce Macdonald operators, we need the notion of a (quasi)minuscule coweight.

Definition. (1) A coweight $\pi \in P^\vee$ is called minuscule if $-1 \leq (\pi, \alpha) \leq 1$ for all $\alpha \in R$.

(2) A coweight $\pi \in P^\vee$ is called quasiminuscule if it belongs to R^\vee and $-1 \leq (\pi, \alpha) \leq 1$ for all $\alpha \in R \setminus \pi^\vee$.

¹To define q^k for arbitrary k we fix the value of $\log q$, so $q^k := e^{k \log q}$.

Using tables from [B] one can check that all root systems except E_8, F_4, G_2 have at least one nonzero minuscule coweight. Meanwhile, for any root system R the coroot $\pi = \theta^\vee$ where θ is the *maximal root* for R , will be quasiminuscule (see [B]).

Now let $\pi \in P^\vee$ be a minuscule coweight for the system R . Then the corresponding Macdonald operator D^π is a difference operator in $x \in V$ defined as follows [M1, M2]:

$$D^\pi = \sum_{\tau \in W\pi} a_\tau T^\tau, \quad a_\tau(x) = \prod_{\substack{\alpha \in R: \\ (\alpha, \tau) > 0}} \frac{t_\alpha q^{(\alpha, x)} - t_\alpha^{-1} q^{-(\alpha, x)}}{q^{(\alpha, x)} - q^{-(\alpha, x)}}. \quad (2.1)$$

For a quasiminuscule coweight π the formula is slightly more complicated:

$$D^\pi = \sum_{\tau \in W\pi} a_\tau (T^\tau - 1) + \sum_{\tau \in W\pi} q^{-2(\rho, \tau)}, \quad (2.2)$$

where

$$a_\tau(x) = \prod_{\substack{\alpha \in R: \\ (\alpha, \tau) > 0}} \frac{t_\alpha q^{(\alpha, x)} - t_\alpha^{-1} q^{-(\alpha, x)}}{q^{(\alpha, x)} - q^{-(\alpha, x)}} \prod_{\substack{\alpha \in R: \\ (\alpha, \tau) = 2}} \frac{t_\alpha q^{1+(\alpha, x)} - t_\alpha^{-1} q^{-1-(\alpha, x)}}{q^{1+(\alpha, x)} - q^{-1-(\alpha, x)}} \quad (2.3)$$

and

$$\rho = \rho_k = \frac{1}{2} \sum_{\alpha \in R_+} k_\alpha \alpha. \quad (2.4)$$

Remark. Note that in the formula (2.3) for a_τ the last product contains one factor only (with $\alpha = \tau^\vee$). Written this way, it formally makes sense for a minuscule coweight, too. Indeed, a formal substitution of a minuscule π into (2.2) leads to (2.1) because the constant term $\sum q^{-2(\rho, \tau)} - \sum a_\tau$ will be zero in that case. In the remainder of the paper, we will refer to the formula (2.2) since it covers both cases.

Remark. The following function $\Delta_k(x)$ plays an important role in Macdonald's theory:

$$\Delta = \Delta_k(x) = q^{-2(\rho, x)} \prod_{\alpha \in R_+} \prod_{i=0}^{\infty} \frac{1 - q^{2i+2(\alpha, x)}}{1 - q^{2k_\alpha + 2i+2(\alpha, x)}}. \quad (2.5)$$

Using it, one can present the coefficients a_τ of the Macdonald operator D^π as follows:

$$a_\pi = T^\pi(\Delta)/\Delta, \quad a_{w\pi}(x) = a_\pi(w^{-1}x). \quad (2.6)$$

Example. In case $R = A_{n-1} = \{\pm(e_i - e_j) \mid i < j\} \subset \mathbb{R}^n$ with $k_\alpha \equiv k$ each fundamental coweight $\pi_s = e_1 + \dots + e_s$ ($s = 1, \dots, n$) is minuscule and the corresponding operator $D_s = D^{\pi_s}$ has the form

$$D_s = \sum_{\substack{I \subset \{1, \dots, n\} \\ |I|=s}} \prod_{\substack{i \in I \\ j \notin I}} \frac{q^{k+x_i-x_j} - q^{-k-x_i+x_j}}{q^{x_i-x_j} - q^{-x_i+x_j}} T^I, \quad (2.7)$$

where T^I stands for $\prod_{i \in I} T^{e_i}$. These operators coincide (up to a certain gauge) with the quantum integrals of the trigonometric Ruijsenaars model introduced in [R].

2.3. Macdonald polynomials. The starting point for Macdonald's theory [M1, M2] is that the operators D^π , introduced above, preserve the space spanned by W -invariant exponents, or *orbitsums* \mathbf{m}_λ :

$$\mathbf{m}_\lambda(x) = \sum_{\tau \in W\lambda} q^{2(\tau, x)}, \quad (2.8)$$

where $\lambda \in P_+$ is a dominant weight and the summation is taken over its W -orbit. Moreover, the action of D^π is lower-triangular:

$$D^\pi \mathbf{m}_\lambda = c_{\lambda\lambda} \mathbf{m}_\lambda + \sum_{\nu \prec \lambda} c_{\lambda\nu} \mathbf{m}_\nu \quad (2.9)$$

where the coefficients $c_{\lambda\nu}$ depend on q, t and $\nu \prec \lambda$ means that $\lambda - \nu$ belongs to Q_+ .

To introduce the Macdonald polynomials, let us first agree about terminology. Throughout the paper by a *polynomial* in x we will always mean a function $f(x)$ of the form

$$f = \sum_{\nu \in P} f_\nu q^{2(\nu, x)}.$$

From algebraic point of view this corresponds to considering a ring of Laurent polynomials in $X_i = q^{(\omega_i, x)}$, where ω_i are the fundamental weights. As well as a standard polynomial ring, it has a unique factorization property with $q^{2(\nu, x)}$ being the only invertible elements.

Now the Macdonald polynomials $P_\lambda = P_\lambda(x; q, t)$ can be defined uniquely as polynomials of the form

$$P_\lambda = \mathbf{m}_\lambda + \sum_{\nu \prec \lambda} a_{\lambda\nu} \mathbf{m}_\nu, \quad \lambda \in P_+, \quad (2.10)$$

which are eigenfunctions of D^π :

$$D^\pi P_\lambda = c_{\lambda\lambda} P_\lambda. \quad (2.11)$$

The coefficients $a_{\lambda\nu}$ in (2.10) are rational in q, t and the polynomials P_λ ($\lambda \in P_+$) have a number of remarkable properties. In particular, they are orthogonal with respect to the following scalar product:

$$\langle f, g \rangle_k = \text{CT} \left(f \bar{g} \Delta_k \bar{\Delta}_k \right), \quad (2.12)$$

where CT means the constant term, Δ_k is the function (2.5) and the bar acts on a function $f(x)$ as $\bar{f}(x) = f(-x)$. This scalar product can be reinterpreted in terms of a certain integral, which makes perfect sense for noninteger k , too. See [M1] for the details.

Remark. It is not difficult to see that the eigenvalues $c_{\lambda\lambda}$ for $\lambda \in P_+$ have the form:

$$c_{\lambda\lambda} = \sum_{\tau \in W\pi} q^{2(\tau, \lambda + \rho)}, \quad (2.13)$$

where $\rho = \rho_k$ is given by (2.4).

Remark. P_λ is correctly defined if the diagonal terms in the action (2.9) are distinct:

$$c_{\lambda\lambda} \neq c_{\nu\nu} \quad \text{for all } \nu \prec \lambda. \quad (2.14)$$

This is true for generic t_α and in this case the polynomials (2.10) are uniquely determined by (2.11). Their coefficients, however, have singularities at certain q, t . For instance, in case $k_\alpha = -m_\alpha$ with $m_\alpha \in \mathbb{Z}_+$ which will be central in further considerations, some first $P_\lambda(x; q, q^{-m})$ are not well-defined. Nevertheless, even in this case the condition (2.14) holds for sufficiently large λ , i.e. if $(\lambda, \alpha) > 2m_\alpha$ for all $\alpha > 0$ (at least, (2.14) will be true for a proper D^π or their linear combination, see [M1, M2]). This means that for such λ the whole set of equations (2.11) for all (quasi)minuscule coweights together with (2.10) determines P_λ correctly.

2.4. Macdonald operators in case $k_\alpha \in \mathbb{Z}_-$. Let us concentrate now on the case of integer multiplicities k_α . It is known that some results in Macdonald theory are easier to achieve for integer k_α , extending then to non-integer values by a proper "analytic continuation" in k_α . However, instead of a common assumption $k_\alpha \in \mathbb{Z}_+$, we will rather consider the case $k_\alpha = -m_\alpha$ where $m_\alpha \in \mathbb{Z}_+$. As we mentioned above, in this situation the Macdonald's theory is not complete. Nevertheless, the structure of the eigenfunctions of the operators D^π can be described quite effectively. Next proposition will be crucial for us.

For given root system R and integer multiplicities $m = \{m_\alpha\}$ introduce a ring \mathfrak{R} which consists of all polynomials $f(x)$ with the following properties: for each $\alpha \in R_+$ and $j = 1, \dots, m_\alpha$

$$f\left(x + \frac{1}{2}j\alpha^\vee\right) \equiv f\left(x - \frac{1}{2}j\alpha^\vee\right) \quad \text{for } q^{2(\alpha, x)} = 1. \quad (2.15)$$

Proposition 2.1. *Let D^π be a Macdonald operator, defined in accordance with the formulas (2.1)–(2.3). Suppose that all t_α have the form $t_\alpha = q^{-m_\alpha}$ with $m_\alpha \in \mathbb{Z}_+$. Then the operator D^π preserves the ring (2.15): $D^\pi(\mathfrak{R}) \subseteq \mathfrak{R}$.*

To prove this, we will look first at the rank-one case, $R \subset V = \mathbb{R}^1$. In this case we will denote by T the unit shift: $(Tf)(x) = f(x+1)$. Similarly, T^s will stand for the shift by a scalar s . Let us consider a difference operator L of the form

$$L = a(x)T + b(x)T^{-1}. \quad (2.16)$$

Suppose that its coefficients a, b are meromorphic with simple poles at $x = 0$ and with no other poles at $x \in \mathbb{Z}$. Further, let us fix an integer $m \in \mathbb{Z}_+$ and impose the following conditions on a, b :

$$\text{res}_{x=0}(a+b) = 0, \quad (2.17)$$

$$a(j) = b(-j) \quad \text{for } j = \pm 1, \dots, \pm m, \quad (2.18)$$

$$a(m) = 0. \quad (2.19)$$

Introduce a ring \mathfrak{R}_0 which consists of all meromorphic functions $f(x)$ with no poles at $x \in \mathbb{Z}$ and such that

$$f(j) = f(-j) \quad \text{for all } j = 1, \dots, m. \quad (2.20)$$

Lemma 2.2. Under conditions (2.17)–(2.19), the operator L preserves the ring (2.20):

$$L(\mathfrak{R}_0) \subseteq \mathfrak{R}_0 .$$

Proof. First of all, for any $f \in \mathfrak{R}_0$ its image Lf will be nonsingular at $x \in \mathbb{Z}$. Indeed, the only apparent pole is $x = 0$. However, it disappears since the residues of a and b are opposite and $(T - T^{-1})f|_{x=0} = 0$ due to (2.20) at $j = 1$.

Now let us prove that Lf still satisfies the conditions (2.20), i.e. that $(T^j - T^{-j})Lf$ is zero at $x = 0$. A simple calculation gives us that

$$\begin{aligned} (T^j - T^{-j})Lf|_{x=0} &= a(j)f(j+1) - b(-j)f(-j-1) + b(j)f(j-1) - a(-j)f(-j+1) \\ &= a(j)(f(j+1) - f(-j-1)) + b(j)(f(j-1) - f(-j+1)) \end{aligned}$$

(here we used the conditions (2.18)). The last expression must be zero for all $j = 1, \dots, m$ due to (2.18) and (2.19). \square

Our next lemma is a modification of the previous one for a three-term difference operator

$$L = a(x)(T^2 - 1) + b(x)(T^{-2} - 1) . \quad (2.21)$$

Now a and b may have simple poles at $x = 0, -1$ and at $x = 0, 1$, respectively, with no other poles allowed at $x \in \mathbb{Z}$. Further, we fix an integer m as before and impose the following conditions on a, b (in case $m > 1$):

$$\begin{aligned} \operatorname{res}_{x=0}(a+b) &= 0, \quad \operatorname{res}_{x=-1}(a) + \operatorname{res}_{x=1}(b) = 0, \\ a(j) &= b(-j) \quad \text{for } j = 1, \pm 2, \dots, \pm m, \\ a(m) &= a(m-1) = 0. \end{aligned}$$

In case $m = 1$ these conditions must be replaced by the following:

$$\begin{aligned} \operatorname{res}_{x=0}(a) = \operatorname{res}_{x=0}(b) &= 0, \quad \operatorname{res}_{x=-1}(a) + \operatorname{res}_{x=1}(b) = 0, \\ a(1) &= b(-1) = 0. \end{aligned}$$

Lemma 2.3. Under the assumptions above, the operator (2.21) preserves the ring (2.20): $L(\mathfrak{R}_0) \subseteq \mathfrak{R}_0$.

Proof. Take any $f \in \mathfrak{R}_0$, then the only possible poles of Lf are $x = \pm 1$ and $x = 0$ (if $m > 1$). We have :

$$\operatorname{res}_{x=0}(Lf) = \operatorname{res}_{x=0}(a)[f(2) - f(0)] + \operatorname{res}_{x=0}(b)[f(-2) - f(0)] .$$

This is zero for $m > 1$ because $f(2) = f(-2)$ and $\operatorname{res}_{x=0}(a+b) = 0$. Further,

$$\begin{aligned} \operatorname{res}_{x=-1}(Lf) &= \operatorname{res}_{x=-1}(a)[f(1) - f(-1)] = 0, \\ \operatorname{res}_{x=1}(Lf) &= \operatorname{res}_{x=1}(b)[f(-1) - f(1)] = 0. \end{aligned}$$

So, Lf has no singularities at $x \in \mathbb{Z}$.

Now let us check that Lf still belongs to \mathfrak{R}_0 , i.e. $(T^j - T^{-j})Lf|_{x=0} = 0$ for $j = 1, \dots, m$. A straightforward calculation gives us the following:

$$\begin{aligned} (T^j - T^{-j})Lf &= a(x+j)[f(x+j+2) - f(x+j)] + b(x+j)[f(x+j-2) - f(x+j)] \\ &\quad - a(x-j)[f(x-j+2) - f(x-j)] - b(x-j)[f(x-j-2) - f(x-j)]. \end{aligned}$$

For $j > 1$ each term in this expression can be evaluated at $x = 0$:

$$\begin{aligned} (T^j - T^{-j})Lf|_{x=0} &= a(j)[f(j+2) - f(j)] + b(j)[f(j-2) - f(j)] \\ &\quad - a(-j)[f(-j+2) - f(-j)] - b(-j)[f(-j-2) - f(-j)] \\ &= a(j)[f(j+2) - f(-j-2) - f(j) + f(-j)] \\ &\quad + b(j)[f(j-2) - f(-j+2) - f(j) + f(-j)]. \end{aligned}$$

Here we used the conditions $a(j) = b(-j)$ and $a(-j) = b(j)$. The resulting expression for all $j = 1, \dots, m$ will be zero due to the properties (2.20) of f and the condition $a(m) = a(m-1) = 0$.

Finally, for $j = 1$ we have:

$$\begin{aligned} (T - T^{-1})Lf &= a(x+1)[f(x+3) - f(x+1)] \\ &\quad - b(x-1)[f(x-3) - f(x-1)] \\ &\quad + (a(x-1) + b(x+1))[f(x-1) - f(x+1)]. \end{aligned}$$

The last term is zero at $x = 0$ because $\text{res}_{x=0}(a(x-1) + b(x+1))$ is zero and $f(1) = f(-1)$. The first two terms after evaluating at $x = 0$ give

$$a(1)[f(3) - f(1)] - b(-1)[f(-3) - f(-1)].$$

This is zero for $m = 1, 2$ since $a(1) = b(-1) = 0$ in this case, and for $m > 2$ since $a(1) = b(-1)$ and $f(3) - f(-3) = f(1) - f(-1) = 0$. \square

The following two lemmas are a direct corollary of the previous two.

Lemma 2.4. Suppose that a difference operator L of the form (2.16) is invariant under the change of variable $x \rightarrow -x$, i.e. $b(x) = a(-x)$. Further, let us suppose that

- (1) a, b have simple poles at $x = 0$ and no other poles at $x \in \mathbb{Z}$,
- (2) $a(m) = 0$.

Then $L(\mathfrak{R}_0) \subseteq \mathfrak{R}_0$.

Lemma 2.5. Suppose that a difference operator L of the form (2.21) is invariant under the change of variable $x \rightarrow -x$, i.e. $b(x) = a(-x)$. Further, let us suppose that

- (1) a, b have simple poles at $x = 0, -1$ and $x = 0, 1$, respectively, and no other poles at $x \in \mathbb{Z}$,
- (2) $a(m) = a(m-1) = 0$.

(In case $m = 1$ we replace it by the condition $a(1) = 0$ but now allow poles at $x = \pm 1$ only.)

Then $L(\mathfrak{R}_0) \subseteq \mathfrak{R}_0$.

One can formulate the following inversions of lemmas 2.2 and 2.3.

Lemma 2.6. Suppose we are in a situation described in lemma 2.2. Moreover, let us impose extra conditions on the coefficients a, b of the operator (2.16) as follows:

$$\begin{aligned} \operatorname{res}_{x=0}(a) &\neq 0, \\ a(j) &\neq 0 \quad \text{for } j = 1, \dots, m-1 \quad (\text{only in case } m > 1). \end{aligned}$$

Let f be an analytic eigenfunction for L : $Lf = \lambda f$. Then f belongs to the ring \mathfrak{R}_0 .

Lemma 2.7. Suppose we are in a situation described in lemma 2.3. Let us impose extra conditions on the coefficients a, b of the operator (2.21) as follows:

$$\begin{aligned} \operatorname{res}_{x=0}(a) &\neq 0 \quad (\text{only in case } m > 1), \\ \operatorname{res}_{x=-1}(a) &\neq 0, \\ a(j) &\neq 0 \quad \text{for } j = 1, \dots, m-2 \quad (\text{only in case } m > 2). \end{aligned}$$

Then each analytic eigenfunction f of the operator L must belong to the ring \mathfrak{R}_0 .

Proof. Both lemmas can be proven by reversing the arguments used to prove lemmas 2.2, 2.3. Indeed, since f is non-singular, $Lf = \lambda f$ must be non-singular, too. Looking at the residues of Lf , we obtain the condition $f(1) = f(-1)$. Thus, $Lf = \lambda f$ must also satisfy this condition, which gives more conditions on f , and so on. □

Let us apply all this to the difference operators

$$D_1 = \frac{q^{x-m} - q^{-x+m}}{q^x - q^{-x}} T + \frac{q^{x+m} - q^{-x-m}}{q^x - q^{-x}} T^{-1} \quad (2.22)$$

and

$$\begin{aligned} D_2 = & \frac{(q^{x-m} - q^{-x+m})(q^{x-m+1} - q^{-x+m-1})}{(q^x - q^{-x})(q^{x+1} - q^{-x-1})} (T^2 - 1) \\ & + \frac{(q^{-x-m} - q^{x+m})(q^{-x-m+1} - q^{x+m-1})}{(q^{-x} - q^x)(q^{-x+1} - q^{x-1})} (T^{-2} - 1). \end{aligned} \quad (2.23)$$

which are the Macdonald operators D^π in cases $R = A_1 = \{\pm 1\} \subset \mathbb{R}^1$ and $\pi = 1$ and 2, respectively. It is obvious that D_1 and D_2 satisfy the conditions of lemma 2.4 and 2.5 (provided that q is not a root of unity). Hence, they preserve the properties (2.20). Moreover, instead of $x = 0$ we may consider any point $x = x_0$ with $q^{x_0} - q^{-x_0} = 0$. Indeed, for such x_0 the symmetry $x \rightarrow 2x_0 - x$ still does not change the operators D_1 and D_2 (this reflects their invariance with respect to the corresponding affine Weyl group). So, after shifting the origin to x_0 one gets the operators with the same properties as in lemma 2.4 and 2.5. Thus, the operators D_1, D_2 preserve, in fact, a bigger ring, which is an affine version of the ring (2.20).

Namely, let us consider the ring of all analytic functions $f(x)$ with the following properties:

$$f(x+j) = f(x-j) \quad \text{for each } j = 1, \dots, m \quad \text{and } q^{2x} = 1. \quad (2.24)$$

The arguments above prove the following result.

Proposition 2.8. *The operators (2.22), (2.23) with $m \in \mathbb{Z}_+$ preserve the ring \mathfrak{R} of analytic functions with properties (2.24).*

This is, essentially, the rank-one case of Proposition 2.1. Now we are ready to prove it in full generality.

Proof of Proposition 2.1. We will only consider the operator D^π given by the formulas (2.2)–(2.3), since this covers the case (2.1), see Remark 1. Choose any $\alpha \in R$, then we should prove that D^π preserves the ring \mathfrak{R}_α of functions with the following properties:

$$f\left(x + \frac{1}{2}j\alpha^\vee\right) = f\left(x - \frac{1}{2}j\alpha^\vee\right) \quad \text{for } j = 1, \dots, m_\alpha \quad \text{and } q^{2(\alpha, x)} = 1. \quad (2.25)$$

Let $s = s_\alpha \in W$ be the reflection with respect to α . Since the orbit $W\pi$ of the coweight π is s -invariant, it splits into pairs τ, τ' with $\tau' = s(\tau)$ plus a number of s -invariant τ 's. This defines a splitting of D^π into a sum of difference operators of one of the following three types:

$$D_0 = a_\tau(x)(T^\tau - 1), \quad s(\tau) = \tau, \quad (2.26)$$

$$D_1 = a_\tau(x)(T^\tau - 1) + a_{\tau'}(x)(T^{\tau'} - 1), \quad \tau' = \tau - \alpha^\vee, \quad (2.27)$$

$$D_2 = a_{\alpha^\vee}(T^{\alpha^\vee} - 1) + a_{-\alpha^\vee}(T^{-\alpha^\vee} - 1) \quad (\text{only for quasiminuscule } \pi). \quad (2.28)$$

This follows directly from the fact that π is (quasi)minuscule. Moreover, since D^π was obviously W -invariant, each of D_i will be invariant under the reflection s . In particular, $a_\tau(x) = a_\tau(sx)$ in D_0 , $a_{\tau'}(x) = a_\tau(sx)$ in D_1 , and $a_{-\alpha^\vee}(x) = a_{\alpha^\vee}(sx)$ in D_2 .

We claim that each of D_i preserves the ring (2.25). First, note that in case of D_0 a_τ has no pole at $q^{2(\alpha, x)} = 1$ since it is s -invariant. So, a_τ itself belongs to the ring \mathfrak{R}_α . Also it is clear that operator $T^\tau - 1$ preserves this ring (the shift acts in direction, orthogonal to α). Hence, $D_0(\mathfrak{R}_\alpha) \subset \mathfrak{R}_\alpha$.

Now let us consider D_1 , it has the form

$$a_\tau(x)T^\tau + a_{\tau'}(x)T^{\tau'} - a_\tau(x) - a_{\tau'}(x).$$

The sum $a_\tau + a_{\tau'}$ is s -invariant, hence, it is nonsingular at $q^{2(\alpha, x)} = 1$ and belongs to \mathfrak{R}_α . So, we may ignore it and consider the first two terms only. Further, note that we can present τ, τ' as

$$\tau = \frac{1}{2}\alpha^\vee + v, \quad \tau' = -\frac{1}{2}\alpha^\vee + v$$

for a certain v such that $(\alpha, v) = 0$. Hence, T^v will preserve the ring \mathfrak{R}_α and we can also ignore it, reducing D_1 to

$$a_\tau T^{\frac{1}{2}\alpha^\vee} + a_{\tau'} T^{-\frac{1}{2}\alpha^\vee}. \quad (2.29)$$

This operator is still s -invariant. Moreover, since $(\alpha, \tau) = 1$ in this case, we see from the formula (2.3) for a_τ that

$$a_\tau\left(x + \frac{1}{2}m_\alpha\alpha^\vee\right) = 0 \quad \text{for } q^{2(\alpha, x)} = 1.$$

Now in absolutely the same way as it was for the operator (2.22), we conclude that (2.29) preserves the ring \mathfrak{R}_α .

In the same manner the case of D_2 reduces to (2.23). \square

Remark. One can show that the ring \mathfrak{R} is finitely generated, therefore it can be viewed as the coordinate ring of a certain quite specific affine algebraic variety. For instance, for A_1 case it is a rational curve with m double points.

Remark. In our proof of Proposition 2.1 essential ingredients were the W -invariance of the operator D^π and specific location of poles and zeros of its coefficients. This has certain parallels with the *residue construction* of Hecke algebras from [GKV], [BEG]. Moreover, using the results from [BEG], one can prove that for $k_\alpha = -m_\alpha \in \mathbb{Z}_-$ all *Macdonald–Cherednik operators* D_1, \dots, D_n coming from W -invariant part of the double affine Hecke algebra [C1] will preserve the ring \mathfrak{R} . All these operators will commute with D^π . However, as we will see later, the centralizer of D^π in case $k_\alpha \in \mathbb{Z}_-$ is much bigger, and it contains many *non-symmetric* difference operators, preserving \mathfrak{R} .

One can formulate an analogue of Proposition 2.1 for the case of positive $k_\alpha \in \mathbb{Z}_+$, too. This is because these two cases are related through a simple gauge transformation. Let π be a (quasi)minuscule coweight and D_m denote the corresponding Macdonald operator (2.3) with $t = q^{-m}$. For $m_\alpha \in \mathbb{Z}_+$ introduce a function $\delta_m(x)$ as follows:

$$\delta_m(x) = \prod_{\alpha \in R_+} \prod_{j=-m_\alpha}^{m_\alpha} [(\alpha, x) + j], \quad (2.30)$$

where $[a]$ denotes $[a] = q^a - q^{-a}$. The following fact is well-known and can be checked by a direct calculation.

Lemma 2.9. For $m_\alpha \in \mathbb{Z}_+$ we have the following relation between the Macdonald operators D_m and $D_{m'}$ with $m' = -1 - m$:

$$\delta_m^{-1} \circ D_m \circ \delta_m = D_{m'}.$$

Corollary 2.10. Let $\mathfrak{R} = \mathfrak{R}_m$ be the ring (2.15). Then the Macdonald operator D , given by (2.2)–(2.3) with $t = q^{m+1}$, preserves the \mathfrak{R} -module $U = \delta_m^{-1}\mathfrak{R}$: $D(U) \subseteq U$.

3. BAKER–AKHIEZER FUNCTIONS FOR MACDONALD OPERATORS

We keep mostly the notations of the previous section. So, we consider an arbitrary (reduced, irreducible) root system R and fix a W -invariant set m of *multiplicities* $m_\alpha \in \mathbb{Z}_+$. Our purpose is to construct eigenfunctions of the corresponding Macdonald operators (2.2) with $t_\alpha = q^{-m_\alpha}$ (so $k_\alpha = -m_\alpha$ in notations of section 2.2). Keeping this in mind, we introduce $\rho = \rho(m)$ instead of $\rho = \rho_k = -\rho_m$ from (2.4):

$$\rho = \rho_m = \frac{1}{2} \sum_{\alpha \in R_+} m_\alpha \alpha. \quad (3.1)$$

We will also use its counterpart for the dual root system R^\vee :

$$\rho^\vee = \frac{1}{2} \sum_{\alpha \in R_+} m_\alpha \alpha^\vee. \quad (3.2)$$

(Warning: $\rho^\vee \neq 2\rho/(\rho, \rho)$!)

In this section we often will deal with functions of two variables $x, z \in V$. We will keep calling a sum $f(x) = \sum_{\nu \in P} q^{2(\nu, x)}$ a *polynomial* in x . However, switching to z , we will also switch from the root system R to its dual R^\vee . For instance, by a polynomial in z we will mean a sum $f(z) = \sum_{\nu \in P^\vee} q^{2(\nu, z)}$. Such (perhaps confusing) terminology is caused by our implicit identification of the vector space V and its dual V^* . To distinguish between these spaces, it would be natural to assume that $R \subset V^*$ and $R^\vee \subset V$. In this case x and z would lie in V and V^* , respectively. However, we prefer not to do this, keeping things simple.

We will apply the term *quasipolynomial* (in x or in z) to a function of the form $q^{2(x, z)}f$, where f is polynomial in x or in z , respectively.

For a polynomial $f(x) = \sum_{\nu \in P} a_\nu q^{2(\nu, x)}$ by its **support** $\text{supp}(f)$ we will always mean the **convex hull** of all points ν with $a_\nu \neq 0$. In the same way we define the support of $g(z) = \sum_{\nu \in P^\vee} a_\nu q^{2(\nu, z)}$. For a quasipolynomial in x of the form $\phi = q^{2(x, z)}f(x)$ by its support $\text{supp}(\phi)$ we will simply mean the support of $f(x)$. In case when ϕ is quasipolynomial in x and z at the same time, it usually will be clear which support we are considering (either in x or in z).

3.1. Baker–Akhiezer function: definition and uniqueness. Let $\psi(x, z)$ be a function of two variables $x, z \in V$ of the form

$$\psi = q^{2(x, z)} \sum_{\nu \in \mathcal{N}} \psi_\nu q^{2(\nu, z)}, \quad (3.3)$$

where the coefficients $\psi_\nu = \psi_\nu(x)$ depend on x , (x, z) is the scalar product in V and the summation in (3.3) is taken over all coweights $\nu \in P^\vee$ from the following polytope \mathcal{N} :

$$\mathcal{N} = \left\{ \nu = \rho^\vee - \sum_{\alpha \in R_+} l_\alpha \alpha^\vee \mid 0 \leq l_\alpha \leq m_\alpha \right\}. \quad (3.4)$$

Using our conventions about terminology, these conditions on a function ψ can be rephrased as follows: ψ is quasipolynomial in z with $\text{supp}(\psi) \subseteq \mathcal{N}$.

Suppose that ψ satisfies also the following conditions: for each $\alpha \in R$ and $s = 1, \dots, m_\alpha$

$$\psi \left(x, z + \frac{1}{2}s\alpha \right) \equiv \psi \left(x, z - \frac{1}{2}s\alpha \right) \quad \text{for } q^{2(\alpha^\vee, z)} = 1. \quad (3.5)$$

Definition. A function $\psi(x, z)$ with the properties (3.3)–(3.5) is called a **Baker–Akhiezer (BA) function** associated to the data $\{R, m\}$.

Our terminology is motivated by the fact that in case $R = A_1$ such a ψ is a Krichever’s Baker–Akhiezer function [Kr1, Kr2] associated to a specific singular rational curve. In contrast with the one-dimensional case, in higher dimension the main problem is to prove the existence of such a function. We do this in the next subsection. Let us presume now that such a ψ *does exist*.

Proposition 3.1. *Properties (3.3)–(3.5) determine ψ uniquely up to a factor depending on x .*

Proof is based on the following two lemmas.

Lemma 3.2. Let a quasipolynomial in z $\psi(x, z) = q^{2(x, z)} \sum_{\nu \in P^\vee} \psi_\nu q^{2(\nu, z)}$ satisfy the conditions (3.5). Then for each $\alpha^\vee \in R^\vee$ and for any $\nu \in P^\vee$ the set of integers j such that $\psi_{\nu+j\alpha^\vee} \neq 0$ either is empty or contains at least two integers j_1, j_2 with $|j_1 - j_2| \geq m_\alpha$.

Lemma 3.3. Let l_1, \dots, l_r be a set of non-parallel segments in affine Euclidean space V and $\Omega \subset V$ be a convex domain in V . Suppose that for each l_i and for any line l , which intersects Ω and is parallel to l_i , the intersection $\Omega \cap l$ has the length greater or equal than $|l_i|$. Then Ω can be presented as $\Omega = \Omega' \# \mathcal{N}$ for some convex domain Ω' , where $\mathcal{N} = l_1 \# l_2 \# \dots \# l_r$. Here $\#$ denotes the Minkowski addition in V .

We recall that the Minkowski sum of two subsets A, B of an affine space V is formed by all the points $a + b$, where a and b run over A and B , respectively. The addition of points, of course, depends on the choice of origin, but the resulting set will be the same up to a shift. This operation is relevant to the multiplication of polynomials: if f_1, f_2 are two polynomials in x , and $N_i = \text{supp}(f_i)$ then $N = \text{supp}(f_1 f_2)$ is the Minkowski sum of N_1 and N_2 .

Proof of Lemma 3.2. For a given $\alpha \in R_+$, substitution of the ψ into (3.5) gives the following set of relations:

$$\sum_{\nu \in P^\vee} \psi_\nu q^{2(\nu, z)} \left(q^{(s\alpha, x+\nu)} - q^{-(s\alpha, x+\nu)} \right) = 0 \quad \text{for} \quad q^{2(\alpha^\vee, z)} = 1$$

($s = 1, \dots, m_\alpha$).

These relations split up into separate linear equations for each "α-string" $\nu_j = \nu_0 + j\alpha^\vee$ ($j \in \mathbb{Z}$):

$$\sum_j \psi_j \left((q_j)^s - (q_j)^{-s} \right) = 0 \quad (s = 1 \dots m_\alpha), \quad (3.6)$$

where $\psi_j := \psi_{\nu_0 + j\alpha^\vee}$ and $q_j := q^{2j} q^{(\alpha, x + \nu_0)}$.

Suppose now that among the coefficients ψ_j only $\psi_1, \dots, \psi_{m_\alpha}$ do not vanish. In this situation we would have a homogeneous linear system of m_α equations for m_α unknowns ψ_j . Thus, it would be sufficient to show that this system is non-degenerate for generic x . To see this, we can look at the asymptotic behaviour of its determinant at large x (cf. [ES]). More precisely, we consider the corresponding matrix $A = (a_{ij})_{i,j=1, \dots, m_\alpha}$ with $a_{ij} = (q_j)^i - (q_j)^{-i}$ where $q_j := q^{2j} q^{(\alpha, x + \nu_0)}$. Then for large x such that $q^{(\alpha, x + \nu_0)} \gg 1$ the determinant $\det A$ asymptotically equals the Vandermonde determinant $\det(q_j^i)$ which is nonzero since q is not a root of unity.

Thus, the system is non-degenerate for generic x and all ψ_j must vanish, which proves the lemma. \square

Proof of Lemma 3.3. We will prove the lemma by induction in the number s of the segments. For an easier reference to the assumptions from the lemma, let us say in such a situation that Ω **dominates** over the segments l_1, \dots, l_s .

First, suppose we have just one segment l_1 and a convex domain Ω which dominates over l_1 . Then $\Omega = l_1 \# \Omega'$, where Ω' is the intersection $\Omega \cap T(\Omega)$ of Ω and its

image under the shift T for a vector \vec{l}_1 , associated with the segment l_1 (in either of two possible directions). This proves the lemma in case $s = 1$.

Now suppose that Ω dominates over l_1, \dots, l_s . Take the first segment l_1 and consider the convex domain Ω' constructed above, so we have $\Omega = l_1 \# \Omega'$. We claim that Ω' still dominates over l_2, \dots, l_s .

To prove this, let us take, for instance, l_2 and choose any line l , parallel to it. Now we may consider a two-dimensional section of the Ω passing through l and parallel to l_1 . The resulting two-dimensional domain will, obviously, dominate over l_1 and l_2 . Thus, essentially, we need to check the lemma in dimension two, for $s = 2$. This is very simple, and we leave it to the reader.

So, we have proved that Ω' dominates over l_2, \dots, l_s . Now the statement of the lemma follows by an obvious induction. \square

Proof of Proposition 3.1. First, notice that the polytope \mathcal{N} in (3.4) is exactly the Minkowski sum of the segments $m_\alpha \alpha^\vee$ with $\alpha^\vee \in R_+^\vee$ (abusing notation, we denote a vector and associated segment by the same symbol). Now let ψ be any BA function. By definition, we have an inclusion

$$\text{supp}(\psi) \subseteq \mathcal{N}.$$

On the other hand, Lemma 3.2 implies that the polytope $\text{supp}(\psi)$ must dominate over each of the segments $m_\alpha \alpha^\vee, \alpha^\vee \in R_+^\vee$. Hence, by Lemma 3.3 it must contain (a copy of) the polytope (3.4) which is their Minkowski sum. Altogether this proves that for each (nonzero) BA function ψ one has the equality

$$\text{supp}(\psi) = \mathcal{N}.$$

Let now ψ', ψ'' be two Baker–Akhiezer functions. Consider their linear combination $\psi = \psi' - c(x)\psi''$, which still satisfies the conditions (3.3)–(3.5). We can choose $c(x)$ in such way that the resulting function ψ will have zero coefficient at one of the vertices of the $\text{supp}(\psi') = \text{supp}(\psi'') = \mathcal{N}$. So, we will have a strict inclusion $\text{supp}(\psi) \subset \mathcal{N}$. Thus, the only possibility is that such ψ is zero, hence, $\psi' = c(x)\psi''$. \square

Corollary 3.4. *For a BA function ψ , the nonzero coefficients ψ_ν in (3.3) can appear only for $\nu = \rho^\vee - \sum_{\alpha \in R_+} l_\alpha \alpha^\vee$ with **integer** l_α . In other words, the summation in (3.3) is taken effectively only over the set $\rho^\vee + Q^\vee \subset P^\vee$.*

Proof. Suppose we have other terms, then let us remove them from the sum (3.3). This would not affect the conditions (3.5). Indeed, in the process of proving lemma 3.2 we saw that these conditions split into separate linear equations involving ν 's from the same coset in P^\vee/Q^\vee . But from the uniqueness of ψ it follows that the resulting function must remain the same. Hence, there were no other terms at all. \square

3.2. Existence of BA function. Comparison of Proposition 2.1 and conditions (3.5) suggests the idea to use a Macdonald operator acting in the z -variable in order to construct a Baker–Akhiezer function ψ . Let $\omega \in P$ be a (quasi)minuscule *weight* for the system R and D_z^ω be the Macdonald operator (2.2) corresponding to the *dual system* R^\vee and acting in the z -variable:

$$D_z^\omega = \sum_{\tau \in W_\omega} a_\tau (T_z^\tau - 1) + \sum_{\tau \in W_\omega} q^{-2(\rho^\vee, \tau)}, \quad (3.7)$$

where ρ^\vee is given by (3.2) and

$$a_\tau(z) = \prod_{\substack{\alpha \in R: \\ (\alpha, \tau) > 0}} \frac{[(\alpha^\vee, z) - m_\alpha]}{[(\alpha^\vee, z)]} \prod_{\substack{\alpha \in R: \\ (\alpha^\vee, \tau) = 2}} \frac{[(\alpha^\vee, z) - m_\alpha + 1]}{[(\alpha^\vee, z) + 1]}, \quad (3.8)$$

where $[a] = q^a - q^{-a}$.

Introduce the ring \mathfrak{R}^\vee which is a counterpart of the ring (2.15) and consists of all polynomials $f(z) = \sum_{\nu \in P^\vee} f_\nu q^{2(\nu, z)}$ with the following properties: for each $\alpha \in R_+$ and $j = 1, \dots, m_\alpha$

$$f\left(z + \frac{1}{2}j\alpha\right) \equiv f\left(z - \frac{1}{2}j\alpha\right) \quad \text{for } q^{2(\alpha^\vee, z)} = 1. \quad (3.9)$$

Then, according to Proposition 2.1, the operator D_z^ω will preserve the ring \mathfrak{R}^\vee : $D_z^\omega(\mathfrak{R}^\vee) \subseteq \mathfrak{R}^\vee$.

Now we need one technical lemma which shows that the action of D_z^ω on \mathfrak{R}^\vee is "lower-triangular".

Lemma 3.5. Let $D = D_z^\omega$ be the Macdonald operator (3.7). Suppose that both f and $\tilde{f} = Df$ are polynomials in z :

$$f(z) = \sum_{\nu \in P^\vee} f_\nu q^{2(\nu, z)}, \quad \tilde{f}(z) = \sum_{\nu \in P^\vee} \tilde{f}_\nu q^{2(\nu, z)}. \quad (3.10)$$

Then $\text{supp}(\tilde{f}) \subseteq \text{supp}(f)$. Further, let λ be a vertex of the polytope $N = \text{supp}(f)$, then the ratio $c_\lambda = \tilde{f}_\lambda / f_\lambda$ of the corresponding coefficients in (3.10) can be calculated as follows. First, choose generic $v \in V$ such that $(v, \lambda) \geq (v, \nu)$ for all $\nu \in N$ and put

$$\rho_\lambda^\vee = \frac{1}{2} \sum_{\alpha \in R: (\alpha, v) > 0} m_\alpha \alpha^\vee.$$

Then one has $c_\lambda = \sum_{\tau \in W_\omega} q^{2(\tau, \lambda - \rho_\lambda^\vee)}$.

This can be proven similar to the proof of (2.9) and (2.13) in [M1, M2]. A key point is that inclusion $A\#C \subseteq B\#C$ for convex domains A, B, C implies $A \subseteq B$.

A similar result is true for quasipolynomials. Recall that for a quasipolynomial $\phi = q^{2(x, z)} f(z)$ its support, by our conventions, coincides with the support of f . Thus, we have the following analog of the lemma above.

Lemma 3.6. Suppose that both ϕ and $\tilde{\phi} = D\phi$ are quasipolynomials in z :

$$\phi(z) = q^{2(x, z)} \sum_{\nu \in P^\vee} \phi_\nu q^{2(\nu, z)}, \quad \tilde{\phi}(z) = q^{2(x, z)} \sum_{\nu \in P^\vee} \tilde{\phi}_\nu q^{2(\nu, z)}. \quad (3.11)$$

Then $\text{supp}(\tilde{\phi}) \subseteq \text{supp}(\phi)$. Further, let λ be a vertex of the polytope $N = \text{supp}(\phi)$, then the ratio $c_\lambda = \tilde{\phi}_\lambda / \phi_\lambda$ of the corresponding coefficients in (3.11) can be calculated as

$$c_\lambda = \sum_{\tau \in W_\omega} q^{2(\tau, x + \lambda - \rho_\lambda^\vee)}, \quad (3.12)$$

where ρ_λ^\vee is defined in the lemma above.

Now everything is ready to construct a BA function. The idea is very simple. We start from the quasipolynomial $\phi = q^{2(x, z)} Q(z)$, where

$$Q(z) = q^{2(\rho^\vee, z)} \prod_{\alpha \in R_+} \prod_{j=1}^{m_\alpha} [(\alpha^\vee, z) + j][(\alpha^\vee, z) - j], \quad (3.13)$$

where $[a]$, as usual, denotes $q^a - q^{-a}$. This polynomial is especially chosen to guarantee that ϕ satisfies the conditions (3.9) in z . Thus, applying $D = D_z^\omega$ successively to ϕ , we will always get a quasipolynomial in z which will still satisfy the conditions (3.9) in z . Let us apply at each step an operator $D - c_i$, so $\phi_{i+1} = (D - c_i)\phi_i$, $\phi_0 = \phi$. The coefficients c_i will be adjusted to reduce $\text{supp}(\phi_i)$, see below. Finally, ψ will be obtained after repeating this sufficiently many times. Before proceeding with more details, let us make one more remark. Notice that the formula (3.13) implies that the only nonzero terms in the initial quasipolynomial $\phi = q^{2(x, z)} \sum_{\nu \in P^\vee} \phi_\nu q^{2(\nu, z)}$ are those with $\nu \in \rho^\vee + Q^\vee \subset P^\vee$. To see this, one should rewrite each factor $[(\alpha^\vee, z) + j]$ in (3.13) as $q^{(\alpha^\vee, z)}(q^j - q^{-j-2(\alpha^\vee, z)})$. The same will be true for all successive functions ψ_i . One can see this directly after rewriting in a similar way the coefficients (3.8) of the difference operator D .

Now let us make everything more concrete. At the beginning we have

$$N_0 = \text{supp}(\phi_0) = \{\nu = \rho^\vee + \sum_{\alpha \in R_+} l_\alpha \alpha^\vee \mid -m_\alpha \leq l_\alpha \leq m_\alpha\}.$$

Further, as we mentioned already, each of ϕ_i will satisfy the conditions (3.5). Hence, due to lemmas 3.2 and 3.3, its support $N_i \subseteq N_0$ must be a union of several copies of the polytope \mathcal{N} given by (3.4). Let us fix generic v lying inside the positive Weyl chamber C . The linear functional (v, \cdot) determines the *height function* on V . We will call a vertex of a polytope the *highest* (respectively, the *lowest*) vertex if it has maximal (respectively, minimal) height among all vertices. At each step we will choose the highest vertex λ of the polytope $N_i = \text{supp}(\phi_i)$. For brevity, let us call the corresponding coefficient ϕ_λ the *highest coefficient* of ϕ . Note that the initial polytope N_0 (as well as all successive N_i) will be composed of the images of \mathcal{N} under some of the shifts by $\nu = \sum_{\alpha \in R_+} l_\alpha \alpha^\vee$ with $l_\alpha = 0, \dots, m_\alpha$. If we look at the smaller polytope \mathcal{N} , then its highest (resp. lowest) vertex will be $\lambda = \rho^\vee$ (resp. $\lambda = -\rho^\vee$). Hence, the highest vertex of the polytope N_i must be of the form

$$\lambda = \rho^\vee + \nu, \quad \nu = \sum_{\alpha \in R_+} l_\alpha \alpha^\vee, \quad l_\alpha = 0, \dots, m_\alpha. \quad (3.14)$$

Now we can kill the highest coefficient ϕ_λ by applying $D - c_i$ where $c_i = c_\lambda$ is given by (3.12). Note that the vector ρ_λ^\vee in the formula (3.12) will be simply ρ^\vee (because λ is the highest vertex). Thus, $c_i = c_\lambda$ will be $c_i = \sum_{\tau \in W_\omega} q^{2(\tau, x + \nu)}$, or simply

$$c_i = \mathbf{m}_\omega(x + \nu)$$

in terms of the orbitsum \mathbf{m}_ω ,

$$\mathbf{m}_\omega(x) = \sum_{\tau \in W_\omega} q^{2(\tau, x)}. \quad (3.15)$$

On the other hand, let us look now what is happening with the lowest coefficient $\phi_{-\rho^\vee}$. According to Lemma 3.6, after application of $D - c_i$ it gets the factor $c_{-\rho^\vee} - c_i$. Note that for $\lambda = -\rho^\vee$ the vector ρ_λ^\vee in the formula (3.12) will be simply $-\rho^\vee$. So, the formula (3.12) gives us:

$$c_{-\rho^\vee} = \sum_{\tau \in W_\omega} q^{2(\tau, x)} = \mathbf{m}_\omega(x).$$

Obviously, $c_{-\rho^\vee} - c_i = \mathbf{m}_\omega(x) - \mathbf{m}_\omega(x + \nu)$ is nonzero as soon as $\nu \neq 0$ (recall that we assume that q is not a root of unity).

These considerations imply the existence of BA functions for all root system, which is one of our main results.

Theorem 3.7. *Let $D = D_z^\omega$ denote the Macdonald operator (3.7). Define $\psi(x, z)$ as follows:*

$$\psi = \prod_{\nu} (D - \mathbf{m}_\omega(x + \nu)) \left[q^{2(x, z)} Q(z) \right], \quad (3.16)$$

in accordance with the formulas (3.13), (3.15), where the product is taken over all $\nu \neq 0$ having the form $\nu = \sum_{\alpha \in R_+} l_\alpha \alpha^\vee$ with $l_\alpha = 0, \dots, m_\alpha$. Then

- (i) ψ has the form (3.3)–(3.4);
- (ii) the coefficient $\psi_{-\rho^\vee}$ in its expansion (3.3) equals

$$\psi_{-\rho^\vee} = \prod_{\nu} (\mathbf{m}_\omega(x) - \mathbf{m}_\omega(x + \nu)) \neq 0; \quad (3.17)$$

- (iii) ψ is a Baker–Akhiezer function for the system R with multiplicities $m = \{m_\alpha\}$;
- (iv) as a function of z , ψ is an eigenfunction of the Macdonald operator D :

$$D\psi = \mathbf{m}_\omega(x)\psi.$$

Proof. As we explained above, the constructed function (3.16) will be a quasipolynomial in z satisfying the conditions (3.5). The part (ii) follows immediately from the construction of ψ . It implies that the polytope $\text{supp}(\psi)$ contains the polytope (3.4). On the other hand, the arguments above show that the highest vertex of $\text{supp}\psi$ must be $\lambda = \rho^\vee$, because every "higher" term $\psi_{\rho^\vee + \nu}$ has been killed after applying $D - \mathbf{m}_\omega(x + \nu)$. Altogether, this gives us that $\text{supp}(\psi)$ coincides with the polytope \mathcal{N} in (3.4). Thus, the part (i) is also proven.

Part (iii) of the theorem follows from the previous two and the remark above that ψ satisfies conditions (3.5). Finally, $\tilde{\psi} = (D - \mathbf{m}_\omega(x))\psi$ must be quasipolynomial in z with all the properties (3.3)–(3.5). Hence, it must be proportional to ψ due to Proposition 3.1. However, application of $D - \mathbf{m}_\omega(x)$ kills the highest coefficient ψ_{ρ^\vee} , so $\tilde{\psi}$ must be zero, which proves the last part. \square

Remark. Formula (3.16) is a trigonometric version of a related formula from [Ch1], which, in its turn, is a discrete version of the Berest's formula [Be].

4. BISPECTRAL DUALITY

In this section we will explain how one should normalize BA function to achieve a certain symmetry between x and z variables. We start by looking closely at the rank one case.

4.1. A_1 case. For the rank-one case the existence of a BA function is a very simple fact, since in this case the number of "free" parameters equals the number of conditions (3.5). Let us consider the root system $R = \{\pm 2\} \subset \mathbb{R}$ with $m_\alpha = m \in \mathbb{Z}_+$. It is convenient to fix a scalar product on $V = \mathbb{R}$ as $(u, v) = \frac{1}{2}uv$. In this case we will have $R = R^\vee$, $Q = Q^\vee = 2\mathbb{Z}$ and $P = P^\vee = \mathbb{Z}$. We will fix $R_+ = \{2\}$, so $\rho = \rho^\vee = m$. In accordance with (3.3), the Baker–Akhiezer function ψ depends on two scalar variables x, z and has the following form:

$$\psi = q^{xz} \sum_{\nu=-m}^m \psi_\nu q^{\nu z}. \quad (4.1)$$

By definition, it must satisfy the following conditions:

$$(T_z^j - T_z^{-j}) \psi = 0 \quad \text{for each } j = 1, \dots, m \quad \text{and} \quad q^{2z} = 1. \quad (4.2)$$

Similarly to (3.6), these conditions lead to the following linear system for the coefficients ψ_ν :

$$\sum_{j=0}^m a_{ij} \psi_{-m+2j} = 0, \quad i = 1 \dots m, \quad (4.3)$$

where $a_{ij} = (q_j)^i - (q_j)^{-i}$ with $q_j := q^{-m+2j+x}$. Introduce $m \times (m+1)$ matrix $A = (a_{ij})$, then the linear system above takes the form $Av = 0$, where v is the column $v = (\psi_{-m}, \psi_{-m+2}, \dots, \psi_m)$. We know already that for generic x the matrix A has the maximal rank (equal to m), hence, its kernel is one-dimensional. Using Cramer's rule, we find the values of ψ_ν (up to a common factor):

$$\psi_{-m+2s} = (-1)^j \det A^{(s)},$$

where $A^{(s)}$ is obtained from A by deleting its s -th column. This gives us the ratio ψ_{-m+2s}/ψ_{-m} as

$$\psi_{-m+2s}/\psi_{-m} = (-1)^s \det A^{(s)} / \det A^{(0)}.$$

To calculate this explicitly, we use the following lemma.

Lemma 4.1. For arbitrary q_1, \dots, q_n consider the matrix $A = (a_{ij})_{i,j=1 \dots n}$ whose entries are $a_{ij} = (q_j)^i - (q_j)^{-i}$. Then

$$\det A = \prod_{i < j} \left(q_i^{1/2} q_j^{1/2} - q_i^{-1/2} q_j^{-1/2} \right) \left(q_i^{-1/2} q_j^{1/2} - q_i^{1/2} q_j^{-1/2} \right) \prod_i (q_i - q_i^{-1}).$$

Expanding the determinant, it is easy to see that this formula is equivalent to the Weyl denominator formula for the C_n -root system. There is also a simple direct way of proving it, using that $\frac{q^i - q^{-i}}{q - q^{-1}}$ is polynomial in $q + q^{-1}$.

Applying the lemma, we calculate the determinants and find that

$$\frac{\psi_{-m+2s}}{\psi_{-m}} = \frac{(q_0 - q_0^{-1}) \prod_{j=1}^{m_\alpha} \left(q_0^{1/2} q_j^{1/2} - q_0^{-1/2} q_j^{-1/2} \right) \left(q_0^{-1/2} q_j^{1/2} - q_0^{1/2} q_j^{-1/2} \right)}{(q_s - q_s^{-1}) \prod_{\substack{j \geq 0 \\ j \neq s}} \left(q_s^{1/2} q_j^{1/2} - q_s^{-1/2} q_j^{-1/2} \right) \left(q_s^{-1/2} q_j^{1/2} - q_s^{1/2} q_j^{-1/2} \right)}.$$

Substituting $q_j = q^{-m+2j+x}$ we arrive after simple transformations at the formula

$$\frac{\psi_{-m+2s}}{\psi_{-m}} = \prod_{j=1}^s \frac{(q^{-m+j-1} - q^{m-j+1})(q^{-m+j-1+x} - q^{m-j+1-x})}{(q^j - q^{-j})(q^{j+x} - q^{-j-x})}. \quad (4.4)$$

In particular, for $s = m$ we have :

$$\psi_m/\psi_{-m} = \prod_{j=1}^m \frac{q^{j-x} - q^{-j+x}}{q^{j+x} - q^{-j-x}}. \quad (4.5)$$

Let us fix ψ_m in the following form:

$$\psi_m = \prod_{j=1}^m (q^{j-x} - q^{-j+x}).$$

Then the relation (4.5) gives us that

$$\psi_{-m} = \prod_{j=1}^m (q^{j+x} - q^{-j-x}). \quad (4.6)$$

Note that our choice of ψ_m implies that all ψ_ν will be Laurent polynomials in q^x (see formula (4.4)).

This allows us to prove the following proposition.

Proposition 4.2. (i) The function $\psi(x, z)$ given by (4.1), (4.4) and (4.6) is a Baker–Akhiezer function for $R = A_1$;
(ii) ψ satisfies the difference equation $L\psi = (q^z + q^{-z})\psi$, where L coincides with the operator (2.22):

$$L = \frac{q^{x-m} - q^{-x+m}}{q^x - q^{-x}} T_x + \frac{q^{x+m} - q^{-x-m}}{q^x - q^{-x}} T_x^{-1};$$

(iii) ψ is symmetric in x and z : $\psi(x, z) = \psi(z, x)$.

Proof. Part (i) is proven above. To prove (ii) we apply the standard argument due to Krichever [Kr1]. Namely, let us consider the function $\phi = L\psi - (q^z + q^{-z})\psi$. The first remark is that ϕ still satisfies the conditions (4.2): $(T_z^j - T_z^{-j})\phi = 0$ for each $j = 1, \dots, m$ and $q^{2z} = 1$. Indeed, it is obvious for $L\psi$ since the operator L does not involve z . Further, $f(z) = q^z + q^{-z}$ satisfies the conditions (4.2), hence, $f(z)\psi$ will satisfy them, too.

Our second remark is that ϕ can be presented as follows:

$$\phi = q^{xz} \sum_{j=0}^{m+1} \phi_j q^{(-m-1+2j)z}. \quad (4.7)$$

This follows directly from the formula (4.1) and our definition of ϕ .

Let us calculate now the coefficient ϕ_0 using (4.6) and the definition of ϕ . This gives:

$$\phi_0 = \frac{q^{x+m} - q^{-x-m}}{q^x - q^{-x}} \psi_{-m}(x-1) - \psi_{-m}(x) = 0.$$

In the same way, $\phi_{m+1} = 0$. Thus, the expansion (4.7) contains m terms only, hence, it must be zero due to Lemma 3.2. This proves part (ii).

To prove part (iii), first notice that according to (4.4) and (4.6) ψ has no singularities in the x -variable and it may be presented as

$$\psi = q^{xz} \sum_{\nu=-m}^m a_\nu(z) q^{\nu x}. \quad (4.8)$$

We know that, as a function of x , ψ is an eigenfunction of the operator (2.22). Invoking Lemma 2.6, we conclude that ψ satisfies the following conditions in x :

$$(T_x^j - T_x^{-j}) \psi = 0 \quad \text{for each } j = 1, \dots, m \text{ and } q^{2x} = 1. \quad (4.9)$$

Thus, $\psi(x, z)$ must coincide with $\psi(z, x)$ up to a z -depending factor: $\psi(x, z) = F(z)\psi(z, x)$. Switching x and z , we conclude that $\psi(z, x) = G(x)\psi(x, z)$. This implies that $F = G^{-1}$ is constant. Expanding the coefficients $\psi_\nu(x)$ with the help of the formulas (4.4) and (4.6), we see that $\psi(x, z)$ contains the term

$$(-1)^m q^{-m(m+1)} q^{xz} q^{mx+ mz}.$$

Since this term is symmetric in x and z , we conclude that $F = 1$ and $\psi(x, z) = \psi(z, x)$. □

4.2. Normalized BA-function. Now we are going to extend the results of the previous subsection to the higher rank case. Above we have proved that a Baker–Akhiezer function ψ is determined uniquely (up to a x -depending factor) by its properties (3.3)–(3.5). Let us impose the following normalization condition on ψ , prescribing its leading coefficient ψ_{ρ^\vee} to be the following:

$$\psi_{\rho^\vee} = \prod_{\alpha \in R_+} \prod_{j=1}^{m_\alpha} [j - (\alpha, x)], \quad [a] := q^a - q^{-a}. \quad (4.10)$$

Definition. A **normalized BA function** is a (unique) function $\psi(x, z)$ with the properties (3.3)–(3.5) and normalization (4.10).

Let us discuss briefly the geometry of the polytope $\mathcal{N} = \text{supp}(\psi)$ defined by (3.4). We mentioned already that \mathcal{N} is the Minkowski sum of the segments associated with the vectors $m_\alpha \alpha^\vee$ ($\alpha^\vee \in R_+^\vee$). It is convenient to use a more symmetric definition of \mathcal{N} , which is obviously equivalent to (3.4):

$$\mathcal{N} = \left\{ \nu = \sum_{\alpha \in R_+} l_\alpha \alpha^\vee \mid -\frac{1}{2} m_\alpha \leq l_\alpha \leq \frac{1}{2} m_\alpha \right\}. \quad (4.11)$$

To better understand its structure, let us choose a generic direction v in V and consider the height function (v, \cdot) on V . Then the highest and the lowest among the points of \mathcal{N} will be the points $\frac{1}{2} \sum_{\alpha \in R_+} \pm m_\alpha \alpha^\vee$, where the signs in the sum either coincide with the signs of (α^\vee, v) , or are exactly the opposite. This shows that the vertices of \mathcal{N} have the form $\frac{1}{2} \sum_{\alpha: (\alpha, v) > 0} m_\alpha \alpha^\vee$, where v is a generic vector. For instance, taking v from the *positive Weyl chamber* C , i.e. such that $(v, \alpha) > 0$ for all $\alpha \in R_+$, we obtain ρ^\vee as one of the vertices of \mathcal{N} . Other vertices will correspond to other chambers and will be of the form $w\rho^\vee$ with $w \in W$. If we take now the vertex ρ^\vee , then its adjacent vertices will correspond to the Weyl chambers, adjacent to the positive one. There are exactly n of them, $n = \text{rank} R$, and they are the images $s_i C$ of C under the simple reflections s_1, \dots, s_n . It is known

that a simple reflection s_i leaves the set $R_+ \setminus \alpha_i$ invariant, while sending α_i to $-\alpha_i$. Thus, $s_i \rho^\vee = \rho^\vee - m_i \alpha_i^\vee$, $m_i := m_{\alpha_i}$. By the way, this implies that $(\alpha_i, \rho^\vee) = m_i$, i.e. ρ^\vee can be rewritten in terms of the fundamental coweights b_i as

$$\rho^\vee = \sum_{i=1}^n m_i b_i, \quad m_i = m_{\alpha_i}. \quad (4.12)$$

So, the edges of \mathcal{N} coming out from the vertex ρ^\vee are given by the vectors $-m_i \alpha_i^\vee$ where $\alpha_1, \dots, \alpha_n$ are the simple roots from R_+ . In the same way, the vectors $-m_i w \alpha_i^\vee$ will lead from the vertex $w \rho^\vee$ to its adjacent vertices. Summarizing, we arrive at the following result.

Lemma 4.3. The vertices of the polytope (4.11) have the form $w \rho^\vee$, $w \in W$. If ν, ν' are two adjacent vertices, then ν' equals $\nu + m_\alpha \alpha^\vee$ for a proper $\alpha \in R$. Moreover, in this case we have $\nu' = s_\alpha \nu$ and $(\alpha, \nu') = -(\alpha, \nu) = m_\alpha$.

Our next proposition justifies our choice of normalization (4.10).

Proposition 4.4. *The normalized BA function ψ has the following properties:*

(i) for all $w \in W$ the coefficient ψ_ν in the vertex $\nu = w \rho^\vee$ of the polytope \mathcal{N} has the form

$$\psi_{w \rho^\vee} = \prod_{\alpha \in w R_+} \prod_{j=1}^{m_\alpha} [j - (\alpha, x)]; \quad (4.13)$$

(ii) for all ν lying on the (one-dimensional) edges of \mathcal{N} the corresponding ψ_ν are polynomial in x ;

(iii) ψ is quasipolynomial in both x and z .

Proof. Let ν and ν' be two adjacent vertices of \mathcal{N} , so for a proper $\alpha \in R$ we have $\nu' = s_\alpha \nu = \nu + m_\alpha \alpha^\vee$. Let us introduce $\psi_j = \psi_{\nu + j \alpha^\vee}$ and $q_j = q^{2j - m_\alpha + (\alpha, x)}$ ($j = 0, 1, \dots, m_\alpha$). Introduce also a matrix A with entries $a_{ij} = (q_j)^i - (q_j)^{-i}$ ($1 \leq i \leq m_\alpha, 0 \leq j \leq m_\alpha$). In these notations, the conditions (3.5) can be expressed as $Av = 0$, where v is the column $v = (\psi_0, \dots, \psi_{m_\alpha})$. This system is completely analogous to the system (4.3). Repeating the same arguments, we arrive at the formula

$$\psi_s / \psi_0 = \prod_{j=1}^s \frac{[-m_\alpha + j - 1][(\alpha, x) - m_\alpha + j - 1]}{[j][j + (\alpha, x)]}, \quad (4.14)$$

where $[a]$ denotes $q^a - q^{-a}$. In particular, for $s = m_\alpha$ we have the following expression for the ratio of the coefficients ψ_ν and $\psi_{\nu'}$ at two adjacent vertices of the polytope \mathcal{N} :

$$\psi_{\nu'} / \psi_\nu = \prod_{j=1}^{m_\alpha} \frac{[j - (\alpha, x)]}{[j + (\alpha, x)]}. \quad (4.15)$$

Recall now that ν, ν' can be presented as $w \rho^\vee, w' \rho^\vee$ for proper $w, w' \in W$. Moreover, we have $w' = s_\alpha w$, and $\pm \alpha$ are simple roots in the sets $w' R_+$ and $w R_+$, respectively. This means that if we denote the common part of the sets $w R_+, w' R_+$ by S , then $w R_+ = S \cup \{-\alpha\}$ and $w' R_+ = S \cup \{\alpha\}$. Now the formula (4.13) in question gives us that

$$\psi_{w' \rho^\vee} / \psi_{w \rho^\vee} = \prod_{j=1}^{m_\alpha} \frac{[j - (\alpha, x)]}{[j + (\alpha, x)]}.$$

Since this expression coincides with the formula (4.15), and since (4.13) is valid at one vertex $\nu = \rho^\vee$, we conclude that it must be valid at all other vertices of \mathcal{N} . This proves part (i) of the proposition.

The second part follows directly from the formulas (4.13) and (4.14). To prove part (iii), we have to show that all coefficients ψ_ν are polynomial in x . The BA function in formula (3.16) is clearly quasipolynomial in x , so we only have to prove the absence of singularities in the normalized BA function ψ .

Suppose ψ has a pole along a hypersurface $F(x) = 0$. Without loss of generality, we may assume that it is irreducible. Multiplying ψ by $F(x)$ we will obtain the function $\tilde{\psi}$ which will be quasipolynomial in x sharing with ψ the properties (3.3)–(3.5). Now take generic point x_0 on this hypersurface, so $F(x_0) = 0$. Then all the coefficients $\tilde{\psi}_\nu = F(x)\psi_\nu$ for ν lying on the edges of the polytope \mathcal{N} must vanish at $x = x_0$ (because ψ_ν are polynomial, see part (ii)). However, $\tilde{\psi}$ does not vanish at x_0 (otherwise ψ would be nonsingular at $F = 0$). Let us denote by $\tilde{\mathcal{N}}$ the support of $\tilde{\psi}(x_0, z)$ (i.e. the convex hull of all ν with nonzero $\tilde{\psi}_\nu(x_0)$). We have, therefore, a strict inclusion

$$\tilde{\mathcal{N}} \subsetneq \mathcal{N}. \quad (4.16)$$

Moreover, this shows that the polytope $\tilde{\mathcal{N}}$ has no vertices lying on the edges of \mathcal{N} . This will lead us to a contradiction in a moment.

Indeed, take any $\alpha \in R$ and consider the function $f(x) = q^{2(\alpha, x)}$ restricted to the hypersurface $H : F = 0$. Since H is irreducible, f either is constant along H or takes infinitely many different values. Suppose now that for every $\alpha \in R$ the latter alternative holds. Then we would be able to choose generic $x_0 \in H$ such that our proof of lemma 3.2 would work, i.e. such that all $m_\alpha \times m_\alpha$ determinants arising from considering α -strings inside the polytope \mathcal{N} would be nonzero. However, this would imply lemma 3.3 and this would contradict the inclusion (4.16).

The upshot is that for some $\alpha \in R$ the function $q^{2(\alpha, x)}$ is constant along our hypersurface H . But in this case we can choose generic $x_0 \in H$ such that lemma 3.2 is still applicable to all roots except α . Hence, the support $\tilde{\mathcal{N}}$ of $\tilde{\psi}$ due to lemma 3.3 must contain a polytope \mathcal{N}' , which by definition is the Minkowski sum of all $m_\beta \beta^\vee$ ($\alpha \neq \beta \in R_+$). However, $\mathcal{N} = \mathcal{N}' \# m_\alpha \alpha^\vee$. This means that each copy of \mathcal{N}' lying inside \mathcal{N} , has at least one vertex on the edges of \mathcal{N} , but $\tilde{\mathcal{N}}$ has no vertices on the edges of \mathcal{N} . This contradiction proves that ψ has no singularities, therefore, it is quasipolynomial in x . \square

So, our choice of normalization (4.10) guarantees that ψ will be quasipolynomial in x . In a certain sense, it is the "minimal" quasipolynomial BA function.

Corollary 4.5. *Any BA function $\psi(x, z)$ which is quasipolynomial in x is the normalized BA function multiplied by some polynomial in x .*

This follows immediately from the formula (4.13) which shows that all the coefficients $\psi_{w\rho^\vee}$ have no common divisors (as polynomials in x).

4.3. Duality. A remarkable and important property of the normalized Baker–Akhiezer function is a certain duality between x and z variables. Namely, it turns out that switching x and z leads to the BA function for the *dual* root system. In particular, for A, D, E root systems we have simply $\psi(x, z) = \psi(z, x)$. The next proposition is the main step in establishing this symmetry.

Proposition 4.6. *The normalized BA function ψ satisfies the following conditions in x : for each $\alpha \in R_+$ and $s = 1, \dots, m_\alpha$*

$$\psi\left(x + \frac{1}{2}s\alpha^\vee, z\right) \equiv \psi\left(x - \frac{1}{2}s\alpha^\vee, z\right) \quad \text{for } q^{2(\alpha, x)} = 1. \quad (4.17)$$

Proof. Take any $\alpha \in R$ and consider two adjacent vertices ν, ν' of the polytope \mathcal{N} such that $\nu' = \nu + m_\alpha\alpha^\vee$. As we know, ν and ν' can be presented as $w\rho^\vee, w'\rho^\vee$ for proper $w, w' \in W$. Moreover, denoting the set $w'R_+ \cup wR_+$ by S , we will have:

$$wR_+ = S \cup \{-\alpha\}, \quad w'R_+ = S \cup \{\alpha\}, \quad s_\alpha(S) = S.$$

Consider the function $\tilde{\psi} = \left(T_x^{\frac{1}{2}s\alpha^\vee} - T_x^{-\frac{1}{2}s\alpha^\vee}\right)\psi$. Similar to ψ , it will be quasipolynomial in z ,

$$\tilde{\psi} = q^{2(x, z)} \sum_{\tau \in P^\vee} a_\tau q^{2(\tau, z)}, \quad (4.18)$$

and it is clear that its support lies inside the polytope

$$\tilde{\mathcal{N}} = \mathcal{N}' \# l, \quad (4.19)$$

where l is the segment with the endpoints at $\nu - \frac{1}{2}s\alpha^\vee$ and $\nu' + \frac{1}{2}s\alpha^\vee$ and \mathcal{N}' , in its turn, is the Minkowski sum of all $m_\beta\beta^\vee$ with $\beta \in S$ (we recommend the reader to draw a picture for A_2 case).

Now take x_0 such that $q^{2(\alpha, x_0)} = 1$. We would like to prove that for such a x_0 $\tilde{\psi}$ will be zero. Suppose that it is not the case. We may assume that x_0 is generic enough, i.e. such that we still can apply Lemma 3.2 to all $\beta \in S$. This, together with Lemma 3.3, implies that $\text{supp}(\tilde{\psi})$ can be presented as $\mathcal{N}' \# \Omega$ for some domain Ω . Comparing with (4.19) gives us that $\Omega \subset l$. This implies that at least one of the coefficients a_τ in (4.18) with $\tau \in l$ must be nonzero. So, we will arrive at a contradiction as soon as we prove that all a_τ for $\tau \in l$ vanish. We will see in a moment that this essentially reduces the problem to the rank-one case.

Indeed, notice that $q^{2(x, z)} \sum_{\tau \in l} a_\tau q^{2(\tau, z)}$, by its construction, coincides with $\left(T_x^{\frac{1}{2}s\alpha^\vee} - T_x^{-\frac{1}{2}s\alpha^\vee}\right)\psi^{(0)}$, where

$$\psi^{(0)} := q^{2(x, z)} \sum_{\tau \in [\nu, \nu']} \psi_\tau q^{2(\tau, z)}$$

is the part of ψ , corresponding to the edge $[\nu, \nu']$ of the polytope \mathcal{N} . We recall that we consider $x = x_0$ such that $q^{2(\alpha, x)} = 1$.

So, we need to show that

$$\left(T_x^{\frac{1}{2}s\alpha^\vee} - T_x^{-\frac{1}{2}s\alpha^\vee}\right)\psi^{(0)} = 0 \quad \text{for } q^{2(\alpha, x)} = 1. \quad (4.20)$$

If our $\psi^{(0)}$ were the normalized BA function for the root system $R^{(0)} = \{\pm\alpha\}$, this would follow directly from Proposition 4.2(ii). This is almost the case, as the formulas (4.14)–(4.15) clearly indicate. The only difference is that, due to (4.13), the coefficient ψ_ν with $\nu = w\rho^\vee$ looks as follows:

$$\psi_\nu = \prod_{\beta \in S} \prod_{j=1}^{m_\beta} [j - (\beta, x)] \prod_{j=1}^{m_\alpha} [(\alpha, x) + j]$$

(here we used that $wR_+ = S \cup \{-\alpha\}$). So, the difference with the similar formula (4.6) for the rank one case comes from the factor $\prod_{\beta \in S} \prod_{j=1}^{m_\beta} [j - (\beta, x)]$. However, this factor is invariant under reflection s_α , hence, it does not affect the properties (4.20). \square

Next thing to prove is that the support of ψ in the x -variable is the following polytope \mathcal{N}^\vee :

$$\mathcal{N}^\vee = \left\{ \nu = \rho - \sum_{\alpha \in R_+} l_\alpha \alpha \mid 0 \leq l_\alpha \leq m_\alpha \right\}. \quad (4.21)$$

To prove this we look first at the formula (3.16), which gives an expression for another BA function with ψ_{ρ^\vee} normalized as in (3.17). It is clear from (3.16) that such ψ is also quasipolynomial in x with $\text{supp}_x \psi = \text{supp}_x \psi_{\rho^\vee}$. If we renormalize now ψ in order to get ψ_{ρ^\vee} as in (4.10), then we will still have that the support of ψ equals the support of ψ_{ρ^\vee} which now will coincide with \mathcal{N}^\vee . So, for the normalized BA function $\psi(x, z)$ one has $\text{supp}_x \psi = \mathcal{N}^\vee$.

As a result, we see that the normalized BA function can be presented in a form

$$\psi = q^{2(x,z)} \sum c_{\mu\nu} q^{2(\mu,x)} q^{2(\nu,z)}$$

with the summation taken over $\mu \in \mathcal{N}^\vee$ and $\nu \in \mathcal{N}$, respectively. The highest term in this expression will correspond to $\mu = \rho$, $\nu = \rho^\vee$.

Summarizing, we see that the properties of $\psi(x, z)$ in the x -variable are completely analogous to its properties in z . Thus, we obtain the following important property of the normalized BA function which reflects the symmetry between x and z variables.

Theorem 4.7 (Duality). *Let $\psi(x, z)$ denote the normalized Baker–Akhiezer function associated to a root system R with multiplicities $m = \{m_\alpha\}$, and $\psi^\vee(x, z)$ denote a similar function associated to the dual system R^\vee with $m_{\alpha^\vee} = m_\alpha$. Then $\psi(x, z) = \psi^\vee(z, x)$.*

Corollary 4.8. *Let ψ be the normalized BA function for a given root system R and $m = \{m_\alpha\}$. Let $\omega \in P$ and $\pi \in P^\vee$ be (quasi)minuscule weight and coweight for the root system R . Then $\psi(x, z)$ solves the following bispectral system of difference equations:*

$$\begin{cases} D_z^\omega \psi = \mathbf{m}_\omega(x) \psi, \\ D_x^\pi \psi = \mathbf{m}_\pi(z) \psi. \end{cases} \quad (4.22)$$

Here D_x^π and D_z^ω are the corresponding Macdonald operators (2.2)-(2.3) and (3.7)-(3.8), while \mathbf{m}_λ stands for the orbitsum (2.8).

5. ALGEBRAIC INTEGRABILITY AND APPLICATIONS

5.1. Algebraic integrability. Above we have shown that the normalized Baker–Akhiezer function $\psi(x, z)$ associated to a datum $\{R, m\}$ is an eigenfunction of the Macdonald operators D_x^π , where π is any (quasi)minuscule coweight for the root system R . In fact, ψ is a common eigenfunction of a much bigger commutative ring of difference operators. This follows in a standard way from its analytic properties in the z -variable (cf. [Kr1]). To formulate the result, let us recall the definition of the ring \mathfrak{R}^\vee which consists of all polynomials $f(z)$ of the form $f(z) = \sum_{\nu \in P^\vee} f_\nu q^{2(\nu,z)}$,

satisfying conditions (3.9) for each $\alpha \in R$ and $s = 1, \dots, m_\alpha$. It is easy to see that all W -invariant polynomials belong to the ring \mathfrak{R}^\vee . According to Chevalley theorem, W -invariants form a polynomial ring with $n = \text{rank}R$ generators m_{b_1}, \dots, m_{b_n} , which are the orbitsums for the fundamental coweights b_1, \dots, b_n . However, the ring \mathfrak{R}^\vee is much bigger, for instance, it contains the principal ideal generated by the polynomial Q given by (3.13).

Theorem 5.1 (Algebraic integrability). *For each polynomial $f(z)$ from the ring \mathfrak{R}^\vee there exists a difference operator D_f in x on the lattice P^\vee such that $D_f\Psi = f(z)\Psi$. All these operators commute. For a (quasi)minuscule coweight π of R and $f = \mathfrak{m}_\pi(z)$ the corresponding operator D_f coincides with the Macdonald operator D^π given in (2.1)–(2.3).*

Proof. Everything is based on the following result.

Lemma 5.2. Any quasipolynomial in z of the form $\phi = q^{2(x,z)} \sum_{\nu \in P^\vee} \phi_\nu(x) q^{2(\nu,z)}$ which satisfies the conditions (3.5) can be obtained by applying a proper difference operator in x to the BA function $\psi(x, z)$.

To prove the lemma, we recall that according to lemmas 3.2 and 3.3 the support of ϕ can be presented as $N_0 \# \mathcal{N}$, where \mathcal{N} is the support of ψ and N_0 is some convex polytope. Now choose any vertex ν of the $\text{supp}\phi$ and let τ be the corresponding vertex of N_0 such that $\nu \in \tau + \mathcal{N} \subseteq \text{supp}\phi$. We can "kill" the coefficient ϕ_ν of ϕ by subtracting a function $T_x^\tau \psi$ taken with a proper coefficient $d(x)$. The resulting function $\tilde{\phi} = \phi - d(x)T_x^\tau \psi$ still satisfies the conditions of the lemma, but it has a smaller support. Repeating this, we will eventually get zero, and this proves the lemma.

Now, to prove the theorem, we notice that for $f \in \mathfrak{R}^\vee$ the function $\phi = f(z)\psi$ will satisfy the conditions of the lemma, hence, $\phi = D_f \psi$ for a proper difference operator $D_f = \sum_{\nu \in P^\vee} d_\nu T_x^\nu$. All these operators commute since they have ψ as their eigenfunction. Indeed, for $f, g \in \mathfrak{R}^\vee$ we have $[L_f, L_g]\psi = (fg - gf)\psi = 0$. However, one shows easily that if a difference operator M in x annihilates a (nonzero) quasipolynomial ψ in z , then $M = 0$ (otherwise $\text{supp}M\psi$ would be nonempty). Hence, $[L_f, L_g] = 0$.

Finally, for $f(z) = \mathfrak{m}_\pi(z)$ and the corresponding D_f we will have $D_f \psi = \mathfrak{m}_\pi(z)\psi$. The same is true for the Macdonald operator D_x^π : $D_x^\pi \psi = \mathfrak{m}_\pi(z)\psi$. Hence, these two operators coincide. \square

Remark. The duality between x and z implies that the normalized BA function will be also a common eigenfunction of a "dual" commutative ring of difference operators in z variable, isomorphic to the ring \mathfrak{R} of polynomials with the properties (2.15). Thus, we have a *bispectral pair* of commutative rings, in the spirit of [W].

Remark. In the limit $q \rightarrow 1$ this theorem reduces to the result from [CSV], where the algebraic integrability was established for the quantum trigonometric Calogero–Sutherland–Moser problem and, more generally, for its generalizations [OP] related to the root systems. Proof in [CSV] was based on results by Heckman and Opdam [HO, H], who developed a nice theory of multivariable hypergeometric functions

related to root systems. An independent direct proof was obtained in [Ch1]. For A_{n-1} case, i.e. for the Macdonald–Ruijsenaars operators (2.7) with $t = q^m$ ($m \in \mathbb{Z}$) the algebraic integrability was established by Etingof–Styrkas in [ES]. They constructed ψ in terms of certain intertwiners from representation theory for quantum groups. Their formula for ψ was made more explicit by Felder and Varchenko in [FV].

5.2. Liouville integrability. As a part of the previous theorem, we have constructed $n = \text{rank} R$ commuting difference operators D_i corresponding to the basic W -invariants $f = \mathfrak{m}_{b_i}(z)$, which are the orbitsums (2.8) for the fundamental coweights $b_i \in P^\vee$. Those of b_i which are (quasi)minuscule, will lead to the Macdonald operators (2.1), (2.2) with $t_\alpha = q^{-m_\alpha}$. For others, there is no such simple explicit formula. However, it is easy to evaluate leading terms in D_i and they look similar to the leading terms in Macdonald operators.

Proposition 5.3. *For any coweight $\pi \in P^\vee$ and its orbitsum $f = \mathfrak{m}_\pi(z)$ the operator D_f constructed in Theorem 5.1 will have the form*

$$D_f = \sum_{\tau \in W\pi} a_\tau T^\tau + \dots,$$

where the dots stand for a sum of the terms $a_\nu T^\nu$ with $\nu \in \pi + Q^\vee$ lying inside the convex hull of the orbit $W\pi$, and the leading coefficients a_τ are given by (2.5), (2.6) with $k_\alpha = -m_\alpha$. The operator D_f is W -invariant.

Proof. Formula for a_τ follows directly from the construction of D_f in theorem (5.1) and formula (4.13) for the leading coefficients in ψ . It remains to prove that D_f is W -invariant. To this end we have the following symmetry property of ψ .

Lemma 5.4. The normalized BA function is W -invariant in the following sense: $\psi(wx, wz) = \psi(x, z)$ for any $w \in W$.

To prove this we notice that for the leading coefficients (4.13) of ψ one has $\psi_{w\rho^\vee}(wx) = \psi_{\rho^\vee}(x)$. Since $\tilde{\psi}(x, z) := \psi(wx, wz)$ shares with ψ the properties (3.3)–(3.5), they must coincide due to the uniqueness of ψ .

Using the lemma and W -invariance of $f(z)$, one gets that $[D\psi](wx, wz) = f(wz)\psi(wx, wz) = [D\psi](x, z)$. From this it easily follows the property $a_{w\nu}(wx) = a_\nu(x)$ for the coefficients of D , i.e. its W -invariance. \square

Let us look what happens if we change the multiplicities m_α . The formula (2.6) shows that the leading coefficient a_π in D_f will be rational in $t = \{t_\alpha\}$. In fact, it is not difficult to prove that all the coefficients of the operator D_f with $f = \mathfrak{m}_\pi(z)$ will be rational in t .

Lemma 5.5. All the operators D_f depend rationally on t .

Proof. Recall that the construction of D_f was given in Lemma 5.2. Now let $f(z) = \mathfrak{m}_\pi(z)$ be an orbitsum, $\pi \in P_+$. We can construct D_f as in Lemma 5.2, starting from $\phi = f(z)\psi$ and "killing" at each step the highest coefficient of ϕ . This shows that to calculate D_f in this case it is sufficient to know a fixed number of the coefficients ψ_ν in the normalized BA function ψ , namely, those with $\nu - \rho^\vee \in -\pi + \overline{W\pi}$. Thus, if

all these coefficients were rational in t , then D_f would be rational in t , too. This is not the case, however, because already ψ_{ρ^\vee} is not rational in t . Let us renormalize ψ in such a way that $\psi_{\rho^\vee} = 1$, and let us calculate D_f using this $\tilde{\psi}$. Of course, the resulting operator \tilde{D}_f will differ from D_f , but the relation is simple:

$$\tilde{D}_f = \Delta \circ D_f \circ \Delta^{-1},$$

where Δ is

$$\Delta = \prod_{\alpha \in R_+} \prod_{j=1}^{m_\alpha} \left(q^{-(\alpha, x)} - q^{-2j+(\alpha, x)} \right)^{-1}, \quad (5.1)$$

which is a specialization of (2.5) in case $t = q^{-m}$. An important point is that despite the fact that Δ is not rational in t , a ratio

$$\Delta(x)/\Delta(x + \nu)$$

is rational in t for any $\nu \in P^\vee$. Therefore, it is sufficient to prove that all the coefficients of the operator \tilde{D}_f depend rationally on t .

Summarizing, we showed that it is enough to check that for any fixed $\tau \in Q_+^\vee$ the coefficient $\tilde{\psi}_\nu$ with $\nu = \rho^\vee - \tau$ will be rational in t . Recall now that the renormalized $\tilde{\psi}$, as well as ψ itself, satisfies a difference equation in the z -variable:

$$D_z^\omega \tilde{\psi} = \mathbf{m}_\omega(x) \tilde{\psi},$$

derived in Theorem 3.7. Using this equation, one can calculate the coefficients of $\tilde{\psi}$ recursively, starting from $\tilde{\psi}_{\rho^\vee}$ (one gets this recursion similar to [HO], expanding the coefficients of the operator D_z^ω in a series in z). Since the Macdonald operator is polynomial in t , each $\tilde{\psi}_\nu$ with $\nu = \rho^\vee - \tau$, calculated from this recursion, will be polynomial in t . This completes the proof. \square

As a corollary, we obtain that all the operators D_i related to the fundamental coweights, must commute for *all values* of t (since they commute for $t = q^{-m}$ with any integer m). Let us consider one of the Macdonald operators as a Hamiltonian of the corresponding quantum problem, which can be viewed as a generalization of the trigonometric Ruijsenaars problem to all root systems. Then what we just proved is the *complete (Liouville) integrability* of this quantum problem. So, in this way we recover the result obtained by Cherednik [C1]:

Corollary 5.6. *A quantum problem related to an arbitrary Macdonald operator is completely integrable.*

5.3. Shift operators. Another important result by Cherednik is the construction of the so-called *shift operators*, which are q -versions of the operators constructed by Opdam for the case $q = 1$. Let us explain how they appear in our approach.

Let us consider a root system R and two sets of *integer* multiplicities m, \hat{m} . Suppose that $m \geq \hat{m}$, i.e. $m_\alpha \geq \hat{m}_\alpha$ for all $\alpha \in R$. Denote by $\psi, \hat{\psi}$ corresponding normalized BA functions. Comparing their properties (3.5), we then apply Lemma 5.2 to conclude that there exists a difference operator S^+ in x such that $\psi = S^+ \hat{\psi}$. Let D, \hat{D} be the Macdonald operators (2.2) with $t = q^{-m}, \hat{t} = q^{-\hat{m}}$, so $D^\pi \psi = m_\pi \psi$ and $\hat{D}^\pi \hat{\psi} = \hat{m}_\pi \hat{\psi}$. Hence,

$$(D \circ S^+ - S^+ \circ \hat{D}) \hat{\psi} = D\psi - S^+(m_\pi \hat{\psi}) = 0.$$

Therefore, we arrive at the identity

$$D \circ S^+ = S^+ \circ \widehat{D}.$$

It means that the operator S^+ intertwines two Macdonald operators with different t . In the same way, S^+ intertwines each pair D_i, \widehat{D}_i of the corresponding operators from Proposition 5.3.

Consider now the case $\widehat{t} = qt$ (i.e. $\widehat{m}_\alpha = m_\alpha - 1$ for all α). We will use subscripts denoting by D_m the Macdonald operator with $t = q^{-m}$. So, we have shown that for any integer $m = \{m_\alpha\}$ we have two Macdonald operators D_m, D_{m-1} and their intertwiner S_m^+ with the relation:

$$D_m \circ S_m^+ = S_m^+ \circ D_{m-1}. \quad (5.2)$$

Similar to Lemma 5.5, one proves that the intertwiner S_m^+ will depend algebraically on $t = q^{-m}$, so the relation (5.2) will make sense for *any value* of m . Corresponding S_m^+ is called a *shift operator*, since it sends eigenfunctions of D_{m-1} to eigenfunctions of D_m . From its construction we can calculate easily the leading terms in S_m . Introduce $\varrho^\vee \in P^\vee$ as

$$\varrho^\vee = \frac{1}{2} \sum_{\alpha \in R_+} \alpha^\vee. \quad (5.3)$$

Proposition 5.7. *The shift operator S_m^+ above has the form*

$$S_m^+ = \sum_{\tau \in W_{\varrho^\vee}} b_\tau T^\tau + \dots,$$

where the dots stand for a sum of lower terms $b_\nu T^\nu$ with $\nu \in \varrho^\vee + Q^\vee$ lying inside the convex hull of the orbit W_{ϱ^\vee} . The leading coefficients b_τ are given by the formulas

$$b_{\varrho^\vee}(x) = q^{\sum_{\alpha > 0} m_\alpha} \frac{\Delta_{1-m}(x + \varrho^\vee)}{\Delta_{-m}(x)}, \quad b_{w\varrho^\vee}(x) = b_{\varrho^\vee}(w^{-1}x),$$

where ϱ is given by (5.3) and Δ_{-m} stands for the function (5.1). The operator S_m^+ is W -invariant and it sends the normalized BA function ψ_{m-1} to ψ_m .

For a proof one should compare the supports and the leading coefficients in two BA functions, ψ_m and ψ_{m-1} .

Another shift operator will appear if we consider, as above, two normalized BA functions ψ_m, ψ_{m-1} corresponding to $m = \{m_\alpha\}$ and $\widehat{m} = m - 1$, and multiply ψ_{m-1} by a polynomial $c_m(z)c_m(-z)$ where $c_m(z)$ is defined by the formula:

$$c_m(z) = \prod_{\alpha \in R_+} [m_\alpha - (\alpha^\vee, z)], \quad [a] := q^a - q^{-a}. \quad (5.4)$$

It is easy to see that the resulting function $\phi = c_m(z)c_m(-z)\psi_{m-1}$ will satisfy the conditions (3.5). Hence, due to Lemma 5.2, it has the form $S^- \psi_m$ for a proper difference operator $S^- = S_m^-$ (in the x -variable). Its leading terms, again, can be calculated easily. Similarly, we get the intertwining relation $D_{m-1} \circ S_m^- = S_m^- \circ D_m$ which, again, extends analytically to all values of m . Next proposition summarizes the properties of the constructed S^- .

Proposition 5.8. For integer $m = \{m_\alpha\}$ there exists a difference operator $S^- = S_m^-$ which intertwines the Macdonald operators D_m, D_{m-1} :

$$D_{m-1} \circ S_m^- = S_m^- \circ D_m.$$

It has the form $S^- = \sum_{\tau \in W_{\varrho^\vee}} b_\tau T^\tau + \dots$, where the dots stand for a sum of lower terms $b_\nu T^\nu$ with $\nu \in \varrho^\vee + Q^\vee$ lying inside the convex hull of the orbit W_{ϱ^\vee} . The leading coefficients b_τ are given by the formulas

$$b_{\varrho^\vee}(x) = (-1)^{|R_+|} q^{-\sum_{\alpha > 0} m_\alpha} \frac{\Delta_{-m}(x + \varrho^\vee)}{\Delta_{1-m}(x)}, \quad b_{w_{\varrho^\vee}}(x) = b_{\varrho^\vee}(w^{-1}x),$$

in accordance with (5.3), (5.1) (here $|R_+|$ denotes the total number of positive roots). Application of S_m^- to the normalized BA function ψ_m gives $S_m^- \psi_m = c_m(z) c_m(-z) \psi_{m-1}$, where c_m is defined in (5.4).

Remark. In case when R consists of two orbits, $R = R_1 \cup R_2$, one can apply similar arguments for the case when multiplicities increase for one of the orbits, i.e. $m = \widehat{m} + 1_i$ with 1_i being the characteristic function of $R_i \subset R$. This proves the existence of the intertwiners, shifting from m to $m \pm 1_i$. Corresponding versions of propositions 5.7, 5.8 are straightforward.

Remark. More generally, we can consider a pair of data $\{\widehat{R}, \widehat{m}\}$ and $\{R, m\}$ with $\widehat{R} \subset R$ and with $\widehat{m}_\alpha \leq m_\alpha$ for all $\alpha \in \widehat{R}$. This leads to the intertwiners between Macdonald operators for *different* root systems. Unlike the shift operators S_m above, they exist for *integer* values of m only. As an example, see [CV] where such an intertwiner was constructed explicitly for the case $q = 1$, and $\widehat{R} = A_n \subset R = A_{n+1}$.

We conclude the section by a description of all possible shift operators. First we need a result showing that the commutative ring from Theorem 5.1 is maximal in a certain sense.

Proposition 5.9. Let $D = D_x^\pi$ be a Macdonald operator related to a root system R and $t = q^{-m}$ with $m_\alpha \in \mathbb{Z}_+$ and let \mathfrak{R}^\vee be the ring of polynomials in z with the properties (3.9). Suppose that a difference operator L commutes with D and has rational coefficients. Then L is one of the operators constructed in Theorem 5.1, i.e. $L = D_f$ for some $f \in \mathfrak{R}^\vee$.

Proof. Suppose that $[L, D] = 0$ and consider a function $\Phi = L\psi$ where ψ is the normalized BA function. Under assumptions of the theorem, we have that $\Phi(x, z)$ is of the form $\Phi = q^{2(x, z)} \varphi$ where φ is polynomial in z and a rational function in x . From the commutativity of D and L we get that Φ is also an eigenfunction for the Macdonald operator D : $D\Phi = m_\pi(z)\Phi$. It is not difficult to deduce that in case $t_\alpha = q^{-m_\alpha}$ with $m_\alpha \in \mathbb{Z}_+$ any eigenfunction of $D = D^\pi$ either has infinitely many poles along hyperplanes of the form $q^{(\alpha, x)} = \text{const}$, or has no poles at all. By our assumptions, Φ cannot have an infinite number of poles, hence, it must be quasipolynomial in x . Then from the same equation, using lemma 3.6, we derive that $\text{supp}_x \Phi$ must coincide with the polytope (3.4). Further, with the help of lemmas 2.6, 2.7 we conclude that Φ satisfies the conditions (4.17). Hence, $\Phi(z, x)$ is a Baker–Akhiezer function for the dual root system R^\vee . Now corollary 4.5 implies that $\Phi(z, x) = f(x) \psi^\vee(x, z)$, where $\psi^\vee(x, z)$ is the normalized BA function for R^\vee

and f is some polynomial. Switching x, z and using duality, we obtain that Φ must be proportional to the normalized BA function: $\Phi(x, z) = f(z)\psi(x, z)$. Notice that since Φ was obtained from ψ applying a difference operator in x , it shares with ψ the properties (3.5) in the z -variable. Since this must be valid for any x , we obtain that f itself must be from the ring (3.9). Thus, f belongs to \mathfrak{R}^\vee and $L = D_f$. \square

Proposition 5.10. *For positive integer $m = \{m_\alpha\}$ let D_m and D_{m-1} be two Macdonald operators with $t = q^{-m}$ and $t = q^{-m+1}$, respectively. Let S be any difference operator in x with rational coefficients satisfying the intertwining relation $D_m \circ S = S \circ D_{m-1}$. Then $S = S^+ \circ L$ where S^+ is the shift operator from Proposition 5.7 and L commutes with D_{m-1} .*

Proof. Let ψ_m and ψ_{m-1} be the normalized BA functions for multiplicities m and $\hat{m} = m-1$, respectively. From the intertwining relation we obtain that $\tilde{\psi} := S\psi_{m-1}$ will satisfy the same difference equation as does ψ_m : $D_m \tilde{\psi} = m_\pi(z)\tilde{\psi}$. Due to Proposition 5.9, $\tilde{\psi}$ must have the form $f(z)\psi_m$ for some polynomial $f(z)$. The same arguments as before give that $f(z)$ must belong to the ring \mathfrak{R}^\vee related to $\hat{m} = m-1$. Hence, $S\psi_{m-1} = (S^+ \circ L)\psi_{m-1}$ where $L = D_f$ is such that $L\psi_{m-1} = f(z)\psi_{m-1}$. This implies that $S = S^+ \circ L$. \square

5.4. Relation to Macdonald polynomials. Let $\psi(x, z)$ be the normalized BA function constructed as above starting from the data $\{R, m\}$. Let us consider two functions Φ_\pm obtained from ψ by (anti)symmetrization in x :

$$\Phi_+ = \sum_{w \in W} \psi(wx, z), \quad \Phi_- = \sum_{w \in W} (-1)^w \psi(wx, z).$$

Notice that the same will be the result of (anti)symmetrization in z , due to Lemma 5.4. So, the resulting functions Φ_\pm are (anti)symmetric in z , too. In fact, the functions $\Phi_\pm(x, z)$ are closely related (for special $z \in P_+$) to the Macdonald polynomials $P_\lambda(x; q, t)$. Let $\delta = \delta_m(x)$ be the function defined by (2.30), that is,

$$\delta(x) = \prod_{\alpha \in R_+} \prod_{j=-m_\alpha}^{m_\alpha} [j + (\alpha, x)].$$

Recall also the definition (3.1) of $\rho = \rho_m$:

$$\rho_m = \frac{1}{2} \sum_{\alpha \in R_+} m_\alpha \alpha. \quad (5.5)$$

In particular, in case $m \equiv 1$ the vector ρ_1 coincides with ϱ from (5.3).

Theorem 5.11. *Let us substitute $z = \lambda \in P_+$ into Φ_\pm . Then*

$$\Phi_+(x, \lambda) = c(\lambda, m) P_{\lambda + \rho_m}(x; q, q^{-m})$$

and

$$\Phi_-(x, \lambda) = c(\lambda, m) \delta(x) P_{\lambda - \rho_{m+1}}(x; q, q^{m+1}),$$

where δ is defined above and the factor $c(\lambda, m)$ is given by the formula

$$c(\lambda, m) = \prod_{\alpha \in R_+} \prod_{j=1}^{m_\alpha} [j - (\alpha^\vee, \lambda)]. \quad (5.6)$$

We suppose that $(\lambda, \alpha^\vee) > m_\alpha$ for all $\alpha \in R_+$ to ensure that $c(\lambda, m) \neq 0$. In other words, $\lambda \in \rho_{m+1} + P_+$.

Proof. Let π be any (quasi)minuscule coweight of R and $D = D_x^\pi$ denote the corresponding Macdonald operator (2.2) with $t = q^{-m}$. Each of the functions $\psi(wx, \lambda) = \psi(x, w^{-1}\lambda)$ satisfies the same equation $D\psi = \mathfrak{m}_\pi(\lambda)\psi$, hence we have:

$$D\Phi_\pm = \mathfrak{m}_\pi(\lambda)\Phi_\pm \quad \mathfrak{m}_\pi(\lambda) = \sum_{\tau \in W\pi} q^{2(\tau, \lambda)}. \quad (5.7)$$

Using Lemma 2.9, we conclude that $p(x) = \Phi_+$ and $\widehat{p}(x) = \delta^{-1}\Phi_-$ will satisfy the equations

$$Dp = \mathfrak{m}_\pi(\lambda)p, \quad \widehat{D}\widehat{p} = \mathfrak{m}_\pi(\lambda)\widehat{p},$$

where D and \widehat{D} are the Macdonald operators (2.2) with $t = q^{-m}$ and $t = q^{m+1}$, respectively. These are exactly the defining equations (2.13) for the Macdonald polynomials $P_{\lambda+\rho_m}(x; q, q^{-m})$ and $P_{\lambda-\rho_{m+1}}(x; q, q^{m+1})$. So, to prove the theorem, we only have to check that p and \widehat{p} are symmetric polynomials and to calculate their leading terms.

The W -invariance of p, \widehat{p} is obvious since Φ_\pm is (anti)invariant. Calculating the leading term is quite straightforward since we know the leading terms in $\psi(x, \lambda)$. So the only non-trivial thing to prove is that Φ_- is divisible by $\delta(x)$. Here we can use the properties (4.17) of the BA function. Let us rewrite them after shifting in x by $-\frac{1}{2}s_\alpha^\vee$:

$$\psi(x, z) - \psi(x - j\alpha^\vee, z) = 0 \quad \text{for } q^{2(\alpha, x)} = q^{2j}, \quad j = 1, \dots, m_\alpha.$$

Choose now any $\alpha \in R$ and x such that $(\alpha, x) = j$, then its image $x' = s_\alpha x$ under reflection s_α will be $x' = x - j\alpha^\vee$, so according to the property above, we will have:

$$\psi(x, z) - \psi(s_\alpha x, z) = \psi(x, z) - \psi(x - j\alpha^\vee, z) = 0 \quad \text{for } (\alpha, x) = j.$$

Now let us split the sum $\Phi_-(x, z) = \sum_{w \in W} (-1)^w \psi(x, wz)$ into pairs of terms with $w, w' = s_\alpha w$. As a result, we obtain that

$$\Phi_-(x, z) = 0 \quad \text{for } (\alpha, x) = 1, \dots, m_\alpha.$$

Φ_- also vanishes for $(\alpha, x) = 0$ due to its antiinvariance. Invoking all $\alpha \in R$, we arrive at the following result.

Lemma 5.12. For any $\alpha \in R_+$ the function $\Phi_-(x, z) = \sum_{w \in W} (-1)^w \psi(x, wz)$ vanishes along the hyperplanes $(\alpha, x) = 0, \pm 1, \dots, \pm m_\alpha$.

This is true for any z . However, for $z = \lambda \in P$ the functions Φ_\pm are quasiperiodic with respect to the lattice $\mathcal{L} = \omega P^\vee$ where $\omega = \pi i(\log q)^{-1}$. Indeed, under a shift $x \rightarrow x + l$ ($l \in \mathcal{L}$) each of $\psi(x, w\lambda)$ ($w \in W$) gets the same factor as does the function $q^{(\lambda+\rho, x)}$. These translation properties imply that as soon as Φ_- vanishes for $2(\alpha, x) = 2j$, it will also vanish for $q^{2(\alpha, x)} = q^{2j}$. This proves that Φ_- for $z = \lambda \in P$ is divisible by the polynomial $\delta(x)$, thus, completing the proof of the theorem. □

Remark. The proof shows that $\Phi_-(x, \lambda)$ will be zero for all $\lambda \in P_+$ which do not belong to $\rho_{m+1} + P_+$. Indeed, in this case Φ_- must be divisible by $\delta(x)$ but has a smaller support.

Our expression for $P_{\lambda+\rho_m}(x; q, q^{-m})$ via the function Φ_+ is well-defined for sufficiently large λ , namely for $\lambda \in \rho_{m+1} + P_+$. For smaller λ it doesn't work, which reflects the known fact that in case $t = q^{-m}$ with $m_\alpha \in \mathbb{Z}_+$ some of P_μ are not well-defined. Let us call the Macdonald polynomials $P_\mu(x; q, q^{-m})$ with $(\mu, \alpha) > 2m_\alpha$ for all $\alpha > 0$ the *stable Macdonald polynomials*. They are always well-defined (if q is not a root of unity). The following "localization" property is a direct corollary of Theorem 5.11.

Corollary 5.13. *For $t = q^{-m}$ with $m_\alpha \in \mathbb{Z}_+$ the stable Macdonald polynomials $P_\lambda(x; q, q^{-m})$ are localized, i.e. in their expression (2.10) through the orbitsums*

$$P_\lambda = \mathbf{m}_\lambda + \sum_{\nu \prec \lambda} a_{\lambda\nu} \mathbf{m}_\nu,$$

only the terms with $\nu = \lambda - \sum_{\alpha \in R_+} l_\alpha \alpha$ and $l_\alpha = 0, 1, \dots, m_\alpha$ can appear, all other coefficients $a_{\lambda\nu}$ vanish. As a result, the total number of nonzero terms remains bounded as λ increases.

Below is a simple example which illustrates Theorem 5.11.

Example. Let us consider the case $R = A_1$ with $m = 1$. Using the results of section 4.1, we obtain (switching x, z) the following formula for the normalized BA function:

$$\psi(x, z) = (q^{1+z} - q^{-1-z}) q^{(z-1)x} + (q^{1-z} - q^{-1+z}) q^{(z+1)x}.$$

As a result, we get the following expressions for $\Phi_\pm = \psi(x, \lambda) \pm \psi(-x, \lambda)$:

$$\begin{aligned} \Phi_\pm(x, \lambda) &= (q^{1+\lambda} - q^{-1-\lambda}) \left(q^{(\lambda-1)x} \pm q^{(-\lambda+1)x} \right) \\ &\quad + (q^{1-\lambda} - q^{-1+\lambda}) \left(q^{(\lambda+1)x} \pm q^{(-\lambda-1)x} \right). \end{aligned}$$

Taking integer $\lambda \geq 2$, we obtain according to Theorem 5.11 the following formulas:

$$P_{\lambda+1}(x; q, q^{-1}) = (q^{1-\lambda} - q^{-1+\lambda})^{-1} \Phi_+ = \mathbf{m}_{\lambda+1} + \frac{[1+\lambda]}{[1-\lambda]} \mathbf{m}_{\lambda-1},$$

$$P_{\lambda-2}(x; q, q^2) = \frac{\Phi_-(x, \lambda)}{[1-\lambda][x-1][x][x+1]}.$$

Remark. In the case $m = 0$ the BA function is a pure exponential, $\psi = q^{2(x,z)}$. Theorem 5.11 reduces in this case to Weyl's character formula. Thus, one can think of Theorem 5.11 as a generalized Weyl formula with ψ being a "perturbed" exponent. Note that there is a similar result involving "non-symmetric" Macdonald polynomials (see [M3]). Our $\psi(x, \lambda)$ for $\lambda \in P$ is polynomial, too, but it differs from non-symmetric Macdonald polynomials. In fact, ψ is related to a natural q -analogue of the hypergeometric function by Heckman and Opdam [HO]. However, we will not discuss this relation here.

Remark. In case $R = A_n$ our formula for Φ_- is the formula conjectured by Felder and Varchenko [FV] and proved by Etingof and Styrkas [ES] in the context of the representation theory for quantum groups.

The following proposition is another direct corollary of Theorem 5.11.

Proposition 5.14. *Let $P_\mu^{(m)} = P_\mu(x; q, q^m)$ denote the Macdonald polynomial for a root system R and $t = q^m$, where $m = \{m_\alpha\}$ are positive integers. Consider the shift operators S_m^\pm constructed in Propositions 5.7 and 5.8. Define \widehat{S}_m^\pm as*

$$\widehat{S}_m^+ = \delta_m^{-1} \circ S_m^+ \circ \delta_{m-1}, \quad \widehat{S}_m^- = \delta_{m-1}^{-1} \circ S_m^- \circ \delta_m.$$

Then these operators act in the following simple way onto Macdonald polynomials:

$$\widehat{S}_m^+ P_{\lambda - \rho_m}^{(m)} = c_m(\lambda) P_{\lambda - \rho_{m+1}}^{(m+1)}$$

for all $\lambda \in \rho_{m+1} + P_+$ and $\widehat{S}_m^+ P_{\lambda - \rho_m}^{(m)} = 0$ otherwise. Further,

$$\widehat{S}_m^- P_{\lambda - \rho_{m+1}}^{(m+1)} = c_m(-\lambda) P_{\lambda - \rho_m}^{(m)}$$

for all $\lambda \in \rho_{m+1} + P_+$. In these formulas c_m denotes the function (5.4).

Proof. Let $\psi_m(x, z)$ denote the normalized BA function for the system R with multiplicities m . According to Theorem 5.11 we have:

$$\Phi_-^{(m)}(x, \lambda) = \sum_{w \in W} (-1)^w \psi_m(x, w\lambda) = c(\lambda, m) \delta_m P_{\lambda - \rho_{m+1}}^{(m+1)}(x).$$

As we know, the shift operator S_m^- sends $\psi_m(x, w\lambda)$ to $c_m(\lambda) c_m(-\lambda) \psi_{m-1}(x, w\lambda)$ (here we use W -invariance of $c_m(z) c_m(-z)$). Applying it to Φ_- , we obtain:

$$S_m^- \Phi_-^{(m)}(x, \lambda) = c_m(\lambda) c_m(-\lambda) \Phi_-^{(m)}.$$

After rewriting it in terms of Macdonald polynomials, we get:

$$S_m^- \left[c(\lambda, m) \delta_m P_{\lambda - \rho_{m+1}}^{(m+1)} \right] = c_m(\lambda) c_m(-\lambda) c(\lambda, m-1) \delta_{m-1} P_{\lambda - \rho_m}^{(m)},$$

which leads directly to the last formula from the proposition. Another part, involving S_m^+ , can be obtained in the same way. \square

Later we will apply these results to prove the so-called norm identity for Macdonald polynomials. In order to do this, we need one more result.

Proposition 5.15. *Let \langle, \rangle_k denote the scalar product (2.12), depending on the parameters k_α which are supposed to be **positive** integers. Consider two shift operators \widehat{S}_k^\pm , constructed in the previous proposition. Then \widehat{S}_k^- is "adjoint" to \widehat{S}_k^+ in the following sense:*

$$\langle \widehat{S}_k^+ f, g \rangle_{k+1} = (-1)^{|R_+|} q^{2 \sum_{\alpha > 0} k_\alpha} \langle f, \widehat{S}_k^- g \rangle_k$$

for any two polynomials f, g .

Proof. First, recall that according to [M1] each of the Macdonald operators D_k , being restricted to the space of W -invariant polynomials, is self-adjoint with respect to the scalar product $\langle \cdot, \cdot \rangle_k$. We should warn the reader that this is not true for $k_\alpha \in \mathbb{Z}_-$. Nevertheless, for all k the Macdonald operator D_k is *formally self-adjoint*. This is a purely algebraic statement, related to the following rule of calculating the adjoint:

$$(a_\tau T^\tau)^* = \left(\Delta_k \overline{\Delta}_k \right)^{-1} \circ T^\tau \circ \left(\Delta_k \overline{\Delta}_k \overline{a_\tau} \right), \quad (5.8)$$

where $\overline{f} = f(-x)$. The point is that this definition leads to the relation

$$\langle a_\tau T^\tau f, g \rangle_k = \langle f, (a_\tau T^\tau)^* g \rangle_k$$

as soon as we are sure that $a_\tau \Delta_k \overline{\Delta}_k$ is polynomial.

In a similar way, let us define another type of adjoint by the rule

$$(b_\nu T^\nu)^* = \left(\Delta_k \overline{\Delta}_k \right)^{-1} \circ T^\nu \circ \left(\Delta_{k+1} \overline{\Delta}_{k+1} \overline{b_\nu} \right). \quad (5.9)$$

This definition leads to the relation

$$\langle b_\nu T^\nu f, g \rangle_{k+1} = \langle f, (b_\nu T^\nu)^* g \rangle_k$$

as soon as we know that $b_\nu \Delta_{k+1} \overline{\Delta}_{k+1}$ is polynomial.

Now let us consider the shift operator \widehat{S}_k^+ from Proposition 5.14. As we know, it satisfies the intertwining relation

$$D_{k+1} \widehat{S}_k^+ = \widehat{S}_k^+ D_k,$$

where D_k now stands for the Macdonald operator with $t = q^k$. Taking the adjoints according to the definition (5.9), we obtain that

$$\left(\widehat{S}_k^+ \right)^* D_{k+1} = D_k \left(\widehat{S}_k^+ \right)^*$$

(here we used that D_k and D_{k+1} are formally self-adjoint in the sense of (5.8)).

Notice that this coincides with the intertwining relation for the shift operator \widehat{S}_k^- . One easily compares their leading terms, using the formulas from Propositions 5.7 and 5.8 and the definition (5.9) of the adjoint (in doing this it is useful to present the function (2.30) in terms of (2.5) as $\delta_m = (-1)^{|R_+|} \Delta_{m+1} / \Delta_{-m}$). A straightforward calculation gives that the leading terms coincide up to a factor $(-1)^{|R_+|} q^{2 \sum k_\alpha}$. Thus, the intertwiners should also coincide, due to Proposition 5.10. This proves that

$$\left(\widehat{S}_k^+ \right)^* = (-1)^{|R_+|} q^{2 \sum k_\alpha} \widehat{S}_k^-,$$

where the adjoint is understood in the formal sense (5.9).

Now, in order to derive the relation between the scalar products, it remains to check that if we present the operator \widehat{S}_k^+ as a sum $\sum b_\nu T^\nu$ then each coefficient b_ν after multiplying by $\Delta_{k+1} \overline{\Delta}_{k+1}$ becomes polynomial. Clearly, it is enough to prove that the operator $A = S_m^+ \circ \delta_{m-1}$ has polynomial coefficients. To prove this, we consider a quasipolynomial $\phi(x, z) = q^{2(x, z)} \delta_{m-1}(x)$ which obviously satisfies the conditions (4.17) for $j = 1, \dots, m_\alpha - 1$. By Lemma 5.2, ϕ can be presented in the form $\phi = L_z(\psi_{m-1})$ for some difference operator L in the z -variable, where ψ_{m-1} is the normalized BA function. Using that $S_m^+ \psi_{m-1} = \psi_m$, we see that $S_m^+ \phi = L_z \psi_m$, i.e. it is polynomial in x . This proves that the operator $A = S_m^+ \circ \delta_{m-1}$ maps polynomials into polynomials, thus, it must have polynomial coefficients. This completes the proof of the proposition. \square

5.5. Macdonald–Cherednik identities. Our results imply immediately three Macdonald’s conjectures about polynomials P_λ , which were first proved (for all reduced root systems) by Cherednik [C1, C2].

First, let us derive the norm identity in a traditional way, following the idea first used by Opdam [O] in case $q = 1$. Let $P_\mu^{(k)}$, as above, denote the Macdonald polynomial $P_\mu(x; q, q^k)$ related to the system R and parameters $k_\alpha \in \mathbb{Z}_+$, and $\langle \cdot, \cdot \rangle_k$ stands for the corresponding scalar product (2.12). Now we take an arbitrary weight $\lambda \in \rho_{m+1} + P_+$ and rewrite the scalar product $\langle P_{\lambda-\rho_{m+1}}^{(m+1)}, P_{\lambda-\rho_{m+1}}^{(m+1)} \rangle_{m+1}$ with the help of propositions 5.14 and 5.15:

$$\begin{aligned} \langle P_{\lambda-\rho_{m+1}}^{(m+1)}, P_{\lambda-\rho_{m+1}}^{(m+1)} \rangle_{m+1} &= [c_m(\lambda)]^{-2} \langle \widehat{S}_m^+ P_{\lambda-\rho_m}^{(m)}, \widehat{S}_m^+ P_{\lambda-\rho_m}^{(m)} \rangle_{m+1} = \\ &= (-1)^{|R_+|} q^{\sum m_\alpha} [c_m(\lambda)]^{-2} \langle P_{\lambda-\rho_m}^{(m)}, \widehat{S}_m^- \widehat{S}_m^+ P_{\lambda-\rho_m}^{(m)} \rangle_m = \\ &= (-1)^{|R_+|} q^{\sum m_\alpha} \frac{c_m(-\lambda)}{c_m(\lambda)} \langle P_{\lambda-\rho_m}^{(m)}, P_{\lambda-\rho_m}^{(m)} \rangle_m. \end{aligned}$$

This reduces the problem of calculating the norms at $t = q^{m+1}$ to the same problem but for $t = q^m$. For instance, in the simplest case when all m_α are equal, we may descend to $m = 0$. Macdonald polynomials $P_\lambda^{(0)}$ are simply the orbitsums $m_\lambda(x)$, so in this case $\langle P_\lambda^{(0)}, P_\lambda^{(0)} \rangle_0 = |W|$. This gives the norm identity:

$$\langle P_{\lambda-\rho_k}^{(k)}, P_{\lambda-\rho_k}^{(k)} \rangle_k = |W| q^{\sum_{\alpha>0} k_\alpha(k_\alpha-1)} \prod_{\alpha \in R_+} \prod_{j=1}^{k_\alpha-1} \frac{[(\alpha^\vee, \lambda) + j]}{[(\alpha^\vee, \lambda) - j]}.$$

This result extends easily to the two-orbit case by using the shift operators which lower the multiplicity m_α at one of the orbits only.

Now let us return to Theorem 5.11 and apply the duality. Recall that starting from a root system R and multiplicities $m_\alpha \in \mathbb{Z}_+$ we have constructed a function

$$\Phi_-(x, z) = \sum_{w \in W} (-1)^w \psi_m(x, wz) = \sum_{w \in W} (-1)^w \psi_m(wx, z).$$

We can consider also a similar function Φ_-^\vee for the dual root system R^\vee . Then Theorem 4.7 implies that these two functions are related through interchanging the arguments:

$$\Phi_-(x, z) = \Phi_-^\vee(z, x).$$

Now take $x = \lambda \in \rho_{m+1} + P_+$ and $z = \mu \in \rho_{m+1}^\vee + P_+^\vee$ and use Theorem 5.11. This gives that

$$c(\lambda, m) \delta_m(\mu) P_{\lambda-\rho_{m+1}}(\mu; q, q^{m+1}) = c^\vee(\mu, m) \delta_m^\vee(\lambda) P_{\mu-\rho_{m+1}^\vee}^\vee(\lambda; q, q^{m+1}),$$

where we used \vee to distinguish the objects related to the dual root system R^\vee . Taking into account formulas (5.6), (2.30) and their natural counterparts for the system R^\vee , we arrive after simple transformations to the relation

$$\begin{aligned} P_{\lambda-\rho_{m+1}}(\mu; q, q^{m+1}) \prod_{\alpha \in R_+} \prod_{j=0}^{m_\alpha} [(\alpha^\vee, \lambda) + j]^{-1} = \\ P_{\mu-\rho_{m+1}^\vee}^\vee(\lambda; q, q^{m+1}) \prod_{\alpha \in R_+} \prod_{j=0}^{m_\alpha} [(\alpha, \mu) + j]^{-1}. \end{aligned} \quad (5.10)$$

Put now $\mu = \rho_{m+1}^\vee$, then the right-hand side will not depend on λ (since $P_0(x) \equiv 1$). This gives us that

$$P_{\lambda - \rho_{m+1}}(\rho_{m+1}^\vee; q, q^{m+1}) = \text{const} \prod_{\alpha \in R_+} \prod_{j=0}^{m_\alpha} [(\alpha^\vee, \lambda) + j].$$

To determine the constant, we substitute $\lambda = \rho_{m+1}$ which leads to the evaluation identity:

$$P_{\lambda - \rho_{m+1}}(\rho_{m+1}^\vee; q, q^{m+1}) = \prod_{\alpha \in R_+} \prod_{j=0}^{m_\alpha} \frac{[(\alpha^\vee, \lambda) + j]}{[(\alpha^\vee, \rho_{m+1}) + j]}.$$

Denoting $\lambda - \rho_{m+1}$ by ν and $m_\alpha + 1$ by k_α , we can rewrite it as follows:

$$P_\nu(\rho_k^\vee; q, q^k) = q^{-2(\rho_k^\vee, \nu)} \prod_{\alpha \in R_+} \prod_{j=0}^{k_\alpha - 1} \frac{1 - q^{2j+2(\alpha^\vee, \nu + \rho_k)}}{1 - q^{2j+2(\alpha^\vee, \rho_k)}}.$$

This can be presented as

$$P_\nu(\rho_k^\vee; q, q^k) = q^{-2(\rho_k^\vee, \nu)} \prod_{\alpha \in R_+} \prod_{j=0}^{(\alpha^\vee, \nu) - 1} \frac{1 - q^{2j+2k_\alpha+2(\alpha^\vee, \rho_k)}}{1 - q^{2j+2(\alpha^\vee, \rho_k)}}.$$

An advantage of this form is that it works for all (e.g. non-integer) k_α . Indeed, it is clearly rational function of $t_\alpha = q^{k_\alpha}$. On the other hand, it is easy to see that for all k_α the value $P_\nu(\rho_k^\vee; q, q^k)$ must be rational in $t = q^k$. Since this identity is valid for integer k , it remains valid for all k .

Using this identity (and its counterpart for the system R^\vee) one can rewrite the relation (5.10) in a more compact form, known as the symmetry identity:

$$\frac{P_{\lambda - \rho_{m+1}}(\mu; q, q^{m+1})}{P_{\lambda - \rho_{m+1}}(\rho_{m+1}^\vee; q, q^{m+1})} = \frac{P_{\mu - \rho_{m+1}}^\vee(\lambda; q, q^{m+1})}{P_{\mu - \rho_{m+1}}^\vee(\rho_{m+1}; q, q^{m+1})}.$$

Changing notations, we can rewrite it as follows:

$$\frac{P_\lambda(\mu + \rho_k^\vee; q, q^k)}{P_\lambda(\rho_k^\vee; q, q^k)} = \frac{P_\mu^\vee(\lambda + \rho_k; q, q^k)}{P_\mu^\vee(\rho_k; q, q^k)} \quad \forall \lambda \in P_+, \quad \forall \mu \in P_+^\vee.$$

Again, it is easy to see that both sides are rational functions of $t = q^k$. Thus, in this form this identity is valid for all (e.g. non-integer) k_α .

This last identity is very important, since it leads directly (see [C2]) to the recurrence relations between Macdonald polynomials P_λ with different λ , which is a higher analogue of the three-term relation for classical orthogonal polynomials.

5.6. Limiting case $q = 1$. Let us describe briefly what happens in the limit $q, t \rightarrow 1$ if we keep t_α to be of the form $t_\alpha = q^{-m_\alpha}$ with fixed m_α . It is convenient to present q as $q = e^\epsilon$ and rescale the x -variable: $x \rightarrow \epsilon^{-1}x$. The normalized BA function constructed in previous sections is a common eigenfunction of the Macdonald operators, acting in x and z . Let us rewrite these operators in new notations.

First, in the x -variable we have the operators D^π related to (quasi)minuscule coweights of the root system R and given by formulas (2.1)–(2.3) with $k_\alpha = -m_\alpha$.

To avoid non-essential details, let us consider minuscule coweights only. Then in new notations the formula (2.1) takes the form:

$$D_x^\pi = \sum_{\tau \in W\pi} a_\tau T_x^{\epsilon\tau}, \quad a_\tau(x) = \prod_{\substack{\alpha \in R: \\ (\alpha, \tau) > 0}} \frac{\sinh(-\epsilon m_\alpha + (\alpha, x))}{\sinh(\alpha, x)}. \quad (5.11)$$

Here $T_x^{\epsilon\tau}$ denotes the shift in the x -variable by the vector $\epsilon\tau$.

Similarly, in the z -variable we have the operators D_z^ω related to minuscule weights ω of the system R :

$$D_z^\omega = \sum_{\tau \in W\omega} a_\tau T_z^\tau, \quad a_\tau(z) = \prod_{\substack{\alpha \in R: \\ (\alpha, \tau) > 0}} \frac{\sinh \epsilon(-m_\alpha + (\alpha^\vee, z))}{\sinh \epsilon(\alpha^\vee, z)}. \quad (5.12)$$

The normalized BA function $\psi(x, z)$ related to the root system R and multiplicities $m_\alpha \in \mathbb{Z}_+$, is a common eigenfunction of all these operators. It can be characterized uniquely in terms of its analytic properties in the z -variable, as it was done above or, alternatively, in terms of its properties in x (due to duality). Similarly, one can construct it using the operator D_z^ω as in Theorem 3.7 or, alternatively, by using in a similar way the operator D_x^π .

Now let us look what happens in the limit $\epsilon \rightarrow 0$. It is quite clear that the operator (5.12) in this limit takes the form:

$$D_z^\omega = \sum_{\tau \in W\omega} a_\tau T_z^\tau, \quad a_\tau(z) = \prod_{\substack{\alpha \in R: \\ (\alpha, \tau) > 0}} \frac{(\alpha^\vee, z) - m_\alpha}{(\alpha^\vee, z)}. \quad (5.13)$$

On the other hand, it is slightly more difficult to see what is a proper limit of the operator (5.11) as ϵ goes to zero. In order to do this one should expand D_x^π in a series in ϵ and pick up the first nontrivial term of this expansion. It turns out to be the following second-order differential operator:

$$L_m = \Delta - \sum_{\alpha \in R_+} 2m_\alpha \coth(\alpha, x) \partial_\alpha + 4(\rho_m, \rho_m), \quad (5.14)$$

where Δ is the Laplacian in $V = \mathbb{R}^n$, ∂_α stands for the derivative in α -direction, and ρ_m is given by (3.1). This operator plays the central role in Heckman–Opdam theory of multivariable hypergeometric functions [HO], and it is gauge-equivalent to the generalized Calogero–Sutherland operator from [OP].

From this it is natural to expect that a proper limit of the BA function $\psi(x, z)$ must give a common eigenfunction both for the rational Macdonald operators (5.13) and the operator L_m . However, it is quite difficult to see this directly, looking at our formula for ψ . The best way is to repeat the main constructions independently for this degenerate case. Since now we have no symmetry between x and z variables, we may choose two different ways to describe ψ : either in terms of x -properties, or in terms of its properties in the z -variable. The x -part of the story is very similar to what we had before: ψ has the form

$$\psi(x, z) = e^{2(x, z)} \sum_{\nu \in P} \psi_\nu e^{2(\nu, x)}$$

with the summation taken over the polytope (4.21). The main difference is that the conditions (4.17) are replaced by the following: for each $\alpha \in R_+$ and $s = 1, \dots, m_\alpha$

$$(\partial_\alpha)^{2s-1} \psi \equiv 0 \quad \text{for} \quad q^{2(\alpha, x)} = 1. \quad (5.15)$$

Here ∂_α denotes the derivative in α -direction in the x -variable. One can check that the operator (5.14) with $k_\alpha = -m_\alpha$ preserves these properties, which gives an analog of Proposition 2.1. After that everything becomes more or less a straightforward modification of the previous constructions.

The z -properties of ψ in this case change more significantly: it becomes quasi-polynomial in z in a standard sense:

$$\psi(x, z) = P(x, z) \exp 2(x, z), \quad P = C(x) \prod_{\alpha \in R_+} (\alpha^\vee, z)^{m_\alpha} + \dots,$$

where $C(x) = \prod_{\alpha \in R_+} (\sinh(\alpha, x))^{m_\alpha}$ and the dots stand for lower terms in z . A detailed exposition of this latter approach is given in [Ch1], where the relation of the ψ -function to the hypergeometric function and Jacobi polynomials by Heckman and Opdam is also discussed in detail.

To illustrate the difference between these two approaches, let us compare how the formula (3.16) will look for each of them. First, one can construct ψ using $L = L_m$. Let us introduce a trigonometric polynomial $Q(x)$ as

$$Q(x) = e^{2(\rho, x)} \prod_{\alpha \in R_+} (2 \sinh(\alpha, x))^{2m_\alpha},$$

where $\rho = \rho_m$ is given by (3.1). Then ψ , up to a certain z -depending factor, is given by the formula

$$\psi \sim \prod_{\nu} (L - 4(z + \nu)^2) [Q(x) e^{2(x, z)}],$$

where the product is taken over all $\nu \neq 0$ having the form $\nu = \sum_{\alpha > 0} l_\alpha \alpha$ with $l_\alpha = 0, 1, \dots, m_\alpha$.

Alternatively, to construct ψ one can use $D = D_z^\omega$ given by (5.13). Then, up to a certain x -depending factor, ψ is given by the expression (see [Ch1])

$$\psi \sim (D - m_\omega(x))^M [q(z) e^{2(x, z)}], \quad M = \sum_{\alpha \in R_+} m_\alpha,$$

where $m_\omega(x) = \sum_{\tau \in W_\omega} e^{2(\tau, x)}$ and the polynomial $q(z)$ looks as follows:

$$q(z) = \prod_{\alpha \in R_+} \prod_{j=1}^{m_\alpha} ((\alpha^\vee, z)^2 - j^2).$$

The first expression for ψ can be viewed as a trigonometric generalization of the original formula by Berest [Be], while the second one is its difference version. As we see, they both are specializations of the same formula (3.16).

6. BC_n CASE AND KOORNWINDER POLYNOMIALS

In this section we consider the case of the non-reduced root system $R = BC_n$. A proper generalization of the Macdonald theory in this case was proposed by Koornwinder [Ko2]. It depends on five parameters (apart from q) and generalizes Askey–Wilson polynomials [AW] to higher dimensions. The one-dimensional case will be essential for us, so we start considering $R = BC_1$.

6.1. Rank-one case. In this case we have 4 parameters a, b, c, d (apart from q), and the corresponding one-dimensional difference operator D , suggested by Askey and Wilson, looks as follows [AW]:

$$D = v^+(z)(T - 1) + v^-(z)(T^{-1} - 1) + q(abcd)^{-1} + q^{-1}abcd, \quad (6.1)$$

where $T^{\pm 1}$ denotes the shift by ± 1 in z and the coefficients v^{\pm} are given by the following formula:

$$v^+(z) = \frac{(aq^z - a^{-1}q^{-z})(bq^z - b^{-1}q^{-z})(cq^z + c^{-1}q^{-z})(dq^z + d^{-1}q^{-z})}{(q^z - q^{-z})(q^{\frac{1}{2}+z} - q^{-\frac{1}{2}-z})(q^z + q^{-z})(q^{\frac{1}{2}+z} + q^{-\frac{1}{2}-z})}, \quad (6.2)$$

$$v^-(z) = v^+(-z). \quad (6.3)$$

Our notations differ from those of Askey and Wilson [AW]: what they denote by (q, a, b, c, d) is $(q^2, a^2, b^2, -c^2, -d^2)$ in our notations.

The following function $\Delta(z) = \Delta(z; a, b, c, d, q)$ plays an important role in Askey–Wilson’s theory:

$$\Delta(z) = (abcd/q)^{-z} \frac{(q^{4z}, q^{2+4z}; q^4)_{\infty}}{(a^2q^{2z}, b^2q^{2z}, -c^2q^{2z}, -d^2q^{2z}; q^2)_{\infty}}. \quad (6.4)$$

Here we used the standard notations:

$$(t; q)_{\infty} := \prod_{i \geq 0} (1 - tq^i), \quad (t_1, \dots, t_n; q)_{\infty} := \prod_{s=1}^n (t_s; q)_{\infty}.$$

Using it, one can present the coefficients v^{\pm} of the Askey–Wilson operator D as

$$v^+(z) = \Delta(z+1)/\Delta(z), \quad v^-(z) = \Delta(-z-1)/\Delta(-z). \quad (6.5)$$

Introduce the *dual parameters* $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ as follows:

$$\tilde{a} = (abcd/q)^{\frac{1}{2}}, \quad \tilde{b} = (abq/cd)^{\frac{1}{2}}, \quad \tilde{c} = (acq/bd)^{\frac{1}{2}}, \quad \tilde{d} = (adq/bc)^{\frac{1}{2}}. \quad (6.6)$$

We will denote by $\tilde{\Delta}$ the function (6.4) with the dual parameters:

$$\tilde{\Delta}(z) = \Delta(z; \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, q). \quad (6.7)$$

Now let us make some special choice of parameters a, b, c, d in (6.2). Namely, we put

$$a = q^{-l}, \quad b = q^{-l'}, \quad c = q^{-m}, \quad d = q^{-m'}, \quad (6.8)$$

where $l, l', m, m' \in \frac{1}{2}\mathbb{Z}$ are some (half)integers with the requirement that

$$\frac{1}{2} + l + l' \in \mathbb{Z} \quad \text{and} \quad \frac{1}{2} + m + m' \in \mathbb{Z}. \quad (6.9)$$

We will assume that l, l', m, m' are positive. Introduce N as

$$N = 1 + l + l' + m + m' \in \mathbb{Z}. \quad (6.10)$$

Using it, one can present the constant term in (6.1) as

$$q(abcd)^{-1} + q^{-1}abcd = q^N + q^{-N}.$$

Below we will denote by $(\tilde{l}, \tilde{l}', \tilde{m}, \tilde{m}')$ the dual set of multiplicities, determined in accordance with (6.6):

$$\begin{pmatrix} \tilde{l} \\ \tilde{l}' \\ \tilde{m} \\ \tilde{m}' \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/2 \\ -1/2 \\ -1/2 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} l \\ l' \\ m \\ m' \end{pmatrix}. \quad (6.11)$$

Notice that this transformation, as well as (6.6), is *involutive*. Geometrically, it reduces to the orthogonal reflection with respect to the hyperplane $l - l' - m - m' = 1$ in \mathbb{R}^4 .

Now we are going to formulate an analogue of Proposition 2.1 for the Askey–Wilson operator. First, introduce the following shorthand notation $u \preccurlyeq (v, w)$ in the situation when for 3 given numbers u, v, w at least one of the differences $v - u, w - u$ belongs to $\mathbb{Z}_{\geq 0}$. Now let us consider a ring \mathfrak{R} which consists of all polynomials

$$f(z) = \sum_{j \in \mathbb{Z}} f_j q^{jz},$$

satisfying the following $N = 1 + l + l' + m + m'$ conditions:

$$f(z - s) = f(z + s) \quad \text{for} \quad q^{2z} = 1 \quad \text{and} \quad 0 < s \preccurlyeq (l, l'), \quad (6.12)$$

$$f(z - s) = f(z + s) \quad \text{for} \quad q^{2z} = -1 \quad \text{and} \quad 0 < s \preccurlyeq (m, m'). \quad (6.13)$$

Proposition 6.1. *The Askey–Wilson operator (6.1)–(6.3) with the parameters of the form (6.8)–(6.9) preserves the ring \mathfrak{R} above: $D(\mathfrak{R}) \subseteq \mathfrak{R}$.*

Proof can be found along the lines of our proof of Proposition 2.8. We need also the following inverse result.

Lemma 6.2. Let D be the Askey–Wilson operator (6.1) with a, b, c, d as in (6.8)–(6.9). Let f be an eigenfunction of D of the form $f = r(x, z)q^{2xz}$ where r is rational in z . Then f is, in fact, non-singular and satisfies conditions (6.12)–(6.13).

Proof. First, it is easy to see from equation $Df = \lambda f$ that in the case (6.8)–(6.9) any eigenfunction either has infinite number of poles, or has no poles at all. This proves that f is nonsingular in z . Then the conditions (6.12)–(6.13) can be derived similar to lemma 2.7. \square

All this suggests the following definition of a Baker–Akhiezer function for the Askey–Wilson operator with multiplicities (l, l', m, m') .

Definition. A function $\psi(x, z)$ of the form

$$\psi = q^{2xz} \sum_{|j| \leq N} \psi_j q^{jz}, \quad \psi_j = \psi_j(x), \quad (6.14)$$

is called a Baker–Akhiezer function for the Askey–Wilson operator if it satisfies $N = 1 + l + l' + m + m'$ conditions (6.12)–(6.13) in z for all x .

In the same way as in case $R = A_1$, one proves that such a ψ does exist and it is unique up to an x -depending factor. Also it is quite clear that the only nonzero coefficients ψ_j in (6.14) will be those with $j + N \in 2\mathbb{Z}$. However, we were not able to calculate ψ_j explicitly. This makes the following result somewhat less trivial.

Proposition 6.3. *Suppose that the multiplicities $l, l', m, m' \in \frac{1}{2}\mathbb{Z}_+$ satisfy (6.9) and are such that the dual multiplicities (6.11) are positive, too. Then if one puts $\psi_N = \tilde{\Delta}^{-1}(x)$ in accordance with the formulas (6.4)–(6.8), then all the coefficients ψ_j will be polynomial in x with $\psi_{-N}(x) = \psi_N(-x)$.*

As we already said, we cannot prove this proposition directly. So we use the following strategy: we will construct an eigenfunction ψ of the Askey–Wilson operator which has the form (6.14) and satisfies the requirements of the proposition. The constructed ψ will be automatically a BA function due to lemma 6.2.

To construct eigenfunctions of the operator (6.1) for the case (6.8)–(6.9), we will use shift operators, which in rank-one case can be computed directly. We need the operators which shift the parameters a, b, c, d in (6.1). We say that an operator S shifts from (a, b, c, d) to $(\hat{a}, \hat{b}, \hat{c}, \hat{d})$ if the following intertwining relation holds:

$$\widehat{D} \circ S = S \circ D, \quad (6.15)$$

where D and \widehat{D} are the Askey–Wilson operators (6.1) which correspond to (a, b, c, d) and $(\hat{a}, \hat{b}, \hat{c}, \hat{d})$, respectively.

Proposition 6.4. *Define 4 operators S_1, S_2, S_3, S_4 as*

$$S_i = A_i^+(z)T^{1/2} + A_i^-(z)T^{-1/2}, \quad i = 1, \dots, 4,$$

where $A_i^-(z) = A_i^+(-z)$ and A_i^+ are given by the formulas:

$$A_1^+ = \frac{(aq^{z-\frac{1}{2}} - a^{-1}q^{\frac{1}{2}-z})(bq^{z-\frac{1}{2}} - b^{-1}q^{\frac{1}{2}-z})(cq^{z-\frac{1}{2}} + c^{-1}q^{\frac{1}{2}-z})(dq^{z-\frac{1}{2}} + d^{-1}q^{\frac{1}{2}-z})}{(q^z - q^{-z})(q^z + q^{-z})}$$

$$A_2^+ = \frac{(aq^{z-\frac{1}{2}} - a^{-1}q^{\frac{1}{2}-z})(bq^{z-\frac{1}{2}} - b^{-1}q^{\frac{1}{2}-z})}{(q^z - q^{-z})(q^z + q^{-z})},$$

$$A_3^+ = \frac{(aq^{z-\frac{1}{2}} - a^{-1}q^{\frac{1}{2}-z})(cq^{z-\frac{1}{2}} + c^{-1}q^{\frac{1}{2}-z})}{(q^z - q^{-z})(q^z + q^{-z})},$$

$$A_4^+ = \frac{(aq^{z-\frac{1}{2}} - a^{-1}q^{\frac{1}{2}-z})(dq^{z-\frac{1}{2}} + d^{-1}q^{\frac{1}{2}-z})}{(q^z - q^{-z})(q^z + q^{-z})}.$$

Then S_i are the shift operators for the Askey–Wilson operator and the corresponding shifts of the parameters are as follows:

$$\begin{aligned} S_1: & (a, b, c, d) \rightarrow (q^{-\frac{1}{2}}a, q^{-\frac{1}{2}}b, q^{-\frac{1}{2}}c, q^{-\frac{1}{2}}d), \\ S_2: & (a, b, c, d) \rightarrow (q^{\frac{1}{2}}a, q^{\frac{1}{2}}b, q^{-\frac{1}{2}}c, q^{-\frac{1}{2}}d), \\ S_3: & (a, b, c, d) \rightarrow (q^{\frac{1}{2}}a, q^{-\frac{1}{2}}b, q^{\frac{1}{2}}c, q^{-\frac{1}{2}}d), \\ S_4: & (a, b, c, d) \rightarrow (q^{\frac{1}{2}}a, q^{-\frac{1}{2}}b, q^{-\frac{1}{2}}c, q^{\frac{1}{2}}d). \end{aligned}$$

Proof. It reduces to a straightforward though tedious calculation. Because of a certain symmetry between a, b, c, d it suffices to check the intertwining relation for S_1 and S_2 only. \square

Proof of Proposition 6.3. Recall that we assume that both l, l', m, m' and their dual (6.11) are positive. Consider the case when \tilde{l} and \tilde{m} are both integers. Then there is a proper composition of shifts S_i which shifts from $(1, q^{1/2}, 1, q^{1/2})$ to $(q^{-l}, q^{-l'}, q^{-m}, q^{-m'})$. Namely, one should apply S_1 \tilde{l} times, S_2 $(\tilde{l}' + 1/2)$ times, S_3 \tilde{m} times and S_4 $(\tilde{m}' + 1/2)$ times. The Askey–Wilson operator with the parameters $(1, q^{1/2}, 1, q^{1/2})$ is a trivial one, $D_0 = T + T^{-1}$, with $\psi_0 = q^{2xz}$ being its eigenfunction. Applying the composition of the shifts above to ψ_0 , we obtain an eigenfunction ψ of the operator D . It is obviously quasipolynomial in x . Moreover, it will be quasipolynomial in z , too (this is not completely obvious, since the shifts have singularities, but follows, for instance, from lemma 6.2). Now in order to find the leading coefficients, it is sufficient to look for each application of S_i at the asymptotics of ψ in z at $\pm\infty$, using the explicit formulas for S_i . After some simple inductive calculations, we arrive directly at the formula for $\psi_{\pm N}$ from the proposition.

The three other possible cases, namely

- (1) $l, m' \in \mathbb{Z}$, (2) $l', m \in \mathbb{Z}$, (3) $l', m' \in \mathbb{Z}$,

can be considered in a similar manner. The only difference is that in cases 2 and 3 one applies shifts starting from D with (a, b, c, d) being $(1, q^{1/2}, q^{1/2}, q)$ or $(1, q^{1/2}, q, q^{1/2})$, respectively. These operators are also almost trivial: they are obtained from $D_0 = T + T^{-1}$ by a simple gauge. For instance, the Askey–Wilson operator with parameters $(1, q^{1/2}, q^{1/2}, q)$ has the form

$$D = \frac{q^{1+x} + q^{-1-x}}{q^x + q^{-x}}(T - 1) + \frac{q^{1-x} + q^{-1+x}}{q^{-x} + q^x}(T - 1) + (q + q^{-1}) \\ = (q^x + q^{-x})^{-1} \circ (T + T^{-1}) \circ (q^x + q^{-x}).$$

Then we apply S_1 $(\tilde{l} + 1/2)$ times, S_2 \tilde{l}' times, S_3 \tilde{m} times and S_4 $(\tilde{m}' + 1/2)$ times, arriving at D with the parameters $(q^l, q^{l'}, q^m, q^{m'})$. Other arguments remain the same. The case 3 is analogous. \square

This leads us directly to the main result of this section.

Theorem 6.5. *Let $\Psi(x, z)$ denote a BA function with the parameters $l, l', m, m' \in \mathbb{Z}_+$ normalized as in Proposition 6.3. Then Ψ solves the following bispectral system:*

$$\begin{cases} D_z \Psi = (q^{2x} + q^{-2x})\Psi, \\ \tilde{D}_x \psi = (q^{2z} + q^{-2z})\Psi. \end{cases} \quad (6.16)$$

Here D_z is the Askey–Wilson operator (6.1) related to l, l', m, m' , and \tilde{D}_x acts in x and is related to the dual multiplicities (6.11). Moreover, if the dual parameters are positive, we will have the duality as follows: $\Psi(z, x) = \tilde{\Psi}(x, z)$ where $\tilde{\Psi}(x, z)$ denotes the normalized BA function related to the dual parameters.

Proof. The first equation (in z) follows from Proposition 6.1 and the uniqueness of ψ . It does not depend on our particular way of normalizing ψ . To obtain the second equation we apply the standard argument: consider the function $\phi = (q^{2z} + q^{-2z})\Psi$. It is quasipolynomial in z , satisfying conditions (6.12)–(6.13). Then it must be obtained from Ψ by applying a proper difference operator:

$$L\Psi = (q^{2z} + q^{-2z})\Psi, \quad L = b_+(x)T_x + b_-(x)T_x^{-1} + b_0(x).$$

The coefficients b_{\pm} can be found easily as soon as we know the leading coefficients $\Psi_{\pm N}$ in Ψ . To calculate b_0 , however, we need one more term, say, Ψ_{-N+2} in Ψ . Unfortunately, we have not found anything better than to calculate it directly from the difference equation $D_z \Psi = (q^{2x} + q^{-2x})\Psi$. This is pretty straightforward and we shall not reproduce this calculation here. As a result, one finds that the operator L is nothing but the dual Askey–Wilson operator \tilde{D}_x .

Finally, the duality between x and z follows similar to the case $R = A_1$, see Proposition 4.2. \square

6.2. BA function for Koornwinder operator. Now let us consider the difference operator by Koornwinder which generalizes the Askey–Wilson operator to higher dimensions. This operator D depends on five parameters a, b, c, d, t apart from q and it looks as follows [Ko1]:

$$D = \sum_{i=1}^n v_i^+(T_i - 1) + v_i^-(T_i^{-1} - 1) + abcdq^{-1} \frac{1 - t^{2n}}{1 - t^2} + (abcd)^{-1} q \frac{1 - t^{-2n}}{1 - t^{-2}}, \quad (6.17)$$

where T_i^s stands for a shift by s in z_i and the coefficients $v_i^{\pm}(z_1, \dots, z_n)$ are given by the formulas:

$$v_i^{\pm}(z) = v^{\pm}(z_i) \prod_{j \neq i} \frac{(tq^{\pm z_i - z_j} - t^{-1}q^{\mp z_i + z_j})(tq^{\pm z_i + z_j} - t^{-1}q^{\mp z_i - z_j})}{(q^{\pm z_i - z_j} - q^{\mp z_i + z_j})(q^{\pm z_i + z_j} - q^{\mp z_i - z_j})}, \quad (6.18)$$

with the functions $v^{\pm}(z_i)$ obtained by substituting $z = z_i$ into the formulas (6.2)–(6.3).

The underlying geometrical structure here is an affine root system $C^{\vee}C_n$ in notations of [M5]. For our purposes, however, it will be enough to consider a usual root system R of B_n -type:

$$R = \{\pm e_i\} \cup \{\pm e_i \pm e_j \mid i \neq j\}.$$

We fix its positive half R_+ as

$$R_+ = \{e_1, \dots, e_n\} \cup \{e_i \pm e_j \mid i < j\}.$$

The root lattice is $Q = \mathbb{Z}^n$ and its positive part Q_+ is defined as:

$$Q_+ = \{\nu \in \mathbb{Z}^n \mid \sum_{i=1}^j \nu_i \geq 0, \quad j = 1, \dots, n\}.$$

Weight lattice P in this case is also the standard lattice \mathbb{Z}^n :

$$P = \{\nu = \nu_1 e_1 + \dots + \nu_n e_n \mid \nu_i \in \mathbb{Z}\},$$

while the cone of the dominant weights P_+ looks as follows:

$$P_+ = \{\nu \in P \mid \nu_1 \geq \nu_2 \geq \dots \geq \nu_n \geq 0\}.$$

The Weyl group W acts by permuting the variables and flipping their signs arbitrarily.

As before, by a polynomial we mean a finite sum $f(z) = \sum_{\nu \in P} q^{2(\nu, z)}$, keeping calling functions like $f(z)q^{2(x, z)}$ *quasipolynomial* in z . Algebra of W -invariant polynomials in this case is a linear span of orbitsums

$$\mathbf{m}_{\lambda}(z) = \sum_{\tau \in W\lambda} q^{2(\tau, z)}, \quad (6.19)$$

and it is a symmetric polynomial algebra of the generators $y_i = q^{2z_i} + q^{-2z_i}$.

Now let us specialize the parameters a, b, c, d, t as follows:

$$t = q^{-k}, \quad a = q^{-l}, \quad b = q^{-l'}, \quad c = q^{-m}, \quad d = q^{-m'}, \quad (6.20)$$

where $k \in \mathbb{Z}_+$ and $l, l', m, m' \in \frac{1}{2}\mathbb{Z}_+$ are some (half)integers with the requirement as in rank-one case that

$$\frac{1}{2} + l + l' \in \mathbb{Z} \quad \text{and} \quad \frac{1}{2} + m + m' \in \mathbb{Z}. \quad (6.21)$$

We will denote by M the whole set

$$M = (k, l, l', m, m').$$

Below we will also use the multiplicities m_α defined in the following way:

$$m_\alpha = k \quad \text{for} \quad \alpha = \pm e_i \pm e_j \quad \text{and} \quad (6.22)$$

$$m_\alpha = 1 + l + l' + m + m' \quad \text{for} \quad \alpha = \pm e_i. \quad (6.23)$$

Introduce a vector ρ depending on $M = (k, l, l', m, m')$ as follows:

$$\rho = \rho_M = \frac{1}{2} \sum_{\alpha \in R_+} m_\alpha \alpha, \quad (6.24)$$

Using it, one can present the constant term in (6.17) in the following way:

$$abcdq^{-1} \frac{1-t^{2n}}{1-t^2} + (abcd)^{-1} q \frac{1-t^{-2n}}{1-t^{-2}} = \sum_i q^{2(\rho, \pm e_i)},$$

which makes it similar to the constant term in (2.2).

Let \mathfrak{R} denote a ring which consists of all polynomials $f(z) = \sum_{\nu \in \mathbb{Z}^n} q^{2(\nu, z)}$ with the following properties:

- (1) for all $i = 1, \dots, n$ and $0 < s \preccurlyeq (l, l')$

$$(T_i^s - T_i^{-s})f = 0 \quad \text{for} \quad q^{2z_i} = 1; \quad (6.25)$$

- (2) for all $i = 1, \dots, n$ and $0 < s \preccurlyeq (m, m')$

$$(T_i^s - T_i^{-s})f = 0 \quad \text{for} \quad q^{2z_i} = -1; \quad (6.26)$$

- (3) for all $1 \leq i < j \leq n$ and $s = 1, \dots, k$

$$(T_i^s - T_j^s)f = 0 \quad \text{for} \quad q^{2z_i} = q^{2z_j}; \quad (6.27)$$

- (4) for all $1 \leq i < j \leq n$ and $s = 1, \dots, k$

$$(T_i^s - T_j^{-s})f = 0 \quad \text{for} \quad q^{2z_i} = q^{-2z_j}. \quad (6.28)$$

We used the same notation $s \preccurlyeq (l, l')$ as above, denoting that at least one of the differences $l - s, l' - s$ is a nonnegative integer.

Proposition 6.6. *The Koornwinder operator (6.17)–(6.18) with the parameters (6.20)–(6.21) preserves the ring \mathfrak{R} as above: $D(\mathfrak{R}) \subseteq \mathfrak{R}$.*

Moreover, in a similar way one can check that all n commuting difference operators D_1, \dots, D_n constructed in [vD1], will preserve the ring \mathfrak{R} in case (6.20)–(6.21).

Now a Baker–Akhiezer function $\psi(x, z)$ is defined similarly to the case of a reduced root system:

(1) ψ has the form

$$\psi = q^{2(x,z)} \sum_{\nu \in \mathcal{N}} \psi_\nu(x) q^{2(\nu,z)}, \quad (6.29)$$

where the summation is taken over $\nu \in P$ lying inside the polytope

$$\mathcal{N} = \left\{ \nu = \sum_{\alpha \in R_+} l_\alpha \alpha^\vee \mid -\frac{1}{2}m_\alpha \leq l_\alpha \leq \frac{1}{2}m_\alpha \right\}; \quad (6.30)$$

(2) $\psi(x, z)$ satisfies the conditions (6.25)–(6.28) in z for all x .

One proves, similar to Section 3, that such ψ does exist and is unique up to an x -depending factor. It can be expressed by a formula, similar to (3.16). Namely, introduce first a function $c(\lambda, M)$ depending on $\lambda \in \mathbb{R}^n$ and the multiplicities $M = (k, l, l', m, m')$ in the following way:

$$c(\lambda, M) = \prod_{\substack{\alpha \in R_+^0 \\ 0 < j \preccurlyeq (l, l')}} [j - (\alpha, \lambda)] \prod_{\substack{\alpha \in R_+^1 \\ 0 < j \preccurlyeq (m, m')}} [j - (\alpha, \lambda)] \prod_{\substack{\alpha \in R_+^1 \\ j=1, \dots, k}} [j - (\alpha, \lambda)], \quad (6.31)$$

where R^0 and R^1 denote the sets of short and long roots, respectively, and $[a]$ stands, as usual, for $q^a - q^{-a}$.

Now define the polynomial $Q(z)$ as follows:

$$Q(z) = (-1)^{n(l+l'+\frac{1}{2})} q^{2(\rho, z)} c(z, M) c(-z, M), \quad (6.32)$$

where $c(z, M)$ is defined above and the vector ρ is given by (6.24). Introduce also the notation \mathbf{m} for the orbitsum

$$\mathbf{m}(x) = \sum_{i=1}^n (q^{2x_i} + q^{-2x_i}). \quad (6.33)$$

Theorem 6.7. *Let D be the Koornwinder operator (6.17) with the parameters as in (6.20)–(6.21). Define $\psi(x, z)$ as follows:*

$$\psi = \prod_{\nu} (D - \mathbf{m}(x + \nu)) \left[q^{2(x,z)} Q(z) \right], \quad (6.34)$$

in accordance with the formulas (6.32), (6.33), where the product is taken over all $\nu \neq 0$ having the form $\nu = \sum_{\alpha \in R_+} l_\alpha \alpha^\vee$ with $l_\alpha = 0, \dots, m_\alpha$. Then

- (i) ψ is a BA function for the Koornwinder operator;
- (ii) the coefficient $\psi_{-\rho}$ in its expansion (6.29) equals

$$\prod_{\nu} (\mathbf{m}(x) - \mathbf{m}(x + \nu)) \neq 0;$$

- (iii) as a function of z , ψ is an eigenfunction of the Koornwinder operator D : $D\psi = \mathbf{m}(x)\psi$.

To normalize ψ , consider the **dual parameters**

$$\widetilde{M} = (\widetilde{k}, \widetilde{l}, \widetilde{l}', \widetilde{m}, \widetilde{m}'),$$

where $\widetilde{k} = k$ while other 4 parameters transform according to (6.11). Let us normalize ψ in the following way:

$$\psi_\rho(x) = c(x, \widetilde{M}) \quad (6.35)$$

in accordance with the formula (6.31).

Theorem 6.8 (Duality). *Let $\psi(x, z)$ be a BA function related to the parameters l, l', m, m', k and normalized as above. Suppose that the dual parameters $\tilde{l}, \tilde{l}', \tilde{m}, \tilde{m}', \tilde{k}$ are positive. Then $\psi(x, z)$ is quasipolynomial in both x and z and has the following duality property:*

$$\psi(z, x) = \tilde{\psi}(x, z),$$

where $\tilde{\psi}$ denotes the normalized BA function related to the dual parameters. In particular, $\psi(x, z)$, as a function of z , is an eigenfunction of the Koornwinder operator (6.17), while in x it satisfies a similar difference equation related to the dual parameters.

The algebraic integrability of the Koornwinder operator in case (6.20) and the existence of the shift operators is a straightforward generalization of the similar results for reduced root systems.

6.3. Koornwinder polynomials. The Koornwinder polynomials

$$P_\lambda(z) = P_\lambda(z; q, a, b, c, d, t)$$

can be defined similar to Macdonald ones, as polynomial eigenfunctions of the Koornwinder operator (6.17), see [Ko2, vD1]. Since our notations are slightly different, we reproduce here their definition for the reader's convenience. Namely, P_λ has the form

$$P_\lambda = \mathfrak{m}_\lambda + \sum_{\nu \prec \lambda} a_{\lambda\nu} \mathfrak{m}_\nu, \quad \lambda \in P_+, \quad (6.36)$$

in notations of the previous section, with $\nu \prec \lambda$ meaning that $\lambda - \nu \in Q_+$. For generic values of the parameters a, b, c, d, t the polynomial P_λ is uniquely determined from the equation:

$$DP_\lambda = c_{\lambda\lambda} P_\lambda, \quad (6.37)$$

where D is the Koornwinder operator (6.17) and the eigenvalue $c_{\lambda\lambda}$ has the form

$$c_{\lambda\lambda} = \sum_{\tau = \pm e_i} q^{2(\tau, \lambda + \rho)}, \quad (6.38)$$

with $\rho = \rho_M$ given by formulas (6.20) and (6.22)–(6.24).

To compare with the notations in [vD1], one should put $q^2 = e^{i\omega}$, then our orbitsums $\mathfrak{m}_\lambda(z)$ correspond to $m_\lambda(\omega z)$ in notations of [vD1]. Our defining equation (6.37) corresponds to the equation

$$\widehat{D}_1 p_\lambda = E_{1,n}(\lambda + \rho) p_\lambda$$

which is the case $r = 1$ of eq. (3.72) from [vD1].

Now the relation of ψ to Koornwinder polynomials (generalized Weyl formula), the norm formula, evaluation identity and duality, they all can be derived similar to the case of a reduced root system. To avoid repetitions, we skip the details (see [vD2] for the formulation of all these identities).

7. INTEGRABLE DEFORMATION OF THE MACDONALD–RUIJSENAARS OPERATORS

In this section we discuss a version of the Macdonald–Ruijsenaars operators related to a certain "deformed" A_n system. This system $R = A_n(m) \subset \mathbb{R}^{n+1}$ was introduced in [CFV1] in the following way: $R = R^0 \cup R^1$ where

$$\begin{aligned} R^0 &= \{\pm(e_i - e_j) \mid 1 \leq i < j \leq n\} \cong A_{n-1} \subset \mathbb{R}^n, \\ R^1 &= \{\pm(e_i - \sqrt{m}e_{n+1}) \mid i = 1, \dots, n\} \quad \text{with} \\ m_\alpha &\equiv m \quad \text{for } \alpha \in R^0, \quad m_\alpha \equiv 1 \quad \text{for } \alpha \in R^1. \end{aligned} \quad (7.1)$$

Here m is a parameter, which at first will be an integer.

7.1. Deformed Macdonald–Ruijsenaars operator. Let us consider the following difference operator related to the deformed A_n system:

$$\tilde{D} = \tilde{a}_1(z)T^{e_1} + \dots + \tilde{a}_n(z)T^{e_n} + \tilde{a}_{n+1}(z)T^{\sqrt{m}e_{n+1}}, \quad (7.2)$$

where e_1, \dots, e_{n+1} is the standard basis in $V = \mathbb{R}^{n+1}$ and the coefficients a_i look as follows:

$$\tilde{a}_i(z) = \frac{[z_i - \sqrt{m}z_{n+1} - \frac{m+1}{2}]}{[z_i - \sqrt{m}z_{n+1}]} \prod_{j \neq i}^n \frac{[z_i - z_j - m]}{[z_i - z_j]}, \quad (7.3)$$

$$\tilde{a}_{n+1}(z) = \frac{[1]}{[m]} \prod_{j=1}^n \frac{[\sqrt{m}z_{n+1} - z_j - \frac{m+1}{2}]}{[\sqrt{m}z_{n+1} - z_j + \frac{m+1}{2}]}.$$

Here, as before, the square brackets are used to denote $[a] := q^a - q^{-a}$.

This operator is a discretization of the deformed Calogero–Moser operator proposed in [CFV1], with m being the deformation parameter. Its rational version was considered in [Ch1, Ch2]. For $m = 1$ it reduces to a special case of the Macdonald–Ruijsenaars operator D_1 in (2.7).

We fix a "positive half" of the system (7.1) as

$$R_+ = \{e_i - e_j \mid 1 \leq i < j \leq n\} \cup \{e_i - \sqrt{m}e_{n+1} \mid i = 1, \dots, n\}.$$

Next, we introduce "weight lattice":

$$P = \mathbb{Z}e_1 \oplus \dots \oplus \mathbb{Z}e_n \oplus \sqrt{m}\mathbb{Z}e_{n+1}.$$

Respectively, we will call any finite sum $f(x) = \sum_{\nu \in P} f_\nu q^{2(\nu, x)}$ a polynomial in x . We don't need coroots and coweights in this case, and there will be no substantial difference between variables $x, z \in \mathbb{R}^{n+1}$ below, so, for instance, polynomials in z are defined in the same way. As before, we will apply the term *quasipolynomial* to a function $\psi(x, z) = q^{2(x, z)}f$ where f is polynomial either in x or in z .

Now let us consider a ring $\tilde{\mathfrak{R}}$ of polynomials $f(z)$ with the properties (3.9) (with $\alpha^\vee := \alpha$). In our case they can be rewritten as follows:

$$f(z + se_i) = f(z + se_j) \quad \text{for } q^{2z_i} = q^{2z_j} \quad (1 \leq s \leq m, \quad 1 \leq i < j \leq n), \quad (7.4)$$

$$f(z + e_i) = f(z + \sqrt{m}e_{n+1}) \quad \text{for } q^{1-m+2z_i} = q^{2\sqrt{m}z_{n+1}} \quad (i = 1, \dots, n). \quad (7.5)$$

Proposition 7.1. *Deformed Macdonald–Ruijsenaars operator (7.2)–(7.3) with $m \in \mathbb{Z}_+$ preserves the ring $\tilde{\mathfrak{R}}$ of the polynomials with the properties (7.4)–(7.5): $\tilde{D}(\tilde{\mathfrak{R}}) \subseteq \tilde{\mathfrak{R}}$.*

Proof. For the conditions (7.4) the arguments repeat those from the proof of Proposition 2.1, because the operator D is symmetric in z_1, \dots, z_n . For the remaining conditions (7.5) everything reduces to the proof that the operators $D_0 = \tilde{a}_j(z)T^{e_j}$ (for $j \neq i$) and $D_1 = \tilde{a}_i(z)T^{e_i} + \tilde{a}_{n+1}(z)T^{\sqrt{m}e_{n+1}}$ preserve the property (7.5). This can be checked straightforwardly, in the spirit of Lemma 2.2. \square

7.2. BA function. Introduce $\rho = \rho(m)$ similar to (3.1):

$$\rho = \frac{1}{2} \sum_{\alpha \in R_+} m_\alpha \alpha. \quad (7.6)$$

In our case we have explicitly:

$$\rho = \frac{m}{2}(n-1, n-3, \dots, -n+1, 0) + \frac{1}{2}(1, \dots, 1, -\sqrt{mn}).$$

We define a polytope \mathcal{N} similar to (3.4):

$$\mathcal{N} = \left\{ \nu = -\rho + \sum_{\alpha \in R_+} l_\alpha \alpha \mid 0 \leq l_\alpha \leq m_\alpha \right\}. \quad (7.7)$$

Now the definition of a BA function repeats our definition in case of a root system.

Definition. A function $\psi(x, z)$ which is quasipolynomial in z ,

$$\psi = q^{2(x,z)} \sum_{\nu \in P} \psi_\nu q^{2(\nu,z)},$$

with $\text{supp}(\psi) \subseteq \mathcal{N}$ and which satisfies the conditions (7.4)–(7.5) in z is called a BA function for the system (7.1).

In exactly the same way as in Section 3, one proves that ψ is defined uniquely up to an x -depending factor. Moreover, analyzing the corresponding linear conditions for the coefficients ψ_ν , we come to the following choice of normalizing ψ :

$$\psi_\rho = \prod_{\alpha \in R_+} \prod_{j=1}^{m_\alpha} [(\alpha, \frac{1}{2}j\alpha - x)], \quad (7.8)$$

with $[a]$ denoting the q -number $[a] := q^a - q^{-a}$. We will call this ψ the *normalized BA function*. One can calculate the leading coefficients ψ_ν at other vertices of the polytope \mathcal{N} . It has $(n+1)!$ vertices which are in one-to-one correspondence with all permutations $\sigma \in S_{n+1}$. Namely, for $\sigma \in S_{n+1}$ let us introduce a vector $t_\sigma \in \mathbb{R}^{n+1}$ as

$$t_\sigma = (\sigma_1, \dots, \sigma_n, \sigma_{n+1}/\sqrt{m}).$$

Now denote by σR_+ the following subset in R :

$$\sigma R_+ = \{ \alpha \in R \mid (\alpha, t_\sigma) < 0 \},$$

and introduce $\sigma \rho$ as

$$\sigma \rho = \frac{1}{2} \sum_{\alpha \in \sigma R_+} m_\alpha \alpha.$$

These are exactly the vertices of the polytope (7.7). For instance, taking $\sigma = e$ we get $\sigma \rho = \rho$, and taking $\sigma = (n+1, n, \dots, 1)$ we get $\sigma \rho = -\rho$.

Similarly to Proposition 4.4, one gets the following result.

Proposition 7.2. *The leading coefficients $\psi_{\sigma\rho}$ of the normalized BA function have the form:*

$$\psi_{\sigma\rho} = \prod_{\alpha \in \sigma R_+} \prod_{j=1}^{m_\alpha} [(\alpha, \frac{1}{2}j\alpha - x)]. \quad (7.9)$$

The normalized BA function is quasipolynomial in both x and z .

The existence of ψ can be proven similar to Theorem 3.7. To formulate the result, introduce a polynomial $Q(z)$ as

$$Q(z) = q^{2(\rho, z)} \prod_{\alpha \in R_+} \prod_{j=1}^{m_\alpha} [(\alpha, z + \frac{1}{2}j\alpha)] [(\alpha, z - \frac{1}{2}j\alpha)], \quad (7.10)$$

with square brackets denoting q -number as before. Define also a "deformed orbit-sum" $\mathbf{m}(x)$ as follows:

$$\mathbf{m}(x) = q^{2x_1} + \dots + q^{2x_n} + \frac{q - q^{-1}}{q^m - q^{-m}} q^{2\sqrt{m}x_{n+1}}. \quad (7.11)$$

Theorem 7.3. *Let \tilde{D} be the deformed Macdonald–Ruijsenaars operator (7.2)–(7.3). Define $\psi(x, z)$ as follows:*

$$\psi = \prod_{\nu} \left(\tilde{D} - \mathbf{m}(x + \nu) \right) \left[q^{2(x, z)} Q(z) \right], \quad (7.12)$$

in accordance with the formulas (7.10), (7.11), where the product is taken over all $\nu \neq 0$ having the form $\nu = \sum_{\alpha \in R_+} l_\alpha \alpha$ with $l_\alpha = 0, \dots, m_\alpha$. We have the following:

- (i) ψ is a BA function for the system (7.1);
- (ii) its coefficient $\psi_{-\rho}$ equals $\prod_{\nu} (\mathbf{m}(x) - \mathbf{m}(x + \nu)) \neq 0$;
- (iii) as a function of z , ψ is an eigenfunction of the operator \tilde{D} : $\tilde{D}\psi = \mathbf{m}(x)\psi$.

Thus, renormalizing the constructed ψ one gets the normalized Baker–Akhiezer function Ψ for the system (7.1).

Now one can derive the duality similar to Theorem 4.7. In this case it is simply the symmetry between x, z .

Theorem 7.4. *The normalized BA function Ψ constructed above, is symmetric under permutation of its arguments: $\Psi(x, z) = \Psi(z, x)$.*

7.3. Quantum integrability. We start from discussing the *algebraic integrability* of the deformed Macdonald–Ruijsenaars operator (7.2)–(7.3) with $m \in \mathbb{Z}_+$, which is a direct corollary of the existence of a BA function for the system (7.1).

Theorem 7.5 (Algebraic integrability). *Let $m \in \mathbb{Z}_+$ and $\Psi(x, z)$ be the normalized BA function for the deformed A_n system (7.1). Then for each polynomial $f(z)$ from the ring $\tilde{\mathfrak{R}}$ there exists a difference operator D_f in x on the "weight" lattice P such that $D_f\Psi = f(z)\Psi$. All these operators commute. For $f = \mathbf{m}(z) \in \tilde{\mathfrak{R}}$ given by (7.11), the corresponding operator D_f is the deformed Macdonald–Ruijsenaars operator \tilde{D} given (2.1)–(2.3).*

Proof is the same as in Theorem 5.1. In this way we obtain a commutative ring of difference operators, isomorphic to the ring (7.4)–(7.5). Moreover, due to the

symmetry between x and z , we obtain a *bispectral pair* of commutative rings with $\Psi(x, z)$ being their common eigenfunction.

The ring \mathfrak{R} is big enough, it contains, for instance, a principal ideal generated by the polynomial (7.10). Now let us choose the following special elements $\mathfrak{m}_1, \dots, \mathfrak{m}_{n+1}$ of the ring \mathfrak{R} :

$$\mathfrak{m}_s(z) = q^{2sz_1} + \dots + q^{2sz_n} + \frac{q^s - q^{-s}}{q^{sm} - q^{-sm}} q^{2s\sqrt{m}z_{n+1}}. \quad (7.13)$$

In particular, for $s = 1$ we obtain the deformed orbitsum from (7.11). Notice that for $m = 1$ these polynomials turn into the Newton basis in the ring of symmetric functions.

In accordance with Theorem 7.5, to each \mathfrak{m}_s corresponds a certain difference operator $D_s = D_{\mathfrak{m}_s}$, and they all commute. Similar to Lemma 5.5, one can show that these operators D_s admit analytic continuation in m and, thus, they give rise to a commutative family for any value of the parameter m . Thus, we arrive at quantum integrability of the deformed Macdonald–Ruijsenaars operator.

Theorem 7.6. *The operator \tilde{D} is completely integrable, i.e. it can be included into a commutative family $D_1 = \tilde{D}, D_2, \dots, D_{n+1}$ of difference operators, which in case $m = 1$ coincide with the Macdonald–Ruijsenaars family (2.7).*

One obtains a natural elliptic version of the operator (7.2) replacing all expressions like $[a] = q^a - q^{-a}$ in its coefficients by their elliptic analogues $\sigma(a)$ with $\sigma(z) = \sigma(z|\omega, \omega')$ being the Weierstrass σ -function. For such an operator a proper version of Proposition 7.1 holds. This indicates that it is (algebraically) integrable, too. As a concluding remark, we mention that in a similar manner one can construct generalized Macdonald operators for other "deformed" root systems, some of which were presented in [CFV2]. Details will appear elsewhere.

REFERENCES

- [AW] Askey, R., Wilson, J.: Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials. Mem. Amer. Math. Soc. 319 (1985)
- [B] Bourbaki, N.: Groups et algebres de Lie, Ch. 4-6. Paris: Hermann 1968.
- [Be] Berest, Yu.: Huygens' principle and the bispectral problem. CRM Proceedings and Lecture Notes 14, 9–28 (1998)
- [BEG] Barannikov, V., Evens, S., Ginzburg, V.: Representations of quantum tori and double-affine Hecke algebras. Preprint: mathRT/0005024 (2000)
- [Ca] Calogero, F.: Solution of the one-dimensional N-body problems with quadratic and/or inversely quadratic pair potentials. J. Math. Phys. 12, 419–436 (1971)
- [C1] Cherednik, I.: Double affine Hecke algebras and Macdonald conjectures. Ann. Math. 141, 191–216 (1995)
- [C2] Cherednik, I.: Macdonald's evaluation conjectures and difference Fourier transform. Invent. Math. 122, 119–145 (1995)
- [Ch1] Chalykh, O.A.: Bispectrality for the quantum Ruijsenaars model and its integrable deformation. J. Math. Phys. 47(8), 5139–5167 (2000)
- [Ch2] Chalykh, O.A.: Duality of the generalized quantum Calogero and Ruijsenaars problems. Russ. Math. Surv. 52(6), 191–192 (1997)
- [CFV1] Veselov, A.P., Feigin, M.V., Chalykh, O.A.: New integrable deformations of quantum Calogero–Moser problem. Russ. Math. Surv. 51(3), 185–186 (1996)
- [CFV2] Chalykh, O.A., Feigin, M.V., Veselov, A.P.: Multidimensional Baker–Akhiezer functions and Huygens' principle. Comm. Math. Phys. 206, 533–566 (1999)

- [CSV] Veselov, A.P., Styrkas, K.L., Chalykh, O.A.: Algebraic integrability for the Schrödinger equation and reflection groups. *Theor. Math. Phys.* 94, 253–275 (1993)
- [CV] Chalykh, O.A., Veselov, A.P.: Commutative rings of partial differential operators and Lie algebras. *Comm. Math. Phys.* 126, 597–611 (1990)
- [DG] Duistermaat, J.J., Grünbaum, F.A.: Differential equations in the spectral parameter. *Comm. Math. Phys.* 103, 177–240 (1986)
- [DMN] Dubrovin, B.A., Matveev, V.B., Novikov, S.P.: Nonlinear equations of the Korteweg–de Vries type, finite-band linear operators and Abelian varieties. *Uspekhi Mat. Nauk* 31, 55–136 (1976)
- [vD1] van Diejen, J.F.: Commuting difference operators with polynomial eigenfunctions. *Compositio Math.* 95(2), 183–233 (1995)
- [vD2] van Diejen, J.F.: Self-dual Koornwinder–Macdonald polynomials. *Invent. Math.* 126(2), 319–339 (1996)
- [EK1] Etingof, P.I., Kirillov, A.A., Jr.: A unified representation-theoretic approach to special functions. *Funct. Anal. and its Appl.* 28(1), 91–94 (1994)
- [EK2] Etingof, P., Kirillov, A., Jr.: Macdonald’s polynomials and representations of quantum groups. *Math. Res. Letters* 1, 279–296 (1994)
- [EK3] Etingof, P., Kirillov, A., Jr.: Representation-theoretic proof of the inner product and symmetry identities for Macdonald’s polynomials. *Compos. Math.* 102, 179–202 (1996)
- [EK4] Etingof, P., Kirillov, A., Jr.: On Cherednik–Macdonald–Mehta identities. *Electr. Res. Announc. Amer. Math. Soc.* 4, 43–47 (1998)
- [ES] Etingof, P.I., Styrkas, K.L.: Algebraic integrability of Macdonald operators and representations of quantum groups. *Compositio Math.* 114, 125–152 (1998)
- [FV] Felder, G., Varchenko, A.: Three formulae for eigenfunctions of integrable Schrödinger operators. *Compos. Math.* 107, 143–175 (1997)
- [GKV] Ginzburg, V., Kapranov, M., Vasserot, E.: Residue construction of Hecke algebras. *Adv. in Math.* 128(1), 1–19 (1997)
- [HO] Heckman, G.J., Opdam, E.M.: Root systems and hypergeometric functions I. *Compos. Math.* 64, 329–352 (1987)
Heckman, G.J.: Root systems and hypergeometric functions II. *Compos. Math.* 64, 353–373 (1987)
Opdam, E.M.: Root systems and hypergeometric functions III,IV. *Compos. Math.* 67, 21–49, 191–209 (1988)
- [H] Heckman, G.J.: An elementary approach to the hypergeometric shift operators of Opdam. *Invent. Math.* 103, 341–350 (1991)
- [Kr1] Krichever, I.M.: Methods of algebraic geometry in the theory of nonlinear equations. *Russ. Math. Surv.* 32, 180–208 (1977)
- [Kr2] Krichever, I.M.: Algebraic curves and nonlinear difference equations. *Russ. Math. Surv.* 33(4), 215–216 (1978)
- [Ki1] Kirillov, A., Jr.: Lectures on affine Hecke algebras and Macdonald’s conjectures. *Bull. Amer. Math. Soc.* 34(3), 251–292 (1997)
- [Ki2] Kirillov, A., Jr.: On inner product in modular tensor categories. *J. Amer. Math. Soc.* 9(4), 1135–1169 (1997)
- [Ko1] Koornwinder, T.H.: unpublished manuscript (1988)
- [Ko2] Koornwinder, T.H.: Askey–Wilson polynomials for root systems of type BC. In: Richards, D.St.P (ed.), *Hypergeometric functions on domains of positivity, Jack polynomials, and applications.* *Contemp. Math.* 138, pp. 189–204 (1992)
- [M1] Macdonald, I.G.: Orthogonal polynomials associated with root systems, preprint (1987); mathQA/0011046
- [M2] Macdonald, I.G.: *Symmetric functions and orthogonal polynomials.* University Lecture Series, 12. AMS, Providence, RI, 1998.
- [M3] Macdonald, I.G.: Constant term identities, orthogonal polynomials, and affine Hecke algebras. *Documenta Mathematica, Extra Volume ICM 1998*, 303–317 (1998)
- [M4] Macdonald, I.G.: *Symmetric functions and Hall polynomials* (2nd ed.). Oxford: Clarendon Press 1995
- [M5] Macdonald, I.G.: Affine root systems and Dedekind’s η -function. *Invent. Math.* 15, 91–143 (1972)

- [O] Opdam, E.M.: Some applications of hypergeometric shift operators. *Invent. Math.* 98, 1–18 (1989)
- [OP] Olshanetsky, M.A., Perelomov A.M.: Quantum integrable systems related to Lie algebras. *Phys. Rep.* 94, 313–404 (1983)
- [R] Ruijsenaars, S.N.M.: Complete integrability of relativistic Calogero–Moser systems and elliptic functions identities. *Comm. Math. Phys.* 110, 191–213 (1987)
- [S] Sahi, S.: Nonsymmetric Koornwinder polynomials and duality. *Ann. Math.* 150, 267–282 (1999)
- [Su] Sutherland, B.: Exact results for a quantum many-body problem in one dimension, *Phys. Rev. A* 4, 2019 (1971)
- [TV] Tarasov, V., Varchenko, A.: Difference equations compatible with trigonometric KZ differential equations. *Int. Math. Res. Notices* 15, 801–829 (2000)
- [W] Wilson, G.: Collisions of Calogero–Moser particles and an adelic Grassmannian (with appendix by I.G.Macdonald). *Invent. Math.* 133, 1–41 (1998)

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