IDEALS OF RINGS OF DIFFERENTIAL OPERATORS ON ALGEBRAIC CURVES (WITH AN APPENDIX BY GEORGE WILSON)

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Abstract. Let X be a smooth affine irreducible curve over $\mathbb{C}$, and let $\mathcal{D} = \mathcal{D}(X)$ be the ring of global differential operators on X. In this paper, we give a geometric classification of left ideals in $\mathcal{D}$ and study the natural action of the Picard group of $\mathcal{D}$ on the space $\mathcal{I}(\mathcal{D})$ of isomorphism classes of such ideals.

We recall that, up to isomorphism in the Grothendieck group $K_0(\mathcal{D})$, the ideals of $\mathcal{D}$ are classified by the Picard group of X: there is a natural fibration $\gamma : \mathcal{I}(\mathcal{D}) \to \mathrm{Pic}(X)$, whose fibres are the stable isomorphism classes of ideals of $\mathcal{D}$ (see [BW]). In this paper, we refine this classification by describing the fibres of γ as unions of finite-dimensional algebraic varieties $\mathcal{C}_n(X, \mathcal{I})$, which we call the Calogero-Moser spaces. We define these varieties as representation varieties of deformed preprojective algebras over a certain extension of the ring of regular functions on X. As in the classical case (see [Wi]), we prove that $\mathcal{C}_n(X, \mathcal{I})$ are smooth affine irreducible varieties of dimension $2n$. Our results generalize the description of left ideals of the first Weyl algebra $A_1(\mathbb{C})$ in [BW1, BW2]; however, our methods are different.

1. Introduction

Let X be a smooth affine irreducible curve over $\mathbb{C}$, and let $\mathcal{D} = \mathcal{D}(X)$ be the ring of global differential operators on X. In this paper, we give a geometric classification of left ideals in $\mathcal{D}$ and study the natural action of the Picard group of $\mathcal{D}$ on the space of isomorphism classes of such ideals. Our results generalize the classification of left ideals of the first Weyl algebra $A_1(\mathbb{C})$ in [BW1] and [BW2]; however, our methods are rather different.

As shown in [BW1, BW2], the ideal classes of A_1 are parametrized by finite-dimensional algebraic varieties $\mathcal{C}_n$ called the *Calogero-Moser spaces*. The starting point for the present paper was the observation of Crawley-Boevey (see [CB1]) that the same varieties $\mathcal{C}_n$ parametrize finite-dimensional irreducible representations of certain (infinite-dimensional) algebras associated to graphs. Specifically, the algebras in question are the *deformed preprojective algebras* $\Pi^{\boldsymbol{\lambda}}(Q)$ of [CBH]; the corresponding graph Q is the framed affine Dynkin diagram of simplest type $\tilde{\mathsf{A}}_0$.

Trying to understand the relation between the ideals of A_1 and irreducible representations of $\Pi^{\boldsymbol{\lambda}}(Q)$, we came up with a new construction of the Calogero-Moser correspondence, which, besides the Weyl algebra, applied to noncommutative Kleinian singularities corresponding to other Dynkin diagrams (see [BCE]). In this paper, we develop a geometric version of this construction in which graphs are replaced by algebraic curves.

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We begin with an informal survey of our results. Let $\mathcal{I}(\mathcal{D})$ be the set of isomorphism classes of left ideals in $\mathcal{D}$. Since $\mathcal{D}$ is a Noetherian hereditary domain, every ideal of $\mathcal{D}$ is a projective $\mathcal{D}$-module of rank 1, so $\mathcal{I}(\mathcal{D})$ can be equivalently defined as the set of isomorphism classes of such modules. The Grothendieck group $K_0(\mathcal{D})$ of finite rank projective $\mathcal{D}$-modules is isomorphic to the algebraic K-group $K_0(X)$ of X, while $K_0(X) \cong \mathbb{Z} \oplus \mathrm{Pic}(X)$, where $\mathrm{Pic}(X)$ is the Picard group of X. Combining these isomorphisms, we may assign to each ideal class $[M] \in \mathcal{I}(\mathcal{D})$ an element of $\mathrm{Pic}(X)$ which determines $[M]$ up to equivalence in $K_0(\mathcal{D})$. In other words, there is a natural map $\gamma : \mathcal{I}(\mathcal{D}) \to \mathrm{Pic}(X)$, whose fibres are precisely the *stable* isomorphism classes of ideals in $\mathcal{D}$ (see [BW]). To classify the ideals of $\mathcal{D}$ we thus need to describe the fibres of γ.

We approach this problem in two steps. First, we introduce the Calogero-Moser spaces $\mathcal{C}_n(X, \mathcal{I})$ for an arbitrary curve X and a line bundle $\mathcal{I}$ on X, building on the observation of Crawley-Boevey. For any associative algebra B, there is a "universal" construction of deformed preprojective algebras $\Pi^{\boldsymbol{\lambda}}(B)$ over B, with parameters $\boldsymbol{\lambda} \in \mathbb{C} \otimes_{\mathbb{Z}} K_0(B)$ (see [CB] and Section 2.1 below). Using this construction, we define $\mathcal{C}_n(X, \mathcal{I})$ as representation varieties of $\Pi^{\boldsymbol{\lambda}}(B)$ over a triangular matrix extension of the ring $A = \mathcal{O}(X)$ of regular functions on X by the line bundle $\mathcal{I}$. This extension $B = A[\mathcal{I}]$ abstracts the idea of "framing" a quiver by adding a distinguished new vertex '∞' and arrows from ∞; geometrically, it can be thought of as a noncommutative "thickening" of $\mathrm{Spec}(A \times \mathbb{C}) = X \bigsqcup \mathrm{pt}$, the trivial one-point extension of X. We note that $\mathcal{C}_n(X, \mathcal{I})$ behave functorially with respect to $\mathcal{I}$, so the quotient spaces $\overline{\mathcal{C}}_n(X, \mathcal{I}) := \mathcal{C}_n(X, \mathcal{I})/\mathrm{Aut}_X(\mathcal{I})$ depend only on the class of $\mathcal{I}$ in $\mathrm{Pic}(X)$.

Next, by functoriality of $\Pi^{\boldsymbol{\lambda}}$-construction, there is a natural map of deformed preprojective algebras $\Pi^{\boldsymbol{\lambda}}(B) \to \Pi^1(A)$, lifting the extension $B \to A$, while $\Pi^1(A)$ can be identified with the ring of differential operators on X (see [CB]). The resulting homomorphism $\boldsymbol{\theta} : \Pi^{\boldsymbol{\lambda}}(B) \to \mathcal{D}$ allows one to relate the representation categories of $\Pi^{\boldsymbol{\lambda}}(B)$ and $\mathcal{D}$ in a fairly interesting way. To be precise, we will prove that the canonical functors $(L\boldsymbol{\theta}^*, \boldsymbol{\theta}_*, R\boldsymbol{\theta}^!)$ induced by $\boldsymbol{\theta}$ between the derived categories $\mathcal{D}^-(\mathtt{Mod}\,\Pi)$ and $\mathcal{D}^-(\mathtt{Mod}\,\mathcal{D})$ satisfy the *recollement* axioms for abstract triangulated categories in the sense of [BBD]. Originally, these axioms were designed to imitate a natural structure on the derived category $\mathcal{D}(\mathcal{S}h_X)$ of abelian sheaves arising from the stratification of a topological space into a closed subspace and its open complement; in [BBD], they were used to construct nontrivial t-structures in $\mathcal{D}(\mathcal{S}h_X)$ associated with perverse sheaves. In an algebraic setting similar to ours, the recollement axioms were first studied in [CPS].

The functor $\boldsymbol{\theta}_*$ yields a fully faithful embedding of $\mathcal{D}^-(\mathtt{Mod}\,\mathcal{D})$ into $\mathcal{D}^-(\mathtt{Mod}\,\Pi)$ as a "closed stratum", while the induction functor $L\boldsymbol{\theta}^* : \mathcal{D}^-(\mathtt{Mod}\,\Pi) \to \mathcal{D}^-(\mathtt{Mod}\,\mathcal{D})$ is an algebraic substitute for the restriction of a sheaf to that stratum. This last functor plays a key role in our construction: it transforms irreducible $\Pi^{\boldsymbol{\lambda}}(B)$-modules (viewed as 0-complexes in $\mathcal{D}^-(\mathtt{Mod}\,\Pi)$) to projective $\mathcal{D}$-modules (located in degree -1 in $\mathcal{D}^-(\mathtt{Mod}\,\mathcal{D})$), inducing natural maps $\omega_n : \overline{\mathcal{C}}_n(X, \mathcal{I}) \to \gamma^{-1}[\mathcal{I}]$. The amalgamation of these maps for all $n \geq 0$ turns out to be a *bijection*

$$\omega : \bigsqcup_{n \geq 0} \overline{\mathcal{C}}_n(X, \mathcal{I}) \xrightarrow{\sim} \gamma^{-1}[\mathcal{I}] \ ,$$

which gives a desired geometric description of the fibration γ over a given $[\mathcal{I}] \in \mathrm{Pic}(X)$. In the special case when X is the affine line, $\mathrm{Pic}(X)$ is trivial: there is only one fibre, and as shown in [BCE], the map ω agrees with the Calogero-Moser map constructed in [BW1, BW2].

By analogy with geometry, we can view finite-dimensional irreducible $\Pi^{\boldsymbol{\lambda}}(B)$-modules as "perverse sheaves" in $\mathcal{D}^-(\mathtt{Mod}\,\Pi)$, with projective $\mathcal{D}$-modules being their spaces of sections in the "closed stratum" $\mathcal{D}^-(\mathtt{Mod}\,\mathcal{D})$. The Calogero-Moser correspondence amounts then to the observation that such "sheaves" are determined by their restriction to $\mathcal{D}^-(\mathtt{Mod}\,\mathcal{D})$, which resembles the behavior of locally constant sheaves in the geometric setting.

Instead of focusing on individual fibres, one can also define the space $\overline{\mathcal{C}}_n(X)$ by amalgamating $\overline{\mathcal{C}}_n(X,\mathcal{I})$'s over the entire $\mathrm{Pic}(X)$. We will show then that, for each $n \geq 0$, there is a natural action on $\overline{\mathcal{C}}_n(X)$ of the Picard group $\mathrm{Pic}(\mathcal{D})$ of the category of $\mathcal{D}$-modules, and the maps $\omega_n : \overline{\mathcal{C}}_n(X) \to \mathcal{I}(\mathcal{D})$ are equivariant under this action. This generalizes another aspect of the Calogero-Moser correspondence for the Weyl algebra. Finally, for an arbitrary curve X, we will give an explicit ring-theoretic description of ω in terms of determinants of generalized Calogero-Moser matrices, extending our earlier calculations in [BC] for A_1.

We now proceed with a detailed summary of the contents of the paper.

Section 2 is preparatory: it introduces notation and reviews the material needed for the rest of the paper. In Section 2.1, we recall the definition of deformed preprojective algebras of [CB] and prove some general results on their representations. In Section 2.2, we define one-point extensions of associative algebras and, in Section 2.3, review the construction of representation varieties. While most results in these sections are known, some are (apparently) new. In particular, Theorem 2.2 and Proposition 2.1 on lifting representations of Π^{λ} as well as Proposition 2.3 on representation varieties did not appear in the literature (although for the path algebras of quivers, the last two propositions can be found in [CB2]). We prove these results in a greater generality than needed in the present paper with a hope that they may be useful in other applications.

In Section 3, after some recollections on differential operators on curves (Section 3.1) and K-theoretic classificaion of ideals of $\mathcal{D}$ (Section 3.2), we define the Calogero-Moser spaces $\mathcal{C}_n(X,\mathcal{I})$ and establish their basic geometric properties (Sections 3.3 and 3.4). In particular, Proposition 3.1 implies that the varieties $\mathcal{C}_n(X,\mathcal{I})$ are non-empty and parametrize the isomorphism classes of *simple* representations of $\Pi^{\boldsymbol{\lambda}}(B)$. Lemma 3.3 shows that $\mathcal{C}_n(X,\mathcal{I})$ and $\mathcal{C}_n(X,\mathcal{I})$ are (non-canonically) isomorphic for any line bundles $\mathcal{I}$ and $\mathcal{J}$. Finally, Theorem 3.2 – the main result of this section – says that $\mathcal{C}_n(X,\mathcal{I})$ are smooth irreducible varieties of dimension $2n$. This theorem is a generalization of a well-known result of Wilson [Wi].

The main results of the paper are gathered in Section 4. First, in Section 4.1, we explain the relation between the algebras $\Pi^{\boldsymbol{\lambda}}(B)$ and $\mathcal{D}$ and their representation categories. Proposition 4.1 asserts that the derived category $\mathcal{D}^-(\mathtt{Mod}\,\Pi)$ admits an algebraic stratification (*recollement* in the sense of [BBD]), with $\mathcal{D}^-(\mathtt{Mod}\,\mathcal{D})$ being its 'closed' stratum. Although, strictly speaking, this result is not used in the main part of the paper, it provides a conceptual explanation for our construction. Next, in Section 4.2, we describe a natural action of the Picard group $\mathrm{Pic}(\mathcal{D})$ on the (reduced) Calogero-Moser spaces $\overline{\mathcal{C}}_n(X)$. This construction heavily relies on earlier results of Cannings-Holland [CH1] and Wilson [BW]. Finally, in Section 4.3, we state the main theorem of this paper (Theorem 4.2), which describes the key properties of our Calogero-Moser map ω: the injectivity of $\omega_n : \overline{\mathcal{C}}_n(X,\mathcal{I}) \to \gamma^{-1}[\mathcal{I}]$ for each $n \geq 0$, the bijectivity of $\omega = \bigsqcup_{n\geq 0} \omega_n$ over each fibre $\gamma^{-1}[\mathcal{I}]$, and the equivariance of ω under the action of $\mathrm{Pic}(\mathcal{D})$ across the fibres.

The proof of Theorem 4.2 occupies the whole of Section 5. We refer the reader to the introduction of that section for a summary of the proof. Here we only mention a few results that play a role in our arguments. First of all, in many calculations of Section 5, we use the explicit presentation of the algebras $\Pi^{\boldsymbol{\lambda}}(B)$ given in Proposition 5.1. This presentation can be viewed as a generalization of Crawley-Boevey's description of deformed preprojective algebras over the path algebras of quivers (see [CB], Theorem 3.1). To prove the surjectivity of the map ω we use a certain 'projective' completion of filtered $\mathcal{D}$-modules. In Section 5.3, we define this operation in purely algebraic (categorical) terms; however, it can also be described in the language of noncommutative projective geometry (see [BC], Appendix). Another important result in Section 5.3 is Proposition 5.3. The proof of this proposition is based on a general homological principle for lifting representations of B to $\Pi^{\boldsymbol{\lambda}}(B)$, which can be summarized by saying that such liftings are controlled by the Hochschild homology of B. We use this principle repeatedly in the present paper (see, e.g., Theorem 2.2). The injectivity of ω is proved in Section 5.4: the key result here is Lemma 5.12 on uniqueness of the induced filtrations on ideals of $\mathcal{D}$. In a weaker form, this lemma first appeared in [BW2] (Lemma 10.1) and was further refined in [NS] (Lemma 3.2). Finally, in Section 5.5, we show that the action of $\mathrm{Pic}(\mathcal{D})$ on $\overline{\mathcal{C}}_n(X)$ is well-defined, and the map ω is equivariant. Our proofs of these results depend on the structure of the group $\mathrm{Pic}(\mathcal{D})$ and do not apply automatically to all curves: the affine line $\mathbb{A}^1$ has to be excluded. The reason for excluding $\mathbb{A}^1$ is that $\mathrm{Pic}(\mathcal{D})$ has a richer structure in that case and a different description (cf. [CH1]). Still, by a theorem of Stafford [St], we know that $\mathrm{Pic}(\mathcal{D})$ is isomorphic to $\mathrm{Aut}(\mathcal{D})$ in the case of the Weyl algebra, so the equivariance of ω for $\mathbb{A}^1$ follows from the results of [BW1, BW2].

In Section 6, we give an alternative description of the map ω by constructing some distinguished representatives for the isomorphism classes in $\mathcal{I}(\mathcal{D})$. The main result of this section (Theorem 6.1) clarifies and generalizes calculations of [BC] in the case of the Weyl algebra (see *loc. cit.*, Section. 4). In the end of Section 6, we consider a number of explicit examples: perhaps, the most interesting one is a general plane curve (see Section 6.2.3). For such curves, the representation varieties $\mathcal{C}_n(X,\mathcal{I})$ can be naturally described in terms of matrices, generalizing the classical Calogero-Moser matrices, and the map ω is given by an explicit formula involving characteristic polynomials of these matrices (see 6.2.3, (6.20)). This formula can be viewed as a generalization of Wilson's formula for the (rational) Baker function of the KP hierarchy in the theory of integrable systems (see [Wi]).

The last section of the paper is an appendix written by G. Wilson. It clarifies the relation between deformed preprojective algebras and rings of differential operators on curves, which, strictly speaking, we did not use in this paper but probably should have. As explained above, our Calogero-Moser map ω is naturally induced by the algebra extension $\boldsymbol{\theta} : \Pi^{\boldsymbol{\lambda}}(B) \to \mathcal{D}$. Unfortunately, this extension is not entirely canonical: it depends on the choice of an identification of $\Pi^1(A)$ with $\mathcal{D}$, the ring of differential operators on the algebra of regular functions on X. By a theorem of Crawley-Boevey (see [CB], Theorem 4.7), $\Pi^1(A)$ is indeed isomorphic to $\mathcal{D}$ as a filtered algebra, but, in general, there seems to be no natural isomorphism between these algebras. To remedy this problem, one should replace $\mathcal{D}$ by the ring $\mathcal{D}(\Omega_X^{1/2})$ of *twisted* differential operators on half-forms. As it was first observed by V. Ginzburg (see [G], Sect. 13.4), $\Pi^1(A)$ is *canonically* isomorphic to $\mathcal{D}(\Omega_X^{1/2})$; however, the construction of the isomorphism depends on a fact (Proposition A.1 below) whose proof in [G] is not quite satisfactory. A complete proof can be found in the Appendix, which may be read independently of the rest of the paper.

In the end, we mention a few open questions. First of all, in the existing literature, there are several different definitions of the Calogero-Moser spaces for algebraic curves. One is proposed by Etingof (see [E], Example 2.19) as a generalization of the ordinary Calogero-Moser spaces associated to Cherednik algebras (see [EG]). Another, due to Ginzburg (see [BN], Definition 1.2), employs a Hamiltonian reduction and is close in spirit to the original definition in [KKS]. It is more or less clear that these definitions agree with ours (and each other), but it might be interesting to construct *canonical* isomorphisms between them. This would clarify the question (originally posed in [EG], see also [BGK]) on the relation between projective $\mathcal{D}$-modules on curves and irreducible representations of generalized Cherednik algebras. In the Weyl algebra case, this question was addressed in [BCE].

Next, there is an alternative description of torsion-free $\mathcal{D}$-modules on curves, due to Ben-Zvi and Nevins (see [BN]). They use methods of noncommutative projective geometry (specifically, a noncommutative version of Beilinson's resolution), which is by now a conventional approach to this kind of moduli problems (see [LeB], [KKO], [BW2], [BGK], [NvdB], [NS] and references therein). In [BN], a more general (and slightly different) question on classifying "$\mathcal{D}$-bundles" of an arbitrary rank is addressed; however, in the rank one case, the resulting classification appears to be more complicated than ours (in particular, no explicit map similar to ω arises in this classification). Comparing our construction to that of [BN] and, more generally, to other approaches to moduli problems in noncommutative geometry, seem to be an interesting problem.

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Notation and Conventions

Throughout this paper, we work over the base field $\mathbb{C}$. Unless otherwise specified, an algebra means an associative algebra over $\mathbb{C}$, a module over an algebra A means a *left* module over A, and $\mathtt{Mod}(A)$ denotes the category of such modules. All bimodules over algebras are assumed to be symmetric over $\mathbb{C}$, and we use the abbreviation $\otimes$ for $\otimes_{\mathbb{C}}$ whenever it is convenient.

2. Preliminaries

2.1. **Deformed preprojective algebras.** If A is an algebra, its tensor square $A^{\otimes 2}$ has two commuting bimodule structures: one is defined by $a.(x \otimes y).b = ax \otimes yb$ and the other by $a.(x \otimes y).b = xb \otimes ay$, where $a, b \in A$. We will refer to these structures as *outer* and *inner* respectively. Any bimodule over A can be viewed as either left or right module over the enveloping algebra $A^{\mathrm{e}} := A \otimes A^{\mathrm{o}}$ of A. Now, if we interpret the outer bimodule structure on $A^{\otimes 2}$ as a left A^{e}-module structure and the inner as a right one, then the canonical map $A^{\otimes 2} \to A^{\mathrm{e}}$, $x \otimes y \mapsto x \otimes y^{\circ}$, is an isomorphism of A^{e}-bimodules. We will often use this isomorphism to identify $A^{\otimes 2} \cong A^{\mathrm{e}}$.

The space $\mathrm{Der}(A, A^{\otimes 2})$ of linear derivations $A \to A^{\otimes 2}$ taken with respect to the outer bimodule structure on $A^{\otimes 2}$ is naturally a bimodule with respect to the inner structure; thus, we can form the tensor algebra $T_A\mathrm{Der}(A, A^{\otimes 2})$. If A is unital, there is a canonical element in $\mathrm{Der}(A, A^{\otimes 2})$: namely the derivation $\Delta = \Delta_A : A \to A^{\otimes 2}$, sending $x \in A$ to $(x \otimes 1 - 1 \otimes x) \in A^{\otimes 2}$. For any $\lambda \in A$, we can consider then the 2-sided ideal $\langle \Delta - \lambda \rangle$ in $T_A\mathrm{Der}(A, A^{\otimes 2})$ and define the quotient algebra

$$\Pi^{\lambda}(A) := T_A\mathrm{Der}(A, A^{\otimes 2})/\langle \Delta - \lambda \rangle \ . \tag{2.1}$$

It turns out that, up to isomorphism, $\Pi^{\lambda}(A)$ depends only on the class of λ in the Hochschild homology $\mathrm{HH}_0(A) := A/[A, A]$ (see [CB], Lemma 1.2). Moreover, instead of elements of $\mathrm{HH}_0(A)$, it is convenient to parametrize the algebras (2.1) by the elements of $\mathbb{C} \otimes_{\mathbb{Z}} K_0(A)$, relating this last vector space to $\mathrm{HH}_0(A)$ via a Chern character map. To be precise, let $\mathrm{Tr}_A : K_0(A) \to \mathrm{HH}_0(A)$ be the map, sending the class of a projective module P to the class of the trace of any idempotent matrix $e \in \mathtt{Mat}(n, A)$, satisfying $P \cong A^n e$. By additivity, this extends to a linear map $\mathbb{C} \otimes_{\mathbb{Z}} K_0(A) \to \mathrm{HH}_0(A)$ to be denoted also Tr_A. Following [CB], we call the elements of $\mathbb{C} \otimes_{\mathbb{Z}} K_0(A)$ *weights* and define the *deformed preprojective algebra of weight* $\boldsymbol{\lambda} \in \mathbb{C} \otimes_{\mathbb{Z}} K_0(A)$ by

$$\Pi^{\boldsymbol{\lambda}}(A) := T_A\mathrm{Der}(A, A^{\otimes 2})/\langle \Delta - \lambda \rangle \ , \tag{2.2}$$

where $\lambda \in A$ is any lifting of $\mathrm{Tr}_A(\boldsymbol{\lambda})$ to A. Note, if A is commutative, then $\mathrm{HH}_0(A) = A$, and λ is uniquely determined by $\mathrm{Tr}_A(\boldsymbol{\lambda})$.

We should mention that the algebras $\Pi^{\boldsymbol{\lambda}}(A)$ are usually ill-behaved unless one imposes some 'smoothness' conditions on A. In this paper, following [KR], we will refer to an algebra A as being *smooth* if it is quasi-free and finitely generated. Technically, this implies that $\Omega^1 A$ — the kernel of the multiplication map of A — is a finitely generated projective bimodule.

For basic properties and examples of the algebras $\Pi^{\boldsymbol{\lambda}}(A)$ the reader is referred to [CB]. Here, we state only one important theorem from [CB], which will play a role in our construction. We recall that a ring homomorphism $\theta : B \to A$ is called *pseudo-flat* if $\mathrm{Tor}_1^B(A, A) = 0$ (see [BD]). We also recall that any ring homomorphism $\theta : B \to A$ induces a homomorphism of abelian groups $\theta^* : K_0(B) \to K_0(A)$, which extends (by linearity) to a map of $\mathbb{C}$-vector spaces $\theta^* : \mathbb{C} \otimes_{\mathbb{Z}} K_0(B) \to \mathbb{C} \otimes_{\mathbb{Z}} K_0(A)$.

Theorem 2.1 ([CB], Theorem 9.3 and Corollary 9.4)**.** *Let $\theta : B \to A$ be a pseudo-flat ring epimorphism. Then, for any $\boldsymbol{\lambda} \in \mathbb{C} \otimes_{\mathbb{Z}} K_0(B)$, there is a canonical algebra map $\boldsymbol{\theta} : \Pi^{\boldsymbol{\lambda}}(B) \to \Pi^{\boldsymbol{\lambda}'}(A)$, where $\boldsymbol{\lambda}' = \theta^*(\boldsymbol{\lambda})$. If B is smooth, then $\boldsymbol{\theta}$ is also a pseudo-flat epimorphism, and the diagram*

$$\begin{array}{ccc} B & \xrightarrow{\ \theta\ } & A \\ \big\downarrow & & \big\downarrow \\ \Pi^{\boldsymbol{\lambda}}(B) & \xrightarrow{\ \boldsymbol{\theta}\ } & \Pi^{\boldsymbol{\lambda}'}(A) \end{array} \tag{2.3}$$

is a push-out in the category of rings.

We now prove a few general results on representations of deformed preprojective algebras, which may be of independent interest. Our first observation is probably well-known to the experts (see, for example, [CEG]); we record it to fix the notation.

Lemma 2.1. *If A is smooth, then the derivation Δ_A lies in the commutator subspace $[A,\, \mathrm{Der}(A, A^{\otimes 2})]$ of the bimodule $\mathrm{Der}(A, A^{\otimes 2})$.*

Proof. The multiplication map $\mu : A^{\otimes 2} \to A$ is a bimodule homomorphism, provided $A^{\otimes 2}$ is taken with its outer bimodule structure. Composing μ with derivations $A \to A^{\otimes 2}$ yields a linear map $\mu_* : \mathrm{Der}(A, A^{\otimes 2}) \to \mathrm{Der}(A, A)$ with $\Delta_A \in \mathrm{Ker}(\mu_*)$. A simple calculation shows that μ_* factors through the canonical projection $\mathrm{Der}(A, A^{\otimes 2}) \twoheadrightarrow \mathrm{Der}(A, A^{\otimes 2})/[A,\, \mathrm{Der}(A, A^{\otimes 2})]$. We claim that, when A is smooth, the induced map $\bar{\mu}_* : \mathrm{Der}(A, A^{\otimes 2})/[A,\, \mathrm{Der}(A, A^{\otimes 2})] \to \mathrm{Der}(A, A)$ is an isomorphism. In fact, identifying $A^{\otimes 2} \cong A^{\mathrm{e}}$ and writing $\Omega^1 A \subseteq A^{\mathrm{e}}$ for the kernel of μ, we have $\mathrm{Der}(A, A) \cong \mathrm{Hom}_{A^{\mathrm{e}}}(\Omega^1 A, A)$ and $\mathrm{Der}(A, A^{\otimes 2}) \cong \mathrm{Hom}_{A^{\mathrm{e}}}(\Omega^1 A, A^{\mathrm{e}})$. Under the last isomorphism, the (inner) bimodule structure on $\mathrm{Der}(A, A^{\otimes 2})$ corresponds to the natural right A^{e}-module structure on $(\Omega^1 A)^\star := \mathrm{Hom}_{A^{\mathrm{e}}}(\Omega^1 A, A^{\mathrm{e}})$ and

$$\mathrm{Der}(A, A^{\otimes 2})/[A,\, \mathrm{Der}(A, A^{\otimes 2})] \cong (\Omega^1 A)^\star \otimes_{A^{\mathrm{e}}} A\ .$$

The quotient map $\bar{\mu}_*$ now becomes the canonical map

$$(\Omega^1 A)^\star \otimes_{A^{\mathrm{e}}} A \to \mathrm{Hom}_{A^{\mathrm{e}}}(\Omega^1 A, A)\ . \tag{2.4}$$

Since A is smooth, $\Omega^1 A$ is a finitely generated projective A^{e}-module, and hence (2.4) is an isomorphism. Now, $\bar{\mu}_*$ being an isomorphism implies that $\mathrm{Ker}(\mu_*) = [A,\, \mathrm{Der}(A, A^{\otimes 2})]$, so $\Delta_A \in [A,\, \mathrm{Der}(A, A^{\otimes 2})]$, as required. □

For any $\lambda \in A$, the algebra $\Pi^\lambda(A)$ is an A-ring: it comes equipped with a canonical algebra homomorphism $A \to \Pi^\lambda(A)$. Every representation of $\Pi^\lambda(A)$ can thus be regarded as a representation of A. Conversely, given a representation of A, one can ask whether it lifts to a representation of $\Pi^\lambda(A)$. The following theorem provides a simple homological criterion for the existence and uniqueness of such liftings.

Theorem 2.2. *Let A be a smooth algebra, and let $\varrho : A \to \mathrm{End}(V)$ be a representation of A on a (not necessarily finite-dimensional) vector space V. Then ϱ can be extended to a representation of $\Pi^\lambda(A)$ if and only if the homology class of $\varrho(\lambda)$ in $\mathrm{H}_0(A, \mathrm{End}\, V)$ is zero, i.e. $\varrho(\lambda) \in [\varrho(A), \mathrm{End}(V)]$. If it exists, an extension of ϱ to $\Pi^\lambda(A)$ is unique if and only if $\mathrm{H}_1(A, \mathrm{End}\, V) = 0$.*

Proof. We will use the same notation as in the proof of Lemma 2.1. Thus, for a fixed $\lambda \in A$, we identify $\Pi^\lambda(A) = T_A(\Omega^1 A)^\star/\langle \Delta_A - \lambda \rangle$, with $\Delta_A \in (\Omega^1 A)^\star$ corresponding to the natural inclusion $\Omega^1 A \hookrightarrow A^{\mathrm{e}}$. A representation $\varrho : A \to \mathrm{End}(V)$ can be extended then to a representation of $\Pi^\lambda(A)$ if and only if there is an A-ring map $\tilde{\varrho} : T_A(\Omega^1 A)^\star \to \mathrm{End}(V)$, such that $\tilde{\varrho}(\Delta_A) = \varrho(\lambda)$. By the universal property of tensor algebras, such a map is uniquely determined by its restriction to $(\Omega^1 A)^\star$. Thus, regarding $\mathrm{End}(V)$ as a bimodule over A via ϱ, we conclude that ϱ lifts to $\Pi^\lambda(A)$ iff there is $\tilde{\varrho} \in \mathrm{Hom}_{A^{\mathrm{e}}}((\Omega^1 A)^\star, \mathrm{End}\, V)$, mapping Δ_A to $\varrho(\lambda)$. Here, the bimodule $\mathrm{End}(V)$ is interpreted as a *right* A^{e}-module.

Now, since A is smooth, the canonical map $\Omega^1 A \to (\Omega^1 A)^{\star\star}$ is an isomorphism, and we can identify

$$\mathrm{Hom}_{A^{\mathrm{e}}}((\Omega^1 A)^{\star}, \,\mathrm{End}\,V) \cong \mathrm{End}(V) \otimes_{A^{\mathrm{e}}} \Omega^1 A \ .$$

Under this identification, the condition $\tilde{\varrho}(\Delta_A) = \varrho(\lambda)$ becomes

$$\exists\, f_i \otimes d_i \in \mathrm{End}(V) \otimes_{A^{\mathrm{e}}} \Omega^1 A \ : \ \sum_i f_i\, \Delta_A(d_i) = \varrho(\lambda) \ . \tag{2.5}$$

Tensoring the exact sequence of A^{e}-modules $0 \to \Omega^1 A \to A^{\mathrm{e}} \to A \to 0$ with $\mathrm{End}(V)$, we get

$$0 \to \mathrm{H}_1(A,\,\mathrm{End}\,V) \to \mathrm{End}(V) \otimes_{A^{\mathrm{e}}} \Omega^1 A \xrightarrow{\partial} \mathrm{End}(V) \xrightarrow{p} \mathrm{H}_0(A,\,\mathrm{End}\,V) \to 0\,, \tag{2.6}$$

with map in the middle given by $\partial : \, f \otimes d \mapsto f\,\Delta_A(d)$, and p being the canonical projection. The condition (2.5) now says that $\varrho(\lambda) \in \mathrm{Im}(\partial)$, and, by exactness of (2.6), this is equivalent to $\varrho(\lambda) \in \mathrm{Ker}\, p$. Thus, ϱ can be extended to $\Pi^{\lambda}(A)$ if and only $\varrho(\lambda)$ vanishes in $\mathrm{H}_0(A,\,\mathrm{End}\,V)$.

Note that the fibre of the map ∂ over $\varrho(\lambda) \in \mathrm{End}(V)$ consists of the different liftings of the given action ϱ to $\Pi^{\lambda}(A)$. Again, by exactness of (2.6), this fibre can be identified with $\mathrm{H}_1(A,\,\mathrm{End}\,V)$. In particular, if ϱ admits an extension to $\Pi^{\lambda}(A)$, this extension is unique if and only if $\mathrm{H}_1(A,\,\mathrm{End}\,V) = 0$. □

As an immediate corollary of Theorem 2.2, we get

Corollary 2.1. *If* $\boldsymbol{\lambda} \in \mathbb{C} \otimes_{\mathbb{Z}} K_0(A)$, *then* $\varrho : \, A \to \mathrm{End}(V)$ *can be extended to* $\Pi^{\boldsymbol{\lambda}}(A)$ *if and only if* $\varrho_* \,\mathrm{Tr}_A(\boldsymbol{\lambda}) = 0$, *where* $\varrho_* : \, \mathrm{HH}_0(A) \to \mathrm{H}_0(A,\,\mathrm{End}\,V)$ *is the map induced by* ϱ *on Hochschild homology.*

We now apply Theorem 2.2 to finite-dimensional representations. The next result is a generalization of [CB2], Theorem 3.3, which deals with path algebras of quivers.

Proposition 2.1. *Let A be a smooth algebra, and let $\varrho : \, A \to \mathrm{End}(V)$ be a representation of A on a finite-dimensional vector space V. Then, ϱ lifts to a representation of $\Pi^{\lambda}(A)$ if and only if the trace of $\varrho(\lambda)$ on any A-module direct summand of V is zero. Moreover, if $\varrho \in \mathrm{Rep}(A, V)$ lifts, then the fibre $\pi^{-1}(\varrho)$ of the canonical map $\pi : \, \mathrm{Rep}(\Pi^{\lambda}(A), V) \to \mathrm{Rep}(A, V)$ is isomorphic to $\mathrm{Ext}^1_A(V, V)^*$.*

Proof. If V is finite-dimensional, the trace pairing on $\mathrm{End}(V)$ yields the linear isomorphism

$$\mathrm{End}(V) \xrightarrow{\sim} \mathrm{End}(V)^* \ , \quad e \mapsto [\, f \mapsto \mathrm{tr}_V(ef)\,] \ , \tag{2.7}$$

where "$*$" stands for the usual duality with $\mathbb{C}$. Now, if we make $\mathrm{End}(V)^*$ an A-A-bimodule in the natural way (i. e., letting $\langle a.\varphi.b,\ f\rangle := \langle \varphi,\ \varrho(b) f \varrho(a)\rangle$ for $a, b \in A$ and $\varphi \in \mathrm{End}(V)^*$), then (2.7) becomes an isomorphism of bimodules. In particular, it maps bijectively the center of the bimodule $\mathrm{End}(V)$ onto the center of the bimodule $\mathrm{End}(V)^*$. The center of $\mathrm{End}(V)$ is equal, by definition, to $\mathrm{End}_A(V)$, while the center of $\mathrm{End}(V)^{\star}$ consists of all functionals on $\mathrm{End}(V)$, vanishing on $[A,\,\mathrm{End}(V)]$, and thus can be identified with $(\mathrm{End}(V)/[A,\,\mathrm{End}(V)])^*$. In this way, (2.7) restricts to an isomorphism $\mathrm{End}_A(V) \xrightarrow{\sim} \mathrm{H}_0(A,\,\mathrm{End}\,V)^*$, which, upon dualizing with $\mathbb{C}$, becomes

$$\mathrm{H}_0(A,\,\mathrm{End}\,V) \xrightarrow{\sim} \mathrm{End}_A(V)^* \ , \quad \bar{f} \mapsto [\, e \mapsto \mathrm{tr}_V(ef)\,] \ . \tag{2.8}$$

Now, let $\varrho : A \to \mathrm{End}(V)$ be a representation of A on V that lifts to $\Pi^\lambda(A)$, and suppose that V has a direct A-linear summand, say W. By Theorem 2.2, the class of $\varrho(\lambda)$ in $\mathrm{H}_0(A, \mathrm{End}\, V)$ is zero, and hence so is its image under (2.8). In other words, we have $\mathrm{tr}_V[e\varrho(\lambda)] = 0$ for all $e \in \mathrm{End}_A(V)$. Taking $e \in \mathrm{End}_A(V)$ to be a projection onto W, we get $\mathrm{tr}_V[e\varrho(\lambda)] = \mathrm{tr}_W[\varrho(\lambda)] = 0$, which proves the first implication of the theorem.

For the converse, it suffices to consider only indecomposable representations of A. Thus, let $\varrho : A \to \mathrm{End}(V)$ be such a representation on V. By Fitting's Lemma, $\mathrm{End}_A(V)$ is then a local ring: every $e \in \mathrm{End}_A(V)$ can be written as $e = c\,\mathrm{Id}_V + \theta$, with $c \in \mathbb{C}$ and $\theta \in \mathrm{End}_A(V)$ being nilpotent. Now, if we assume that $\mathrm{tr}_V[\varrho(\lambda)] = 0$, then $\mathrm{tr}_V[e\varrho(\lambda)] = 0$ for any $e \in \mathrm{End}_A(V)$. The class of $\varrho(\lambda)$ in $\mathrm{H}_0(A, \mathrm{End}\, V)$ lies thus in the kernel of (2.8) and hence is zero. By Theorem 2.2, we conclude that ϱ lifts to a representation of $\Pi^\lambda(A)$.

To prove the last statement, we note that $\pi^{-1}(\varrho) \cong \mathrm{H}_1(A, \mathrm{End}\, V)$ in view of the exact sequence (2.6). Thus, it remains only to show that

$$\mathrm{H}_1(A, \mathrm{End}\, V) \cong \mathrm{Ext}^1_A(V, V)^* \ . \tag{2.9}$$

Since V is finite-dimensional, there is a canonical linear isomorphism $\mathrm{End}(V) \cong V \otimes V^*$, which is also a bimodule map if V and V^* are equipped with natural A-module structures, via ϱ. Using this isomorphism, we identify (see [CE], Corollary 4.4, p. 170)

$$\mathrm{H}_1(A, \mathrm{End}\, V) \cong \mathrm{H}_1(A, V \otimes V^*) \cong \mathrm{Tor}_1^{A^{\mathrm{e}}}(A, V \otimes V^*) \cong \mathrm{Tor}_1^A(V^*, V) \ .$$

Now, since V has a resolution by finitely generated projective modules (for example, we may take $P^\bullet = [\,0 \to \Omega^1 A \otimes_A V \to A \otimes V \to 0\,]$), the isomorphism $\mathrm{Tor}_1^A(V^*, V) \cong \mathrm{Ext}^1_A(V, V)^*$ is standard homological algebra, see Lemma 2.2 below. □

Lemma 2.2. *Let A be any $\mathbb{C}$-algebra, and let U be an A-module having a resolution $P^\bullet \to U$ by finitely generated projective modules. Then, for any A-module V of finite dimension over $\mathbb{C}$ and for all $n \geq 0$, we have*

$$\mathrm{Tor}_n^A(V^*, U) \cong \mathrm{Ext}^n_A(U, V)^* \ .$$

Proof. Observe that if $\dim_{\mathbb{C}}(V) < \infty$, then $\dim_{\mathbb{C}}[\,V^* \otimes_A P\,] < \infty$ for any finitely generated A-module P. Hence, the natural maps

$$V^* \otimes_A P \hookrightarrow (V^* \otimes_A P)^{**} = \mathrm{Hom}_{\mathbb{C}}(V^* \otimes_A P, \mathbb{C})^* \cong \mathrm{Hom}_A(P, V^{**})^* \cong \mathrm{Hom}_A(P, V)^*$$

are isomorphisms. Applying this to the projective resolution $P^\bullet \to U$, we get

$$\mathrm{Tor}_n^A(V^*, U) \cong H_n(V^* \otimes_A P^\bullet) \cong H_n[\,\mathrm{Hom}_A(P^\bullet, V)^*\,] \cong H^n[\,\mathrm{Hom}_A(P^\bullet, V)\,]^* \ .$$

Whence $\mathrm{Tor}_n^A(V^*, U) \cong \mathrm{Ext}^n_A(U, V)^*$ for all $n \geq 0$, as required. □

Remark. In the special case, when A is the path algebra of a quiver, Proposition 2.1 was proven earlier, by a different method, by Crawley-Boevey (see [CB2], Theorem 3.3). With identifications (2.8) and (2.9), our basic exact sequence (2.6) becomes

$$0 \to \mathrm{Ext}^1_A(V, V)^* \to \mathrm{End}(V) \otimes_{A^{\mathrm{e}}} \Omega^1 A \to \mathrm{End}(V) \to \mathrm{End}_A(V)^* \to 0 \ , \tag{2.10}$$

which, in the quiver case, agrees with [CB2], Lemma 3.1.

2.2. **One-point extensions.** If A is a unital associative algebra, and I a left module over A, we define the *one-point extension* of A by I to be the ring of triangular matrices

$$A[I] := \begin{pmatrix} A & I \\ 0 & \mathbb{C} \end{pmatrix} \tag{2.11}$$

with matrix addition and multiplication induced from the module structure of I. Clearly, $A[I]$ is a unital associative algebra, with identity element being the identity matrix. There are two distinguished idempotents in $A[I]$: namely

$$e := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad e_\infty := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\ . \tag{2.12}$$

If A is indecomposable (e.g., A is a commutative integral domain), then (2.12) form a complete set of primitive orthogonal idempotents in $A[I]$.

A module over $A[I]$ can be identified with a triple $\boldsymbol{V} = (V,\, V_\infty,\, \varphi)$, where V is an A-module, V_∞ is a $\mathbb{C}$-vector space and $\varphi :\ I \otimes V_\infty \to V$ is an A-module map. Using the standard matrix notation, we will write the elements of $\boldsymbol{V}$ as column vectors $(v, w)^T$ with $v \in V$ and $w \in V_\infty$; the action of $A[I]$ is then given by

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} a.v + \varphi(b \otimes w) \\ cw \end{pmatrix}\ .$$

Now, if $\boldsymbol{U} = (U,\, U_\infty,\, \varphi_U)$ and $\boldsymbol{V} = (V,\, V_\infty,\, \varphi_V)$ are two $A[I]$-modules, a homomorphism $\boldsymbol{U} \to \boldsymbol{V}$ is determined by a pair of maps (f, f_∞), with $f \in \mathrm{Hom}_A(U, V)$ and $f_\infty \in \mathrm{Hom}_{\mathbb{C}}(U_\infty, V_\infty)$, making the following diagram commutative

$$\begin{array}{ccc} I \otimes U_\infty & \xrightarrow{\varphi_U} & U \\ {\scriptstyle \mathrm{Id} \otimes f_\infty}\downarrow & & \downarrow{\scriptstyle f} \\ I \otimes V_\infty & \xrightarrow{\varphi_V} & V \end{array} \tag{2.13}$$

If $\boldsymbol{V}$ is finite-dimensional, with $\dim_{\mathbb{C}} V = n$ and $\dim_{\mathbb{C}} V_\infty = n_\infty$, we call $\boldsymbol{n} = (n,\, n_\infty)$ the *dimension vector* of $\boldsymbol{V}$.

The next lemma gathers together a few basic properties of one-point extensions.

Lemma 2.3. (1) $A[I]$ *is canonically isomorphic to* $T_{\tilde{A}}(I)$, *where* $\tilde{A} := A \times \mathbb{C}$.
(2) *If A is smooth and I is a f. g. projective A-module, then $A[I]$ is smooth.*
(3) $I \mapsto A[I]$ *is a functor from* $\mathtt{Mod}(A)$ *to the category of associative algebras.*
(4) *The natural projection* $\theta :\ A[I] \to A$ *is a flat ring epimorphism.*
(5) *There is an isomorphism of abelian groups* $K_0(A[I]) \cong K_0(A) \oplus \mathbb{Z}$.

Proof. (1) We identify $\tilde{A}$ with the subalgebra of diagonal matrices in $A[I]$ and I with the complementary nilpotent ideal $\tilde{I} \subset A[I]$:

$$\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & \mathbb{C} \end{pmatrix}\ , \quad \tilde{I} := \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}\ . \tag{2.14}$$

By the universal property of tensor algebras, the inclusions $\tilde{A} \hookrightarrow A[I]$ and $\tilde{I} \hookrightarrow A[I]$ can then be extended to an algebra map $\phi : T_{\tilde{A}}(\tilde{I}) \to A[I]$, which is a required isomorphism. Indeed, as $A[I] = \tilde{A} \oplus \tilde{I}$, it is clear that ϕ is surjective, its kernel being $\bigoplus_{n\ge 2} \tilde{I}^{\otimes n}$. To see that $\mathrm{Ker}(\phi) = 0$, it suffices to see that $\tilde{I} \otimes_{\tilde{A}} \tilde{I} = 0$. For this, we notice that $e_\infty \in \tilde{A}$ acts on $\tilde{I}$ as identity on the right and as zero on the left: thus, if $b_1 \otimes b_2 \in \tilde{I} \otimes_{\tilde{A}} \tilde{I}$, we have $b_1 \otimes b_2 = b_1 e_\infty \otimes b_2 = b_1 \otimes e_\infty b_2 = 0$.

(2) If I is a f. g. module over A, then $\tilde{I}$ is a f. g. bimodule over $\tilde{A}$. Hence, if A is a f. g. algebra, so is $A[I]$ by (1). So it suffices to show that $A[I]$ is quasi-free if A is quasi-free and I is projective. Again, in view of (1), it suffices to show that $\tilde{I}$ is a projective $\tilde{A}$-bimodule (see [CQ], Prop. 5.3). Now, if I is a projective A-module, then it is isomorphic to a direct summand of a free module $A \otimes V$ and $\tilde{I}$ is isomorphic to a direct summand of $\tilde{A}e \otimes V \otimes e_\infty \tilde{A}$. The latter is a projective $\tilde{A}$-bimodule, since it is a direct summand of $\tilde{A} \otimes V \otimes \tilde{A}$.

(3) Any A-module map $f : I_1 \to I_2$ gives rise to an $\tilde{A}$-bimodule map $\tilde{f} : \tilde{I}_1 \to \tilde{I}_2$. Identifying $A[I_1] = T_{\tilde{A}}(\tilde{I}_1)$ and $A[I_2] = T_{\tilde{A}}(\tilde{I}_2)$, we may extend $I \mapsto A[I]$ to morphisms by $A[f] := T_{\tilde{A}}(\tilde{f})$. As $T_{\tilde{A}}$ is a functor on bimodules, the result follows.

(4) The projection θ is given by

$$\theta \, : \, A[I] \to A \, , \quad \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mapsto a \, . \tag{2.15}$$

It is immediate from (2.15) that $A \cong A[I]\, e$ as a left $A[I]$-module. Since e is an idempotent, $A[I]\, e$ is projective and hence flat.

(5) The algebra map

$$\tilde{\theta} \, : \, A[I] \to \tilde{A} \, , \quad \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mapsto (a, \, c) \, . \tag{2.16}$$

has a nilpotent kernel (equal to $\tilde{I}$). By [B], Proposition IX.1.3, it then induces isomorphisms $K_i(A[I]) \cong K_i(\tilde{A})$ for all $i = 0, 1, 2, \ldots$ In particular, we have $K_0(A[I]) \cong K_0(\tilde{A}) \cong K_0(A) \oplus \mathbb{Z}$. □

We will also need the following result relating homological properties of A and $A[I]$.

Proposition 2.2. *Let A be a finitely generated hereditary algebra, and let $B := A[I]$ be the one-point extension of A by a f. g. projective A-module. Then, for any finite-dimensional B-modules $\boldsymbol{U} = (U, U_\infty)$ and $\boldsymbol{V} = (V, V_\infty)$, we have*

$$\chi_B(\boldsymbol{U}, \boldsymbol{V}) = \chi_A(U, V) + \dim(U_\infty)\,[\,\dim(V_\infty) - \dim\, \mathrm{Hom}_A(I, V)\,] \, , \tag{2.17}$$

where χ_A and χ_B denote the Euler characteristics for the Ext-groups over the algebras A and B respectively.

Proof. Let $S := \mathbb{C}e \oplus \mathbb{C}e_\infty \subseteq B$. Decomposing $B = Be \oplus Be_\infty$ and $B = eB \oplus e_\infty B$, we observe that $B \otimes_S B \cong (Be \otimes eB) \oplus (Be_\infty \otimes e_\infty B)$ as bimodules over B. The standard resolution of B then reads

$$0 \to \Omega^1_S B \to (Be \otimes eB) \oplus (Be_\infty \otimes e_\infty B) \to B \to 0 \, , \tag{2.18}$$

where $\Omega^1_S B$ is the kernel of multiplication map $B \otimes_S B \to B$. Tensoring (2.18) with $\boldsymbol{U}$ on the right and dualizing with $\boldsymbol{V}$, we get the exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_B(\boldsymbol{U}, \boldsymbol{V}) \quad &\to \quad \mathrm{Hom}_{\mathbb{C}}(U, V) \,\oplus\, \mathrm{Hom}_{\mathbb{C}}(U_\infty, V_\infty) \to \\ &\to \quad \mathrm{Hom}_B(\Omega^1_S B \otimes_B \boldsymbol{U}, \boldsymbol{V}) \to \mathrm{Ext}^1_B(\boldsymbol{U}, \boldsymbol{V}) \to 0 \, . \end{aligned} \tag{2.19}$$

To compute the Euler characteristics $\chi_B(\boldsymbol{U}, \boldsymbol{V})$ we thus need to compute $\Omega^1_S B$. To this end, we will use Lemma 2.3(1) to identify $B \cong T_{\tilde{A}}(\tilde{I}) = \tilde{A} \oplus \tilde{I}$, see notation (2.14). Tensoring then the exact sequence $0 \to \Omega^1_S \tilde{A} \to \tilde{A} \otimes_S \tilde{A} \to \tilde{A} \to 0$ with B on both sides and restricting to the bimodules of differentials, we get

$$0 \to B \otimes_{\tilde{A}} \Omega^1_S \tilde{A} \otimes_{\tilde{A}} B \to \Omega^1_S B \to \Omega^1_{\tilde{A}} B \to 0 \, . \tag{2.20}$$

Now, observe that $\Omega^1_{\tilde{A}} B \cong B \otimes_{\tilde{A}} \tilde{I} \otimes_{\tilde{A}} B$ (see [BD], (17)) and $\Omega^1_S \tilde{A} \cong \Omega^1_{\mathbb{C}\times\mathbb{C}}(A \times \mathbb{C}) \cong \Omega^1 A \oplus \Omega^1\mathbb{C} = \Omega^1 A$ (see [BD], (54)). Hence, tensoring (2.20) again with $\boldsymbol{U}$, dualizing with $\boldsymbol{V}$ and identifying terms, we get

(2.21)

$$0 \to \mathrm{Hom}_A(I \otimes U_\infty, V) \to \mathrm{Hom}_B(\Omega^1_S B \otimes_B \boldsymbol{U}, \boldsymbol{V}) \to \mathrm{Hom}_A(\Omega^1 A \otimes_A U, V) \to 0$$

Note that no Ext-terms occur in the exact sequence (2.21), since $\tilde{I}$ is a projective $\tilde{A}$-bimodule, and $\Omega^1_{\tilde{A}} B$ is then a projective B-bimodule.

Finally, the same procedure applied to $0 \to \Omega^1 A \to A \otimes A \to A \to 0$ yields

(2.22)

$$0 \to \mathrm{Hom}_A(U, V) \to \mathrm{Hom}_{\mathbb{C}}(U, V) \to \mathrm{Hom}_A(\Omega^1 A \otimes_A U,\, V) \to \mathrm{Ext}^1_A(U, V) \to 0\ .$$

Now, since A is finitely generated, $\Omega^1 A$ is finitely generated, and the space

$$\mathrm{Hom}_A(\Omega^1 A \otimes_A U,\, V) \cong \mathrm{Hom}_{A^e}(\Omega^1 A,\, \mathrm{Hom}_{\mathbb{C}}(U, V))$$

is finite-dimensional. From (2.22), it follows then that $\chi_A(U, V)$ is well-defined. Also, since I is finitely generated, $\mathrm{Hom}_A(I \otimes U_\infty, V) \cong \mathrm{Hom}_A(I, V) \otimes \mathrm{Hom}_{\mathbb{C}}(U_\infty, \mathbb{C})$ is finite-dimensional, and hence so is $\mathrm{Hom}_B(\Omega^1_S B \otimes_B \boldsymbol{U}, \boldsymbol{V})$, the middle term of the exact sequence (2.21). It follows from (2.19) that $\chi_B(\boldsymbol{U}, \boldsymbol{V})$ is well-defined. Now, summing up the Euler characteristics of all the three sequences (2.19), (2.21) and (2.22), we get the desired formula (2.17). □

2.3. **Representation varieties.** Let R be a finitely generated associative algebra. Fix S, a finite-dimensional semisimple subalgebra of R, and V, a finite-dimensional S-module. By definition, the *representation variety* $\mathrm{Rep}_S(R, V)$ of R over S parametrizes all R-module structures on the vector space V extending the given S-module structure on it. The S-module structure on V determines an algebra homomorphism $S \to \mathrm{End}(V)$ making $\mathrm{End}(V)$ an S-algebra. The points of $\mathrm{Rep}_S(R, V)$ can thus be interpreted as S-algebra maps $R \to \mathrm{End}(V)$.

If $S = \mathbb{C}$, we simply write $\mathrm{Rep}(R, V)$ for $\mathrm{Rep}_{\mathbb{C}}(R, V)$. Choosing a basis in V and a presentation of R, say $R \cong \mathbb{C}\langle x_1, \dots, x_m\rangle / I$, we can identify in this case

$$\mathrm{Rep}(R, V) \cong \{(X_1, \dots, X_m) \in \mathtt{Mat}(n, \mathbb{C})^m : p(X_1, \dots, X_m) = 0\,,\ \forall p \in I\}\ .$$

Thus $\mathrm{Rep}(R, V)$ is an affine variety[1]. In general, for any semisimple $S \subseteq R$, $\mathrm{Rep}_S(R, V)$ can be identified with a fibre of the canonical morphism of affine varieties $r : \mathrm{Rep}(R, V) \to \mathrm{Rep}(S, V)$, and hence it is an affine variety as well.

The group $\mathrm{Aut}_S(V)$ of S-linear automorphisms of V acts on $\mathrm{Rep}_S(R, V)$ in the natural way, with scalars $\mathbb{C}^* \subseteq \mathrm{Aut}_S(V)$ acting trivially. We set $\mathrm{G}_S(V) := \mathrm{Aut}_S(V)/\mathbb{C}^*$. Since V is semisimple, $V \cong \bigoplus_i V_i^{\oplus n_i}$, with V_i non-isomorphic simple S-modules, and $\mathrm{Aut}_S(V) \cong \prod_i \mathrm{GL}(n_i, \mathbb{C})$. Thus $\mathrm{G}_S(V)$ is reductive. The orbits of $\mathrm{G}_S(V)$ on $\mathrm{Rep}_S(R, V)$ are in 1-1 correspondence with isomorphism classes of R-modules, which are isomorphic to V as S-modules. The stabilizer of a point $\varrho : R \to \mathrm{End}(V)$ in $\mathrm{Rep}_S(R, V)$ is canonically isomorphic to $\mathrm{Aut}_R(V_\varrho)/\mathbb{C}^* \subseteq \mathrm{G}_S(V)$, where V_ϱ is the left R-module corresponding to ϱ. Now, the closure of any orbit $\mathcal{O}_M$ contains a unique closed orbit, corresponding to a semisimple R-module with the same composition factors and multiplicities as M. Thus the space $\mathrm{Rep}_S(R, V) /\!/ \mathrm{G}_S(V)$ of closed orbits in $\mathrm{Rep}_S(R, V)$ is an affine variety, whose (closed) points are in bijection with isomorphism classes of semisimple R-modules M isomorphic to V as S-modules.

[1] Here, by an affine variety we mean an affine scheme of finite type over $\mathbb{C}$.

Typically, representation varieties of R are defined over subalgebras spanned by idempotents. For example, let $\{e_i\}_{i\in I}$ be a complete set of orthogonal idempotents in R. Set $S := \bigoplus_{i\in I} \mathbb{C}\, e_i \subseteq R$. A finite-dimensional S-module is then isomorphic to a direct sum $\mathbb{C}^{\boldsymbol{n}} := \bigoplus_{i\in I} \mathbb{C}^{n_i}$, each e_i acting as the projection onto the i-th component. The corresponding representation variety $\mathrm{Rep}_S(R, \mathbb{C}^{\boldsymbol{n}})$, which we simply denote by $\mathrm{Rep}_S(R, \boldsymbol{n})$ in this case, parametrizes all algebra maps $R \to \mathrm{End}(\mathbb{C}^{\boldsymbol{n}})$, sending e_i to the projection onto $\mathbb{C}^{n_i}$. The group $\mathrm{G}_S(\mathbb{C}^{\boldsymbol{n}})$ (to be denoted $\mathrm{G}_S(\boldsymbol{n})$) is isomorphic to $\prod_{i\in I} \mathrm{GL}(n_i, \mathbb{C})/\mathbb{C}^*$, with $\mathbb{C}^*$ embedded diagonally.

We will need a few general results on geometry of representation varieties. First, we recall the following well-known fact.

Theorem 2.3. *If R is a smooth algebra, then* $\mathrm{Rep}(R, V)$ *is a smooth variety.*

Proof. See, for example, [G], Proposition 19.1.4. □

The next lemma implies that Theorem 2.3 holds also in the relative setting: for the varieties $\mathrm{Rep}_S(R, V)$ over any semisimple subalgebra $S \subseteq R$. We recall that $\mathrm{Rep}_S(R, V)$ can be identified with a fibre of the canonical map $r : \mathrm{Rep}(R, V) \to \mathrm{Rep}(S, V)$, $\varrho \mapsto \varrho|_S$, restricting the representations of R to S.

Lemma 2.4. *If $\varrho \in \mathrm{Rep}_S(R, V)$ and $\mathrm{Rep}(R, V)$ is smooth at ϱ, then so is $\mathrm{Rep}_S(R, V)$.*

Proof. Let $T_\varrho \mathrm{Rep}(R, V)$ denote the tangent space of $\mathrm{Rep}(R, V)$ at a point ϱ. We identify $T_\varrho \mathrm{Rep}(R, V)$ with the space of derivations $\mathrm{Der}(R, \mathrm{End}\, V)$, where $\mathrm{End}(V)$ is equipped with natural R-bimodule structure via ϱ (see, e.g., [G], Sect. 12.4). The differential of the restriction map is then given by

$$dr_\varrho : \mathrm{Der}(R, \mathrm{End}V) \to \mathrm{Der}(S, \mathrm{End}V), \quad d \mapsto d|_S \ . \tag{2.23}$$

We need to show that dr_ϱ is surjective.

To this end, we note that $\mathrm{Der}(R, –)$ and $\mathrm{Der}(S, –)$ are represented by the bimodules $\Omega^1 R$ and $\Omega^1 S$, which are related by the cotangent exact sequence (see [CQ], Prop. 2.9):

$$0 \to R \otimes_S \Omega^1 S \otimes_S R \to \Omega^1 R \to \Omega^1_S R \to 0 \ , \tag{2.24}$$

where $\Omega^1_S R := \mathrm{Ker}[\, R \otimes_S R \to R\,]$. Dualizing (2.24) by $\mathrm{End}(V)$ and identifying terms, we get

$$0 \to \mathrm{Der}_S(R, \mathrm{End}V) \to \mathrm{Der}(R, \mathrm{End}V) \xrightarrow{dr} \mathrm{Der}(S, \mathrm{End}V) \xrightarrow{\delta} \mathrm{Ext}^1_{R^e}(\Omega^1_S R, \mathrm{End}V)$$

with map in the middle being exactly (2.23). The connecting homomorphism δ sends a derivation $d : S \mapsto \mathrm{End}(V)$ (or equivalently, the corresponding bimodule map $\Omega^1 S \to \mathrm{End}(V)$) to the class of the exact sequence (2.24) pushed out along d. Now, since S is a separable algebra (see [CQ], Sect. 4), the sequence (2.24) splits, and the image of δ in $\mathrm{Ext}^1_{R^e}(\Omega^1_S R, \mathrm{End}V)$ is zero. This implies the surjectivity of dr, finishing the proof of the lemma. □

Now, given an algebra A, we set $R := T_A \mathrm{Der}(A, A^{\otimes 2})$, see Section 2.1. If A is finitely generated, then so is R, and we consider the variety $\mathrm{Rep}(R, V)$ of representations of R on a vector space V. Recall that R contains a distinguished element: the derivation $\Delta_A : A \to A^{\otimes 2}$ defined by $x \mapsto x \otimes 1 - 1 \otimes x$. We write

$$\mu : \mathrm{Rep}(R, V) \to \mathrm{End}(V) \ , \quad \varrho \mapsto \varrho(\Delta_A) \ ,$$

for the evaluation map at Δ_A and consider its fibre $F_\xi := \mu^{-1}[\mu(\xi)]$ for some fixed representation $\xi \in \mathrm{Rep}(R, V)$.

Proposition 2.3. *Assume that A is a smooth algebra. Then F_ξ is smooth at $\varrho \in \mathrm{Rep}(R, V)$ if and only if $\mathrm{End}_R(V_\varrho) \cong \mathbb{C}$.*

Proof. First of all, since A is smooth, the bimodule $\mathrm{Der}(A, A^{\otimes 2})$ is f. g. projective, and its tensor algebra R is smooth (see [CQ], Prop. 5.3). Hence, $\mathrm{Rep}(R, V)$ is a smooth variety by Theorem 2.3. By Lemma 2.1, we also have $\Delta_A \in [A,\, \mathrm{Der}(A, A^{\otimes 2})]$, so that $\mathrm{tr}_V[\varrho(\Delta)] = 0$ for any $\varrho \in \mathrm{Rep}(R, V)$. Thus

$$\mu : \mathrm{Rep}(R, V) \to \mathrm{End}(V)_0 \ , \tag{2.25}$$

where $\mathrm{End}(V)_0 := \{f \in \mathrm{End}(V) \ : \ \mathrm{tr}_V(f) = 0\}$.

To compute the differential of (2.25) we identify $T_\varrho \,\mathrm{Rep}(R, V) \cong \mathrm{Der}(R, \mathrm{End}\, V)$ as in (the proof of) Lemma 2.4 and also $T_\mu \,\mathrm{End}(V)_0 \cong \mathrm{End}(V)_0$. Then it is easy to see that

$$d\mu_\varrho : \ \mathrm{Der}(R,\, \mathrm{End}\, V) \to \mathrm{End}(V)_0 \ , \quad \delta \mapsto \delta(\Delta_A) \ . \tag{2.26}$$

Now, observe that the map $d\mu_\varrho^*$ dual to (2.26) fits into the commutative diagram

$$\begin{array}{ccc} \mathrm{End}(V)_0^* & \xrightarrow{d\mu_\varrho^*} & \mathrm{Der}(R,\, \mathrm{End}\, V)^* \\ \mathrm{tr}_V \big\uparrow & & \big\uparrow i(\mathrm{Tr}\,\hat{\omega}) \\ \mathrm{End}(V)/\mathbb{C} & \xrightarrow{\mathrm{ad}} & \mathrm{Der}(R,\, \mathrm{End}\, V) \end{array} \tag{2.27}$$

with vertical arrows being *isomorphisms*. Here, the map tr_V comes from the trace pairing on $\mathrm{End}(V)$ (and hence, it is obviously an isomorphism), and ad is induced by the canonical map, assigning to $f \in \mathrm{End}(V)$ the inner derivation $\mathrm{ad}(f) : a \mapsto [f,\, \varrho(a)]$. The crucial isomorphism $i(\mathrm{Tr}\,\hat{\omega})$ is constructed explicitly[2] in [CEG] (see *loc. cit.*, the proof of Theorem 6.4.3). Instead of repeating this construction, we simply notice that (2.25) can be interpreted as a *moment map* for the natural action of $\mathrm{GL}(V)/\mathbb{C}^*$ on $\mathrm{Rep}(R,\, V)$. The commutativity of the diagram (2.27) is then equivalent to the defining equation for moment maps (see [CEG], (6.4.7)).

Now, to finish the proof, it remains to note that F_ξ is smooth at ϱ if and only if $d\mu_\varrho$ is surjective. This is equivalent to $d\mu_\varrho^*$ being injective, and hence, in view of (2.27), to Ker(ad) being zero. Since $\mathrm{Ker(ad)} \cong \mathrm{End}_R(V)/\mathbb{C}$, this last condition holds, and the result follows. □

3. The Calogero-Moser Spaces

3.1. **Rings of differential operators.** Let X be a smooth affine irreducible curve over $\mathbb{C}$, with coordinate ring $\mathcal{O} = \mathcal{O}(X)$, and let $\mathcal{D} = \mathcal{D}(X)$ be the ring of differential operators on X. We recall that $\mathcal{D}$ is a filtered algebra $\mathcal{D} = \bigcup_{k \ge 0} \mathcal{D}_k$, with filtration components $0 \subset \mathcal{D}_0 \subset \ldots \subset \mathcal{D}_{k-1} \subset \mathcal{D}_k \subset \ldots$ defined inductively by

$$\mathcal{D}_k := \{\, D \in \mathrm{End}_{\mathbb{C}}\, \mathcal{O} \ : \ [D, f] \in \mathcal{D}_{k-1} \text{ for all } f \in \mathcal{O} \,\} \ .$$

The elements of $\mathcal{D}_k$ are called *differential operators of order* $\le k$. In particular, $\mathcal{D}_0$ consists of multiplication operators by regular functions on X, i.e. $\mathcal{D}_0 = \mathcal{O}$, and $\mathcal{D}_1$ is spanned by $\mathcal{O}$ and the space $\mathrm{Der}(\mathcal{O})$ of derivations of $\mathcal{O}$ (the algebraic vector fields on X). As X is smooth, $\mathcal{O}$ and $\mathrm{Der}(\mathcal{O})$ generate $\mathcal{D}$ as an algebra, and ring-theoretically $\mathcal{D}$ shares many properties in common with the first Weyl algebra

[2]To avoid confusion, here we use the same notation for this map as in [CEG].

$A_1(\mathbb{C})$. For example, like A_1, $\mathcal{D}$ is a simple Noetherian domain of global dimension 1; however, unlike A_1, $\mathcal{D}$ has a nontrivial K-group (see, e. g. [Bj], Ch. 2).

We write $\bar{\mathcal{D}} := \bigoplus_{k=0}^{\infty} \mathcal{D}_k/\mathcal{D}_{k-1}$ for the associated graded ring of $\mathcal{D}$: this is a commutative algebra isomorphic to the coordinate ring of the cotangent bundle T^*X of X. Given a $\mathcal{D}$-module M with a filtration $\{M_k\}$, we also write $\overline{M} := \bigoplus_{k=0}^{\infty} M_k/M_{k-1}$ for the associated graded $\bar{\mathcal{D}}$-module. Using the standard terminology, we say that $\{M_k\}$ is a *good* filtration if $\overline{M}$ is finitely generated.

3.2. **Stable classification of ideals.** Let $K_0(X)$ and $\mathrm{Pic}(X)$ denote the Grothendieck group and the Picard group of X respectively. By definition, $K_0(X)$ is generated by the stable isomorphism classes of (algebraic) vector bundles on X, while the elements of $\mathrm{Pic}(X)$ are the isomorphism classes of line bundles. As X is affine, we may (and often will) identify $K_0(X)$ with $K_0(\mathcal{O})$, the Grothendieck group of the ring $\mathcal{O}$, and $\mathrm{Pic}(X)$ with $\mathrm{Pic}(\mathcal{O})$, the ideal class group of $\mathcal{O}$. There are two natural maps $\mathrm{rk} : K_0(X) \to \mathbb{Z}$ and $\det : K_0(X) \to \mathrm{Pic}(X)$ defined by taking the rank and the determinant of a vector bundle respectively. In the case of curves, it is well-known that $\mathrm{rk} \oplus \det : K_0(X) \xrightarrow{\sim} \mathbb{Z} \oplus \mathrm{Pic}(X)$ is a group isomorphism.

Now, let $\mathcal{I}(\mathcal{D})$ denote the set of isomorphism classes of (nonzero) left ideals of $\mathcal{D}$. Unlike Pic in the commutative case, $\mathcal{I}(\mathcal{D})$ carries no natural structure of a group. However, since $\mathcal{D}$ is a hereditary domain, $\mathcal{I}(\mathcal{D})$ can be identified with the space of isomorphism classes of rank 1 projective modules, and there is a natural map relating $\mathcal{I}(\mathcal{D})$ to $\mathrm{Pic}(X)$:

$$\gamma : \mathcal{I}(\mathcal{D}) \to K_0(\mathcal{D}) \xrightarrow{\iota_*^{-1}} K_0(X) \xrightarrow{\det} \mathrm{Pic}(X) \ . \tag{3.1}$$

The first arrow in (3.1) is the tautological map from $\mathcal{I}(\mathcal{D})$ to the Grothendieck group $K_0(\mathcal{D})$ of the ring $\mathcal{D}$, assigning to the isomorphism class of an ideal its stable isomorphism class in $K_0(\mathcal{D})$. The second arrow ι_*^{-1} is the inverse of the Quillen isomorphism $\iota_* : K_0(X) \xrightarrow{\sim} K_0(\mathcal{D})$ induced by the inclusion $\iota : \mathcal{O} \hookrightarrow \mathcal{D}$ (see [Q], Theorem 7).

The role of the map γ becomes clear from the following result proved in [BW].

Theorem 3.1 (see [BW], Proposition 2.1)**.** *Let M be a projective $\mathcal{D}$-module of rank 1 equipped with a good filtration such that $\overline{M}$ is torsion-free*[3]*. Then*

(*a*) *there is a unique (up to isomorphism) ideal $\mathcal{I}_M \subseteq \mathcal{O}$, such that $\overline{M}$ is isomorphic to a sub-$\bar{\mathcal{D}}$-module of $\bar{\mathcal{D}}\mathcal{I}_M$ of finite codimension (over $\mathbb{C}$);*

(*b*) *the class of $\mathcal{I}_M$ in $\mathrm{Pic}(X)$ and the codimension $n := \dim_{\mathbb{C}} [\bar{\mathcal{D}}\mathcal{I}_M/\overline{M}]$ are independent of the choice of filtration on M, and we have $\gamma[M] = [\mathcal{I}_M]$;*

(*c*) *if M and N are two projective $\mathcal{D}$-modules of rank* 1*, then*

$$[\,M\,] = [\,N\,] \ \text{in}\ K_0(\mathcal{D}) \quad \Longleftrightarrow \quad [\mathcal{I}_M] = [\mathcal{I}_N] \ \text{in}\ \mathrm{Pic}(X) \ .$$

Theorem 3.1 shows that the fibres of γ are precisely the stable isomorphism classes of ideals of $\mathcal{D}$: thus, up to isomorphism in $K_0(\mathcal{D})$, the ideals of $\mathcal{D}$ are classified by the elements of $\mathrm{Pic}(X)$. Our goal is to refine this classification by describing the fibres of γ in geometric terms. As we will see in Section 4, each fibre $\gamma^{-1}[\mathcal{I}]$ naturally breaks up into a countable union of affine varieties $\overline{\mathcal{C}}_n(X, \mathcal{I})$. In the next section, we introduce these varieties and study their geometric properties.

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[3]To define such a filtration it suffices to embed M in $\mathcal{D}$ as a left ideal and filter it by the induced filtration.

3.3. **The definition of Calogero-Moser spaces.** Given a curve X with a line bundle $\mathcal{I}$, we set $A := \mathcal{O}(X)$ and form the one-point extension $B := A[\mathcal{I}]$ of A by $\mathcal{I}$. By Lemma 2.3(2), B is a smooth algebra, since so is the algebra A and $\mathcal{I}$ is a f. g. projective A-module. As in Section 2.2, we will identify the subalgebra of diagonal matrices in B with $\tilde{A} := A \times \mathbb{C}$, and let $\tilde{\theta} : B \twoheadrightarrow \tilde{A}$ denote the natural projection, see (2.14). Since $\tilde{\theta}$ is a nilpotent extension, it is suggestive to think of 'Spec B' as a (noncommutative) infinitesimal 'thickening' of $\operatorname{Spec} \tilde{A} = X \bigsqcup \mathtt{pt}$.

We now prove two auxiliary lemmas. The first lemma implies that B is determined, up to isomorphism, by the class of $\mathcal{I}$ in $\operatorname{Pic}(X)$ and is independent of $\mathcal{I}$ up to Morita equivalence. The second lemma computes the Euler characteristics for finite-dimensional representations of B, refining the result of Proposition 2.2.

Lemma 3.1. *For line bundles $\mathcal{I}$ and $\mathcal{J}$, the algebras $A[\mathcal{I}]$ and $A[\mathcal{J}]$ are*
(a) *Morita equivalent;*
(b) *isomorphic if and only if $\mathcal{J} \cong \mathcal{I}^\tau$ for some $\tau \in \operatorname{Aut}(X)$, where $\mathcal{I}^\tau := \tau^* \mathcal{I}$.*

Proof. (a) Given $\mathcal{I}$ and $\mathcal{J}$, we set $\mathcal{L} := \operatorname{Hom}_A(\mathcal{I}, \mathcal{J})$, which is a line bundle on X isomorphic to $\mathcal{J}\mathcal{I}^\vee = \mathcal{J} \otimes_A \mathcal{I}^\vee$, where $\mathcal{I}^\vee$ is the dual of $\mathcal{I}$. Then, we extend $\mathcal{L}$ to a line bundle over $\tilde{A}$, letting $\tilde{\mathcal{L}} := \mathcal{L} \times \mathbb{C}$, and define $P := \tilde{\mathcal{L}} \otimes_{\tilde{A}} B$, where $B = A[\mathcal{I}]$. Clearly, P is a f. g. projective B-module. On the other hand, since A is a Dedekind domain, $\mathcal{L} \oplus \mathcal{L} \cong A \oplus \mathcal{L}^2$, where $\mathcal{L}^2 = \mathcal{L} \otimes_A \mathcal{L}$, and hence $\tilde{\mathcal{L}} \oplus \tilde{\mathcal{L}} \cong \tilde{A} \oplus \tilde{\mathcal{L}}^2$. It follows that B is isomorphic to a direct summand of $P \oplus P$, so P is a generator in the category of right B-modules. By Morita's Theorem, the ring B is then equivalent to $\operatorname{End}_B(P)$, and we claim that $\operatorname{End}_B(P) \cong A[\mathcal{J}]$. In fact, $\operatorname{End}_B(P) = \operatorname{Hom}_B(\tilde{\mathcal{L}} \otimes_{\tilde{A}} B,\, P) \cong \operatorname{Hom}_{\tilde{A}}(\tilde{\mathcal{L}},\, P)$, and since $P \cong \tilde{\mathcal{L}} \oplus (0,\, \mathcal{L}\mathcal{I})$ as a (right) $\tilde{A}$-module, we have

$$\operatorname{End}_B(P) \cong \begin{pmatrix} \operatorname{End}_A(\mathcal{L}) & \mathcal{L}\mathcal{I} \\ 0 & \mathbb{C} \end{pmatrix} \cong \begin{pmatrix} A & \mathcal{J} \\ 0 & \mathbb{C} \end{pmatrix} = A[\mathcal{J}] \ .$$

(b) First of all, if $\mathcal{J} \cong \mathcal{I}$, then $A[\mathcal{J}] \cong A[\mathcal{I}]$ by functoriality of one-point extensions (see Lemma 2.3(3)). Without loss of generality, we may therefore identify $\mathcal{I}$ and $\mathcal{J}$ with ideals in A. Given an automorphism $\tau \in \operatorname{Aut}(X) = \operatorname{Aut}(A)$, we have then $\mathcal{I}^\tau = \tau^{-1}(\mathcal{I})$, and the map

$$A[\mathcal{I}] \to A[\mathcal{I}^\tau] \ , \quad \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mapsto \begin{pmatrix} \tau^{-1}(a) & \tau^{-1}(b) \\ 0 & c \end{pmatrix} \ ,$$

is a required isomorphism.

Conversely, if $f : A[\mathcal{I}] \to A[\mathcal{J}]$ is an isomorphism of (unital) algebras, then

$$f(e) = \begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad f(e_\infty) = \begin{pmatrix} 0 & -b \\ 0 & 1 \end{pmatrix} \quad \text{for some } b \in \mathcal{J} \ .$$

Indeed, $f(e)$ and $f(e_\infty)$ are (nontrivial) orthogonal idempotents in $A[\mathcal{J}]$, satisfying $f(e) + f(e_\infty) = 1$, and it is easy to see that the only such idempotents are as above. Replacing now f by its composition with the inner automorphism $A[\mathcal{J}] \to A[\mathcal{J}]$, $x \mapsto u\,x\,u^{-1}$, with $u = f(e) + e_\infty$ and $u^{-1} = f(e_\infty) + e$, we may assume that $f(e) = e$ and $f(e_\infty) = e_\infty$. Then $f(e\,A[\mathcal{I}]\,e) = e\,A[\mathcal{J}]\,e$ and $f(e\,A[\mathcal{I}]\,e_\infty) = e\,A[\mathcal{J}]\,e_\infty$. With natural identifications $e\,A[\mathcal{I}]\,e = A = e\,A[\mathcal{J}]\,e$ and $e\,A[\mathcal{I}]\,e_\infty = \mathcal{I}$, $e\,A[\mathcal{J}]\,e_\infty = \mathcal{J}$, this yields an automorphism $\tau = f^{-1} \in \operatorname{Aut}(A)$ and an isomorphism of A-modules: $\mathcal{J} \cong \tau^*(\mathcal{I})$. □

Lemma 3.2. *For any finite-dimensional B-modules $\boldsymbol{U} = (U, U_\infty)$ and $\boldsymbol{V} = (V, V_\infty)$, the Euler characteristics $\chi_B(\boldsymbol{U}, \boldsymbol{V}) := \sum_{k\geq 0} (-1)^k \dim_{\mathbb{C}} \mathrm{Ext}^k_B(\boldsymbol{U},\, \boldsymbol{V})$ is given by*

$$\chi_B(\boldsymbol{U}, \boldsymbol{V}) = \dim(U_\infty)\,[\,\dim(V_\infty) - \dim(V)\,]\ .$$

Proof. First, observe that $\chi_A(U, V) = 0$ for any pair of finite-dimensional A-modules. Indeed, if U and V have disjoint supports, then

$$\mathrm{Hom}_A(U, V) = \mathrm{Ext}^1_A(U, V) = 0\ ,$$

and certainly $\chi_A(U, V) = 0$. By additivity of χ_A, it thus suffices to see that $\chi_A(U, V) = 0$ for modules U and V supported at one point. If $\mathfrak{m}$ is the maximal ideal of A corresponding to that point, we have $\mathrm{Ext}^i_A(U, V) \cong \mathrm{Ext}^i_{A_\mathfrak{m}}(U, V)$ and, by Lemma 2.2, $\mathrm{Ext}^i_{A_\mathfrak{m}}(U, V) \cong \mathrm{Tor}_i^{A_\mathfrak{m}}(V^*, U)^*$ for all $i \geq 0$. Thus

$$\chi_A(U, V) = \chi_{A_\mathfrak{m}}(U, V) = \sum (-1)^i \dim_{\mathbb{C}} \mathrm{Tor}_i^{A_\mathfrak{m}}(V^*, U)\ .$$

The vanishing of $\chi_A(U, V)$ follows now from standard intersection theory, since $A_\mathfrak{m}$ is a regular local ring of (Krull) dimension 1, while $\dim_{A_\mathfrak{m}}(U) + \dim_{A_\mathfrak{m}}(V^*) = 0$ (see [S], Ch. V, Part B, Theorem 1).

Identifying $\mathcal{I}$ with an ideal in A and dualizing the exact sequence $0 \to \mathcal{I} \to A \to A/\mathcal{I} \to 0$ by V, we get

$$\dim_{\mathbb{C}} \mathrm{Hom}_A(\mathcal{I}, V) = \dim_{\mathbb{C}}(V) - \chi_A(A/\mathcal{I}, V) = \dim_{\mathbb{C}}(V)\ . \tag{3.2}$$

The result the lemma follows now from Proposition 2.2. □

Next, we introduce deformed preprojective algebras over B. For this, we need to compute the Chern character map $\mathrm{Tr}_B : K_0(B) \to \mathrm{H}_0(B)$. Recall that $\mathrm{Tr}_* : K_0 \to \mathrm{H}_0$ is a natural transformation of functors on the category of associative algebras, so the algebra map $\tilde{\theta} : B \to \tilde{A}$ gives rise to the commutative diagram

$$\begin{array}{ccc} K_0(B) & \xrightarrow{\mathrm{Tr}_B} & \mathrm{H}_0(B) \\ \downarrow & & \downarrow \\ K_0(\tilde{A}) & \xrightarrow{\mathrm{Tr}_{\tilde{A}}} & \mathrm{H}_0(\tilde{A}) \end{array} \tag{3.3}$$

The two vertical maps in (3.3) are isomorphisms: the first one is given by Lemma 2.3(6), while the second has the obvious inverse (induced by the natural inclusion $\tilde{A} \hookrightarrow B$). We will use these isomorphisms to identify $\mathrm{H}_0(B) \cong \mathrm{H}_0(\tilde{A}) = \tilde{A} \subset B$ and

$$K_0(B) \cong K_0(\tilde{A}) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathrm{Pic}(X)\ . \tag{3.4}$$

Now, for any commutative algebra (e.g., $\tilde{A}$), the Chern character map factors through the rank. Hence, with above identifications, Tr_B is completely determined by its values on the first two summands in (3.4), while vanishing on the last. Since $\mathrm{Tr}_B[(1,0)] = e$ and $\mathrm{Tr}_B[(0,1)] = e_\infty$, the linear map $\mathrm{Tr}_B : \mathbb{C} \otimes_{\mathbb{Z}} K_0(B) \to \mathrm{H}_0(B)$ takes its values in the two-dimensional subspace S of B spanned by the idempotents e and e_∞. Identifying S with $\mathbb{C}^2$, we may regard the vectors $\boldsymbol{\lambda} := (\lambda,\, \lambda_\infty) = \lambda e + \lambda_\infty e_\infty \in S$ as weights for the family of deformed preprojective algebras associated to B:

$$\Pi^{\boldsymbol{\lambda}}(B) = T_B \mathrm{Der}(B, B^{\otimes 2}) / \langle \Delta_B - \boldsymbol{\lambda} \rangle\ . \tag{3.5}$$

Since A is an integral domain, $\{e,\, e_\infty\}$ is a complete set of primitive orthogonal idempotents in $\Pi^{\boldsymbol\lambda}(B)$, and $S = \mathbb{C}e \oplus \mathbb{C}e_\infty$ is the associated semisimple subalgebra of $\Pi^{\boldsymbol\lambda}(B)$.

Now, for each $\boldsymbol{n} = (n,\, n_\infty) \in \mathbb{N}^2$, we form the variety $\mathrm{Rep}_S(\Pi^{\boldsymbol\lambda}(B), \boldsymbol{n})$ of representations of $\Pi^{\boldsymbol\lambda}(B)$ of dimension vector $\boldsymbol{n}$ and, with notation of Section 2.3, define

$$\mathcal{C}_{\boldsymbol{n},\boldsymbol\lambda}(X,\mathcal{I}) := \mathrm{Rep}_S(\Pi^{\boldsymbol\lambda}(B), \boldsymbol{n})/\!/\, \mathrm{G}_S(\boldsymbol{n})\ . \tag{3.6}$$

Thus, $\mathcal{C}_{\boldsymbol{n},\boldsymbol\lambda}(X,\mathcal{I})$ is an affine scheme, whose (closed) points are in bijection with isomorphism classes of semisimple $\Pi^{\boldsymbol\lambda}(B)$-modules of dimension vector $\boldsymbol{n}$.

Lemma 3.3. *For any line bundles $\mathcal{I}$ and $\mathcal{J}$, the schemes $\mathcal{C}_{\boldsymbol{n},\boldsymbol\lambda}(X,\mathcal{I})$ and $\mathcal{C}_{\boldsymbol{n},\boldsymbol\lambda}(X,\mathcal{J})$ are isomorphic.*

Proof. By Lemma 3.1, the algebras $A[\mathcal{I}]$ and $A[\mathcal{J}]$ are Morita equivalent: the corresponding equivalence is given by

$$\mathtt{Mod}\, A[\mathcal{I}] \xrightarrow{\sim} \mathtt{Mod}\, A[\mathcal{J}]\ , \quad \boldsymbol{V} \mapsto \tilde{\mathcal{L}} \otimes_{\tilde{A}} \boldsymbol{V}\ , \tag{3.7}$$

where $\tilde{\mathcal{L}} = \mathcal{J}\mathcal{I}^\vee \times \mathbb{C}$. The functor (3.7) induces an isomorphism of vector spaces: $\mathrm{H}_0(A[\mathcal{I}]) \xrightarrow{\sim} \mathrm{H}_0(A[\mathcal{J}])$, which restricts to the identity on $S \subset \tilde{A}$. By [CB], Corollary 5.5, it can then be extended (non-canonically) to a Morita equivalence between the algebras $\Pi^{\boldsymbol\lambda}(A[\mathcal{I}])$ and $\Pi^{\boldsymbol\lambda}(A[\mathcal{J}])$ for any $\boldsymbol\lambda \in S$. Now, if $\boldsymbol{V} = (V,\, V_\infty)$ with $\dim_{\mathbb{C}} V < \infty$, we have $\tilde{\mathcal{L}} \otimes_{\tilde{A}} \boldsymbol{V} = (\mathcal{J}\mathcal{I}^\vee \otimes_A V,\, V_\infty)$, so by formula (3.2),

$$\dim_{\mathbb{C}}[\mathcal{J}\mathcal{I}^\vee \otimes_A V] = \dim_{\mathbb{C}} \mathrm{Hom}_A(\mathcal{I}\mathcal{J}^\vee,\, V) = \dim_{\mathbb{C}}(V)\ .$$

This shows that the equivalence (3.7) preserves dimensions, and its extension to Π^λ induces thus an isomorphism: $\mathcal{C}_{\boldsymbol{n},\boldsymbol\lambda}(X,\mathcal{I}) \xrightarrow{\sim} \mathcal{C}_{\boldsymbol{n},\boldsymbol\lambda}(X,\mathcal{J})$. □

The next lemma is a generalization of [CBH], Lemma 4.1: it implies that $\mathcal{C}_{\boldsymbol{n},\boldsymbol\lambda}(X,\mathcal{I})$ is empty unless $\boldsymbol\lambda \cdot \boldsymbol{n} := \lambda\, n + \lambda_\infty n_\infty$ is zero.

Lemma 3.4. *If $\boldsymbol\lambda \cdot \boldsymbol{n} \neq 0$, there are no representations of $\Pi^{\boldsymbol\lambda}(B)$ of dimension $\boldsymbol{n}$.*

Proof. If $\boldsymbol{V} = V \oplus V_\infty$ is a $\Pi^{\boldsymbol\lambda}(B)$-module of dimension vector $\boldsymbol{n}$, then e and e_∞ act on $\boldsymbol{V}$ as projectors onto V and V_∞ respectively. The trace of $\boldsymbol\lambda = \lambda\, e + \lambda_\infty\, e_\infty \in B$ on $\boldsymbol{V}$ is then equal to $\boldsymbol\lambda \cdot \boldsymbol{n}$, and it must be zero by Proposition 2.1. □

Example 3.1. Let X be the affine line $\mathbb{A}^1$. Any line bundle $\mathcal{I}$ on X is trivial. So, choosing a coordinate on X, we may identify $A \cong \mathbb{C}[x]$ and $\mathcal{I} \cong \mathbb{C}[x]$. The one-point extension of A by $\mathcal{I}$ is then isomorphic to the matrix algebra:

$$A[\mathcal{I}] \cong \begin{pmatrix} \mathbb{C}[x] & \mathbb{C}[x] \\ 0 & \mathbb{C} \end{pmatrix}\ ,$$

which is, in turn, isomorphic to the path algebra $\mathbb{C}Q$ of the quiver Q consisting of two vertices $\{0,\, \infty\}$ and two arrows $X : 0 \to 0$ and $\mathrm{v} : \infty \to 0$. In fact, the map sending the vertices 0 and ∞ to the idempotents e and e_∞ in $A[\mathcal{I}]$ and

$$X \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}\ , \qquad \mathrm{v} \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\ ,$$

extends to an algebra isomorphism $\mathbb{C}Q \xrightarrow{\sim} A[\mathcal{I}]$.

Now, let $\bar{Q}$ be the double quiver of Q obtained by adding the reverse arrows $Y := X^*$ and $\mathrm{w} := \mathrm{v}^*$ to the corresponding arrows of Q. Then, for any $\boldsymbol\lambda =$

$\lambda\, e + \lambda_\infty e_\infty$, with $(\lambda,\, \lambda_\infty) \in \mathbb{C}^2$, the algebra $\Pi^{\boldsymbol{\lambda}}(Q)$ is isomorphic to the quotient of $\mathbb{C}\bar{Q}$ modulo the relation $[X, Y] + [\mathrm{v}, \mathrm{w}] = \boldsymbol{\lambda}$ (see [CB], Theorem 3.1). The ideal generated by this last relation is the same as the ideal generated by the elements $[X, Y] + \mathrm{v}\,\mathrm{w} - \lambda\, e$ and $\mathrm{w}\,\mathrm{v} + \lambda_\infty e_\infty$. Thus, the $\Pi^{\boldsymbol{\lambda}}(Q)$-modules can be identified with representations $\boldsymbol{V} = V \oplus V_\infty$ of $\bar{Q}$, in which linear maps $\bar{X}, \bar{Y} \in \mathrm{Hom}(V,\, V)$, $\bar{\mathrm{v}} \in \mathrm{Hom}(V_\infty,\, V)$, $\bar{\mathrm{w}} \in \mathrm{Hom}(V,\, V_\infty)$, given by the action of $X, Y, \mathrm{v}, \mathrm{w}$, satisfy

$$[\bar{X},\, \bar{Y}] + \bar{\mathrm{v}}\,\bar{\mathrm{w}} = \lambda\,\mathrm{Id}_V \quad \text{and} \quad \bar{\mathrm{w}}\,\bar{\mathrm{v}} = -\lambda_\infty\,\mathrm{Id}_{V_\infty}\ . \tag{3.8}$$

Now, taking $\boldsymbol{\lambda} = (1, -n)$, it is easy to see that all representations of $\Pi^{\boldsymbol{\lambda}}(Q)$ of dimension vector $\boldsymbol{n} = (n,\, 1)$ are simple, and the representation varieties $\mathcal{C}_{\boldsymbol{n},\boldsymbol{\lambda}}$ coincide (in this special case) with the classical Calogero-Moser spaces $\mathcal{C}_n$. This coincidence was first noticed by W. Crawley-Boevey (see [CB1], remark on p. 45). For explanations and further discussion of this example we refer the reader to [BCE].

Motivated by the above example, we will be interested in representations of $\Pi^{\boldsymbol{\lambda}}(B)$ of dimension $\boldsymbol{n} = (n, 1)$. By Lemma 3.4, such representations may exist only if $\boldsymbol{\lambda} = 0$ or $\boldsymbol{\lambda} = (\lambda, -n\lambda)$, with $\lambda \neq 0$. In this last case, the algebras $\Pi^{\boldsymbol{\lambda}}(B)$ are all isomorphic to each other, so without loss of generality we may assume $\lambda = 1$.

Proposition 3.1. *Let* $\boldsymbol{\lambda} = (1, -n)$ *and* $\boldsymbol{n} = (n, 1)$ *with* $n \in \mathbb{N}$. *Then, for any* $\mathcal{I}$ *, the algebra* $\Pi^{\boldsymbol{\lambda}}(B)$ *has modules of dimension vector* $\boldsymbol{n}$. *Every such module is simple.*

Proof. On any B-module of dimension $\boldsymbol{n}$, the element $\boldsymbol{\lambda} = e - n\, e_\infty \in B$ will act with zero trace. Thus, by Proposition 2.1, it suffices to see that there exist *indecomposable* B-modules of this dimension vector. Now, as explained in Section 2.2, a B-module structure on $\boldsymbol{V} = V \oplus V_\infty$ is determined by an A-module homomorphism $\varphi_V : \mathcal{I} \otimes V_\infty \to V$. If $\boldsymbol{V}$ is decomposable with $\dim(V_\infty) = 1$, then one of its direct summands must be of the form $\boldsymbol{V}' = V' \oplus V_\infty$, where V' is an A-module summand of V of dimension $< n$. In that case, we have $\mathrm{Im}(\varphi_V) \subseteq V' \subsetneq V$. Thus, for constructing an indecomposable B-module of dimension $\boldsymbol{n}$, it suffices to construct a torsion A-module V of length n together with a *surjective* A-module map $\varphi : \mathcal{I} \to V$. Geometrically, this can be done as follows.

Identify $\mathcal{I}$ with an ideal in A and fix n distinct points $x_1, x_2, \dots x_n$ on X outside the zero locus of $\mathcal{I}$. Let $V := A/\mathcal{J}$, where $\mathcal{J}$ is the product of the maximal ideals $\mathfrak{m}_i \subset A$ corresponding to x_i's. Clearly, $A/\mathcal{J} \cong \oplus_{i=1}^n A/\mathfrak{m}_i$ and $\dim_{\mathbb{C}} V = n$. Now, since A is a Dedekind domain and $\mathcal{I} \not\subset \mathfrak{m}_i$ for any i, we have $(A/\mathcal{J}) \otimes_A (A/\mathcal{I}) \cong \oplus_{i=1}^n (A/\mathfrak{m}_i) \otimes_A (A/\mathcal{I}) = 0$ and $\mathrm{Tor}_1^A(A/\mathcal{J}, A/\mathcal{I}) \cong (\mathcal{I} \cap \mathcal{J})/\mathcal{I}\,\mathcal{J} = 0$, so the canonical map $V \otimes_A \mathcal{I} \to V$ is an isomorphism. On the other hand, as V is a cyclic A-module, $\mathcal{I}$ surjects naturally onto $V \otimes_A \mathcal{I}$. Combining $\mathcal{I} \twoheadrightarrow V \otimes_A \mathcal{I} \xrightarrow{\sim} V$, we get the required homomorphism φ. This proves the first part of the proposition.

Now, let $\boldsymbol{V} = V \oplus V_\infty$ be any $\Pi^{\boldsymbol{\lambda}}(B)$-module of dimension vector $\boldsymbol{n}$. Let $\boldsymbol{V}'$ be a submodule of $\boldsymbol{V}$ of dimension vector $\boldsymbol{k} = (k,\, k_\infty)$ (say). By Lemma 3.4, we have then $\boldsymbol{\lambda} \cdot \boldsymbol{k} = k - n\, k_\infty = 0$. Since $0 \leq k_\infty \leq n_\infty = 1$, there are only two possibilities: either $\boldsymbol{k} = 0$ or $\boldsymbol{k} = \boldsymbol{n}$, i.e. $\boldsymbol{V}'$ is either 0 or $\boldsymbol{V}$. Hence $\boldsymbol{V}$ is a simple module. □

Remark. 1. The above argument shows that a B-module $\boldsymbol{V}$ of dimension vector $\boldsymbol{n} = (n, 1)$ lifts to a module over $\Pi^{\boldsymbol{\lambda}}(B)$ if and *only if* it is indecomposable.
2. If $\boldsymbol{V}$ is a B-module with a surjective structure map $\varphi_V : \mathcal{I} \otimes V_\infty \to V$,

then $\mathrm{End}_B(\boldsymbol{V}) \subseteq \mathrm{End}(V_\infty)$. (This follows immediately from the diagram (2.13), characterizing B-module homomorphisms.) Thus, the modules $\boldsymbol{V}$ constructed in the proof of Proposition 3.1 are actually indecomposables with $\mathrm{End}_B(\boldsymbol{V}) \cong \mathbb{C}$.

Definition 1. The variety $\mathcal{C}_{\boldsymbol{n},\boldsymbol{\lambda}}(X,\mathcal{I})$ with $\boldsymbol{\lambda} = (1,-n)$ and $\boldsymbol{n} = (n,1)$ will be denoted $\mathcal{C}_n(X,\mathcal{I})$ and called the *n-th Calogero-Moser space* of type $(X,\mathcal{I})$.

In view of Proposition 3.1, the varieties $\mathcal{C}_n(X,\mathcal{I})$ parametrize the isomorphism classes of simple $\Pi^{\boldsymbol{\lambda}}(B)$-modules of dimension $\boldsymbol{n} = (n,1)$; they are non-empty for any $[\mathcal{I}] \in \mathrm{Pic}(X)$ and $n \geq 0$. In the special case, when X is the affine line, $\mathcal{C}_n(X,I)$ coincide with the ordinary Calogero-Moser spaces $\mathcal{C}_n$ (see Example 3.1 above).

3.4. **Smoothness and irreducibility.** One of the main results of [Wi] says that each $\mathcal{C}_n$ is a smooth affine irreducible variety of dimension $2n$. Theorem 3.2 below shows that this holds in general, for an arbitrary curve X. To prove the irreducibility we will use the following simple observation due to Crawley-Boevey.

Lemma 3.5 ([CB2], Lemma 6.1)**.** *If X is an equidimensional variety, Y is an irreducible variety and $f : X \to Y$ is a dominant morphism with all fibres irreducible of constant dimension, then X is irreducible.*

Theorem 3.2. *For each $n \geq 0$ and $[\mathcal{I}] \in \mathrm{Pic}(X)$, $\mathcal{C}_n(X,\mathcal{I})$ is a smooth affine irreducible variety of dimension $2n$.*

Proof. The varieties $\mathcal{C}_n(X,\mathcal{I})$ are affine by definition; we need only to show that these are smooth and irreducible.

Fix $n \in \mathbb{N}$ and $[\mathcal{I}] \in \mathrm{Pic}(X)$. To simplify the notation write Π for $\Pi^{\boldsymbol{\lambda}}(B)$ with $\boldsymbol{\lambda} = (1,-n)$. Then, by Proposition 3.1, every Π-module $\boldsymbol{V}$ of dimension $\boldsymbol{n} = (n,1)$ is simple, so, by Schur Lemma, $\mathrm{End}_\Pi(\boldsymbol{V}) \cong \mathbb{C}$ and $\mathrm{Aut}_\Pi(\boldsymbol{V}) \cong \mathbb{C}^*$. The last isomorphism implies that every point of $\mathrm{Rep}_S(\Pi,\boldsymbol{n})$ has trivial stabilizer in $\mathrm{G}_S(\boldsymbol{n})$, i. e. the natural action of $\mathrm{GL}(\boldsymbol{n})$ on $\mathrm{Rep}_S(\Pi,\boldsymbol{n})$ is free. In that case, by Luna's Slice Theorem (see [Lu], Corollaire III.1.1), the quotient variety $\mathcal{C}_n(X,\mathcal{I}) = \mathrm{Rep}_S(\Pi,\boldsymbol{n})/\!/\,\mathrm{G}_S(\boldsymbol{n})$ will be smooth if so is the original variety $\mathrm{Rep}_S(\Pi,\boldsymbol{n})$. Now, to see that $\mathrm{Rep}_S(\Pi,\boldsymbol{n})$ is smooth, it suffices to see, by Lemma 2.4, that $\mathrm{Rep}(\Pi,\boldsymbol{n})$ is smooth, and that follows from Proposition 2.3 of Section 2.3. In fact, let $R := T_B\mathrm{Der}(B, B^{\otimes 2})$, and let $\sigma : R \twoheadrightarrow \Pi$ be the canonical projection. Then σ induces the closed embedding of affine varieties $\sigma_* : \mathrm{Rep}(\Pi,\boldsymbol{n}) \hookrightarrow \mathrm{Rep}(R,\boldsymbol{n})$, whose image is a fibre of the evaluation map (2.25). Since for every $\varrho \in \mathrm{Im}(\sigma_*)$, we have $\mathrm{End}_R(\boldsymbol{V}) \cong \mathrm{End}_\Pi(\boldsymbol{V}) \cong \mathbb{C}$, the assumption of Proposition 2.3 holds, and the result follows.

Now, we show that $\mathcal{C}_n(X,\mathcal{I})$ is irreducible of dimension $2n$. For this, we examine first the varieties $\mathrm{Rep}_S(B,\boldsymbol{n})$ and $\mathrm{Rep}_S(\Pi,\boldsymbol{n})$. Since B is smooth, $\mathrm{Rep}_S(B,\boldsymbol{n})$ is smooth, i. e. for every point $\varrho \in \mathrm{Rep}_S(B,\boldsymbol{n})$, we have

$$\dim_\varrho \mathrm{Rep}_S(B,\boldsymbol{n}) = \dim_{\mathbb{C}} T_\varrho \mathrm{Rep}_S(B,\boldsymbol{n}) \,, \tag{3.9}$$

where $\dim_\varrho$ stands for the local dimension and T_ϱ for the Zariski tangent space of $\mathrm{Rep}_S(B,\boldsymbol{n})$ at ϱ. Now, $T_\varrho \mathrm{Rep}_S(B,\boldsymbol{n})$ is canonically isomorphic to $\mathrm{Der}_S(B,\mathrm{End}\,\boldsymbol{V})$, the space of S-linear derivations from B to $\mathrm{End}(\boldsymbol{V})$ (cf. Lemma 2.4). To evaluate the dimension of this space, we consider the standard exact sequence

$$0 \to \mathrm{End}_B(\boldsymbol{V}) \to \mathrm{End}_S(\boldsymbol{V}) \to \mathrm{Der}_S(B,\,\mathrm{End}\,\boldsymbol{V}) \to \mathrm{H}^1(B,\,\mathrm{End}\,\boldsymbol{V}) \to 0 \,,$$

which can be obtained by dualizing $0 \to \Omega^1_S B \to B \otimes_S B \to B \to 0$ with $\mathrm{End}(\boldsymbol{V})$. Identifying terms $\mathrm{End}_S(\boldsymbol{V}) \cong \mathtt{Mat}(n, \mathbb{C}) \times \mathbb{C}$, $\mathrm{H}^1(B, \mathrm{End}\, \boldsymbol{V}) \cong \mathrm{Ext}^1_B(\boldsymbol{V}, \boldsymbol{V})$ (see [CE], Proposition 4.3, p. 170), and using Lemma 3.2, we get

$$\dim_{\mathbb{C}} \mathrm{Der}_S(B, \mathrm{End}\, \boldsymbol{V}) = n^2 + 1 - \chi_B(\boldsymbol{V}, \boldsymbol{V}) = n^2 + n \ . \tag{3.10}$$

Thus $\mathrm{Rep}_S(B, \boldsymbol{n})$ is a smooth equidimensional variety of dimension $n^2 + n$. To see that it is actually irreducible, we apply Lemma 3.5 to the canonical projection $f : \mathrm{Rep}_S(B, \boldsymbol{n}) \to \mathrm{Rep}(A, n)$. In this case, the assumptions of Lemma 3.5 are easy to verify: since X is irreducible, so is clearly $\mathrm{Rep}(A, n)$, and the fibres of f over each $V \in \mathrm{Rep}(A, n)$ can be identified with the vector spaces $\mathrm{Hom}_A(I, V)$ and, hence, are all irreducible of the same dimension n, by formula (3.2).

Next, we consider the restriction map $\pi : \mathrm{Rep}_S(\Pi, \boldsymbol{n}) \to \mathrm{Rep}_S(B, \boldsymbol{n})$. As remarked above, the image of π consists exactly of indecomposable modules in $\mathrm{Rep}_S(B, \boldsymbol{n})$, while each (non-empty) fibre $\pi^{-1}(\boldsymbol{V})$ is isomorphic, by Proposition 2.1, to a coset of $\mathrm{Ext}^1_B(\boldsymbol{V}, \boldsymbol{V})^*$ and thus, is irreducible of dimension

$$\dim \pi^{-1}(\boldsymbol{V}) = \dim_{\mathbb{C}} \mathrm{End}_B(\boldsymbol{V}) - \chi_B(\boldsymbol{V}, \boldsymbol{V}) = \dim_{\mathbb{C}} \mathrm{End}_B(\boldsymbol{V}) + n - 1 \ . \tag{3.11}$$

Now, let $\mathcal{U}$ be the subset of $\mathrm{Rep}_S(B, \boldsymbol{n})$ consisting of modules $\boldsymbol{V}$ with $\mathrm{End}_B(\boldsymbol{V}) \cong \mathbb{C}$. As explained in Remark 2 (after Proposition 3.1), this subset is non-empty. By Chevalley's Theorem (see, e.g., [CB3], p. 15), the function $\boldsymbol{V} \mapsto \dim_{\mathbb{C}} \mathrm{End}_B(\boldsymbol{V})$ is upper semi-continuous on $\mathrm{Rep}_S(B, \boldsymbol{n})$, i. e.

$$\{\boldsymbol{V} \in \mathrm{Rep}_S(B, \boldsymbol{n}) \, : \, \dim_{\mathbb{C}} \mathrm{End}_B(\boldsymbol{V}, \boldsymbol{V}) \geq n\}$$

are closed sets for all $n \in \mathbb{N}$. Hence $\mathcal{U}$ is open in $\mathrm{Rep}_S(B, \boldsymbol{n})$ and therefore dense, since $\mathrm{Rep}_S(B, \boldsymbol{n})$ is irreducible. As $\mathcal{U} \subseteq \mathrm{Im}(\pi)$, this implies that π is dominant.

Now, $\pi^{-1}(\mathcal{U})$ is an open subset of $\mathrm{Rep}_S(\Pi, \boldsymbol{n})$, whose local dimension at every point $\varrho \in \pi^{-1}(\mathcal{U})$ is equal, by (3.11), to

$$\dim_{\varrho} \pi^{-1}(\mathcal{U}) = \dim \mathcal{U} + \dim \pi^{-1}(\pi(\varrho)) = \dim \mathrm{Rep}_S(B, \boldsymbol{n}) + n = n^2 + 2n \ .$$

Thus $\pi^{-1}(\mathcal{U})$ is equidimensional and therefore, by Lemma 3.5, irreducible. We claim that $\pi^{-1}(\mathcal{U})$ is dense in $\mathrm{Rep}_S(\Pi, \boldsymbol{n})$. Indeed, since $\mathrm{Im}(\pi)$ coincides with the set of indecomposable B-modules in $\mathrm{Rep}_S(B, \boldsymbol{n})$, we have

$$\dim \pi^{-1}(\mathrm{Im}(\pi) \setminus \mathcal{U}) < \dim \pi^{-1}(\mathcal{U}) = n^2 + 2n \ .$$

On the other hand, the variety $\mathrm{Rep}_S(\Pi, \boldsymbol{n})$ can be identified with a fibre of the evaluation map $\mu : \mathrm{Rep}_S(R, \boldsymbol{n}) \to \mathrm{End}_S(\boldsymbol{V})_0$, see (2.25), so any irreducible component of it has dimension at least

$$\dim \mathrm{Rep}_S(R, \boldsymbol{n}) - \dim \mathrm{End}_S(\boldsymbol{V})_0 = 2(n^2 + n) - n^2 = n^2 + 2n \ .$$

(Here, we calculated $\dim \mathrm{Rep}_S(R, \boldsymbol{n})$ by identifying $\mathrm{Rep}_S(R, \boldsymbol{n})$ with the cotangent bundle $T^* \mathrm{Rep}_S(B, \boldsymbol{n})$, see [CEG], Sect. 5.) Thus, $\mathrm{Rep}_S(\Pi, \boldsymbol{n})$ must coincide with the closure of $\pi^{-1}(\mathcal{U})$, and hence is also irreducible of dimension $n^2 + 2n$. This certainly implies the irreducibility of $\mathcal{C}_n(X, \mathcal{I})$, since $\mathcal{C}_n(X, \mathcal{I})$ is a quotient of $\mathrm{Rep}_S(\Pi, \boldsymbol{n})$ by a free action of $\mathrm{G}_S(\boldsymbol{n})$.

Finally, we have $\mathrm{G}_S(\boldsymbol{n}) \cong [\, \mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(1, \mathbb{C}) \,]/\mathbb{C}^* \cong \mathrm{GL}(n, \mathbb{C})$, so

$$\dim \mathcal{C}_n(X, \mathcal{I}) = \dim \mathrm{Rep}_S(\Pi, \, \boldsymbol{n}) - \dim \mathrm{G}_S(\boldsymbol{n}) = n^2 + 2n - n^2 = 2n \ .$$

This completes the proof of the theorem. □

4. The Calogero-Moser Correspondence

4.1. **Recollement.** We begin by relating the algebras $\Pi^{\boldsymbol{\lambda}}(B)$ to the ring $\mathcal{D}$ of differential operators on X.

Lemma 4.1. *There is a canonical algebra map* $\boldsymbol{\theta} : \Pi^{\boldsymbol{\lambda}}(B) \to \Pi^1(A)$, *which is a surjective pseudo-flat ring homomorphism, with* $\mathrm{Ker}(\boldsymbol{\theta}) = \langle\, e_\infty \,\rangle$.

Proof. By Lemma 2.3(4), the canonical projection $\theta : B \to A$, see (2.15), is a pseudo-flat ring epimorphism. By Theorem 2.1, it extends to an algebra map $\boldsymbol{\theta} : \Pi^{\boldsymbol{\lambda}}(B) \to \Pi^{\theta^*(\boldsymbol{\lambda})}(A)$ (which is also a pseudo-flat ring epimorphism, since B is smooth, see Lemma 2.3(2)). Since θ is surjective with $\mathrm{Ker}(\theta) = \langle e_\infty \rangle$, it follows from the push-out diagram (2.3) that $\boldsymbol{\theta}$ is surjective and $\mathrm{Ker}(\boldsymbol{\theta}) = \langle e_\infty \rangle$. Finally, with identifications of Section 3.3, it is easy to see that $\theta^*(\boldsymbol{\lambda}) = 1$. □

Next, we recall the following important observation due to Crawley-Boevey.

Theorem 4.1 ([CB], Theorem 4.7)**.** *If* $A = \mathcal{O}(X)$ *is a coordinate ring of a smooth affine curve, then* $\Pi^1(A)$ *is isomorphic (as a filtered algebra) to* $\mathcal{D} = \mathcal{D}(X)$.

We fix, once and for all, an isomorphism[4] of Theorem 4.1 to identify $\mathcal{D} = \Pi^1(A)$. In combination with Lemma 4.1, this yields an algebra map $\boldsymbol{\theta} : \Pi \to \mathcal{D}$. We will use $\boldsymbol{\theta}$ to relate the (derived) module categories of Π and $\mathcal{D}$, as follows (cf. [BCE]).

First, we let U denote the endomorphism ring of the projective module $e_\infty\Pi$: this ring can be identified with the associative subalgebra $e_\infty\Pi\, e_\infty$ of Π having e_∞ as an identity element. Next, we introduce six additive functors $(\boldsymbol{\theta}^*, \boldsymbol{\theta}_*, \boldsymbol{\theta}^!)$ and $(j_!, j^*, j_*)$ between the module categories of Π, $\mathcal{D}$ and U. We define $\boldsymbol{\theta}_* : \mathtt{Mod}(\mathcal{D}) \to \mathtt{Mod}(\Pi)$ to be the restriction functor associated to the algebra map $\boldsymbol{\theta} : \Pi \to \mathcal{D}$. This functor is fully faithful and has both the right adjoint $\boldsymbol{\theta}^! := \mathrm{Hom}_\Pi(\mathcal{D}, \text{—})$ and the left adjoint $\boldsymbol{\theta}^* := \mathcal{D} \otimes_\Pi \text{—}$, with adjunction maps $\boldsymbol{\theta}^*\boldsymbol{\theta}_* \simeq \mathrm{Id} \simeq \boldsymbol{\theta}^!\boldsymbol{\theta}_*$ being isomorphisms. Now we define $j^* : \mathtt{Mod}(\Pi) \to \mathtt{Mod}(U)$ by $j^*\boldsymbol{V} := e_\infty\boldsymbol{V}$. Since $e_\infty \in \Pi$ is an idempotent, j^* is exact and has also the right and the left adjoint functors: $j_* := \mathrm{Hom}_U(e_\infty\Pi, \text{—})$ and $j_! := \Pi e_\infty \otimes_U \text{—}$ respectively, satisfying $j^*j_* \simeq \mathrm{Id} \simeq j^*j_!$.

The functors $(\boldsymbol{\theta}^*, \boldsymbol{\theta}_*, \boldsymbol{\theta}^!)$ and $(j_!, j^*, j_*)$ defined above induce the six exact functors at the level of derived categories:

$$\mathcal{D}^-(\mathtt{Mod}\,\mathcal{D}) \;\underset{R\boldsymbol{\theta}^!}{\overset{L\boldsymbol{\theta}^*}{\overleftarrow{\xrightarrow{\;\boldsymbol{\theta}_*\;}}}}\; \mathcal{D}^-(\mathtt{Mod}\,\Pi) \;\underset{Rj_*}{\overset{Lj_!}{\overleftarrow{\xrightarrow{\;j^*\;}}}}\; \mathcal{D}^-(\mathtt{Mod}\,U)\ . \tag{4.1}$$

Proposition 4.1. *The diagram* (4.1) *is a recollement of triangulated categories* (*in the sense of* [BBD]).

Proposition 4.1 follows from general results on recollement of module categories (see [Ko]) and the following observation, the proof of which will be given in Section 5.1 (see Lemma 5.4): the multiplication map $\Pi\, e_\infty \otimes_U e_\infty\Pi \to \Pi$ fits into the exact sequence

$$0 \to \Pi\, e_\infty \otimes_U e_\infty\Pi \to \Pi \xrightarrow{\boldsymbol{\theta}} \mathcal{D} \to 0\ , \tag{4.2}$$

[4] See Appendix and remarks in the Introduction.

which is a projective resolution of $\mathcal{D}$ in the category of (left and right) Π-modules. The existence of resolution (4.2) implies that $\mathcal{D}$ has projective dimension 1 in $\mathtt{Mod}(\Pi)$. Hence $\mathrm{Tor}_n^{\Pi}(\mathcal{D},\mathcal{D}) = 0$ for all $n \geq 2$. On the other hand, by Lemma 4.1, $\boldsymbol{\theta}$ is a pseudo-flat ring epimorphism, which means that $\mathrm{Tor}_1^{\Pi}(\mathcal{D},\mathcal{D}) = 0$ as well. Proposition 4.1 follows now from [Ko], Corollary 14.

As another consequence of the above observation, we obtain

Lemma 4.2. *If $\boldsymbol{V}$ is a finite-dimensional Π-module, then $L_n\boldsymbol{\theta}^*(\boldsymbol{V}) = 0$ for $n \neq 1$ and*

$$L_1\boldsymbol{\theta}^*(\boldsymbol{V}) \cong \mathrm{Ker}\left[\Pi\, e_\infty \otimes_U e_\infty \boldsymbol{V} \xrightarrow{\boldsymbol{\mu}} \boldsymbol{V}\right], \tag{4.3}$$

where $\boldsymbol{\mu}$ is the natural multiplication-action map.

Proof. Tensoring (4.2) with $\boldsymbol{V}$ yields the exact sequence

$$0 \to \mathrm{Tor}_1^{\Pi}(\mathcal{D},\boldsymbol{V}) \to \Pi e_\infty \otimes_U e_\infty \boldsymbol{V} \to \boldsymbol{V} \to \mathcal{D} \otimes_\Pi \boldsymbol{V} \to 0\ ,$$

and isomorphisms

$$\mathrm{Tor}_n^{\Pi}(\mathcal{D},\boldsymbol{V}) \cong \mathrm{Tor}_{n-1}^{\Pi}(\Pi\, e_\infty \otimes_U e_\infty \Pi, \boldsymbol{V})\ , \quad n \geq 2\ .$$

Since $\Pi\, e_\infty \otimes_U e_\infty \Pi$ is projective (as a right Π-module), the last Tor's vanish. On the other hand, $\dim_{\mathbb{C}} \boldsymbol{V} < \infty$ implies that $\mathcal{D} \otimes_\Pi \boldsymbol{V} = 0$, since $\mathcal{D}$ has no nonzero finite-dimensional modules. The result follows now from the definition of the derived functors: $L_n\boldsymbol{\theta}^*(\boldsymbol{V}) := \mathrm{Tor}_n^{\Pi}(\mathcal{D},\, \boldsymbol{V})$, $n \geq 0$. □

Remark. Using (4.1), we may define a 'perverse' t-structure on $\mathcal{D}^-(\mathtt{Mod}\,\Pi)$ by

$${}^p\mathcal{D}^{\leq 0}(\mathtt{Mod}\,\Pi) := \{\, K^\bullet \in \mathcal{D}^-(\mathtt{Mod}\,\Pi)\ :\ j^*K^\bullet \in \mathcal{D}^{\leq 0}(U)\ ,\ \boldsymbol{L\theta}^* K^\bullet \in \mathcal{D}^{\leq -1}(\mathcal{D})\,\}\ ,$$

$${}^p\mathcal{D}^{\geq 0}(\mathtt{Mod}\,\Pi) := \{\, K^\bullet \in \mathcal{D}^-(\mathtt{Mod}\,\Pi)\ :\ j^*K^\bullet \in \mathcal{D}^{\geq 0}(U)\ ,\ \boldsymbol{R\theta}^! K^\bullet \in \mathcal{D}^{\geq -1}(\mathcal{D})\,\}\ ,$$

where $\{\mathcal{D}^{\leq 0}(U),\, \mathcal{D}^{\geq 0}(U)\}$ and $\{\mathcal{D}^{\leq 0}(\mathcal{D}),\, \mathcal{D}^{\geq 0}(\mathcal{D})\}$ denote the standard t-structures on $\mathcal{D}^-(\mathtt{Mod}\,U)$ and $\mathcal{D}^-(\mathtt{Mod}\,\mathcal{D})$ respectively. Lemma 4.2 shows that the 0-complexes $[\,0 \to \boldsymbol{V} \to 0\,]$ with $\dim_{\mathbb{C}} \boldsymbol{V} < \infty$ are in the heart of this t-structure. So we may think of finite-dimensional representations of Π as 'perverse sheaves' relative to the stratification (4.1). The functor $\boldsymbol{L\theta}^*$ is then an algebraic analogue of the restriction functor of a (perverse) sheaf to a closed subspace.

4.2. **The action of $\mathrm{Pic}(\mathcal{D})$ on Calogero-Moser spaces.** We first recall some facts about the Picard group $\mathrm{Pic}(\mathcal{D})$ of the algebra $\mathcal{D}$ and its action on the space of ideals $\mathcal{I}(\mathcal{D})$ (see [BW]). It is known that $\mathrm{Pic}(\mathcal{D})$ has rather different descriptions for the affine line $\mathbb{A}^1$ and other curves (see [CH1]). Since the case of $\mathbb{A}^1$ is well studied, we will assume in this section that $X \neq \mathbb{A}^1$. Our main theorem (Theorem 4.2) still holds for all curves X, including $\mathbb{A}^1$.

In general, $\mathrm{Pic}(\mathcal{D})$ can be identified with the group of $\mathbb{C}$-linear auto-equivalences of the category $\mathtt{Mod}(\mathcal{D})$, and thus it acts naturally on $\mathcal{I}(\mathcal{D})$ and $K_0(\mathcal{D})$. To be precise, the elements of $\mathrm{Pic}(\mathcal{D})$ are the isomorphism classes $[\mathcal{P}]$ of invertible $\mathcal{D}$-bimodules (with scalars $\mathbb{C} \subset \mathcal{D}$ acting symmetrically on both sides), and the action of $\mathrm{Pic}(\mathcal{D})$ on $\mathcal{I}(\mathcal{D})$ and $K_0(\mathcal{D})$ is defined by $[M] \mapsto [\mathcal{P} \otimes_{\mathcal{D}} M]$. Clearly, the action of $\mathrm{Pic}(\mathcal{D})$ on $K_0(\mathcal{D})$ preserves rank and hence restricts to $\mathrm{Pic}(X)$ through the identification $K_0(\mathcal{D}) \cong K_0(X) \cong \mathbb{Z} \oplus \mathrm{Pic}(X)$, see Section 3.2. We have

Proposition 4.2 (see [BW], Theorem 1.1)**.** *$\mathrm{Pic}(\mathcal{D})$ acts on $\mathrm{Pic}(X)$ transitively, and the map $\gamma : \mathcal{I}(\mathcal{D}) \to \mathrm{Pic}(X)$ defined by (3.1) is equivariant under this action.*

Explicitly, the action of $\mathrm{Pic}(\mathcal{D})$ on $\mathrm{Pic}(X)$ can be described as follows (cf. [BW], Prop. 3.1). By [CH1], Cor. 1.13, every invertible bimodule over $\mathcal{D}$ is isomorphic to $\mathcal{D}\mathcal{L} = \mathcal{D} \otimes_A \mathcal{L}$ as a left module, while the right action of $\mathcal{D}$ on $\mathcal{D}\mathcal{L}$ is determined by an algebra isomorphism $\varphi : \mathcal{D} \xrightarrow{\sim} \mathrm{End}_{\mathcal{D}}(\mathcal{D}\mathcal{L})$, where $\mathcal{L}$ is a line bundle on X. Following [BW], we write $(\mathcal{D}\mathcal{L})_\varphi$ for this bimodule. Restricting φ to A yields an automorphism of X, and the assignment

$$g : \mathrm{Pic}(\mathcal{D}) \to \mathrm{Pic}(X) \rtimes \mathrm{Aut}(X) , \quad [\,(\mathcal{D}\mathcal{L})_\varphi] \mapsto ([\mathcal{L}],\, \varphi|_A) \ , \tag{4.4}$$

defines then a group homomorphism[5]. On the other hand, $\mathrm{Pic}(X) \rtimes \mathrm{Aut}(X)$ acts on $\mathrm{Pic}(X)$ in the obvious way, via left multiplication:

$$([\mathcal{L}], \tau) : \, [\,\mathcal{I}\,] \mapsto [\,\mathcal{L}\,\tau(\mathcal{I})\,] \ , \tag{4.5}$$

where $([\mathcal{L}], \tau) \in \mathrm{Pic}(X) \rtimes \mathrm{Aut}(X)$ and $[\mathcal{I}] \in \mathrm{Pic}(X)$. Combining (4.4) and (4.5) together, we get an action of $\mathrm{Pic}(\mathcal{D})$ on $\mathrm{Pic}(X)$, which is easily seen to agree with the natural action of $\mathrm{Pic}(\mathcal{D})$ on $K_0(\mathcal{D})$.

Now, given a line bundle $\mathcal{I}$ and an invertible bimodule $\mathcal{P} = (\mathcal{D}\mathcal{L})_\varphi$, we define

$$P := \tilde{\mathcal{L}} \otimes_{\tilde{A}} B_\tau \ ,$$

where $\tilde{\mathcal{L}} := \mathcal{L} \times \mathbb{C}$, $\tau := \varphi|_A$, and $B_\tau := A[\tau(\mathcal{I})]$. By Lemma 3.1$(a)$, P is a progenerator in the category of right B_τ-modules, with endomorphism ring

$$\mathrm{End}_{B_\tau}(P) \cong P \otimes_{B_\tau} P^* \cong \tilde{\mathcal{L}} \otimes_{\tilde{A}} B_\tau \otimes_{\tilde{A}} \tilde{\mathcal{L}}^\vee \cong A[\mathcal{J}] \ ,$$

where $\mathcal{J} := \mathcal{L}\,\tau(\mathcal{I})$. Associated to the bimodule $\mathcal{P}$ is thus the Morita equivalence:

$$\mathtt{Mod}(B_\tau) \xrightarrow{\sim} \mathtt{Mod}(A[\mathcal{J}]) \ , \quad \boldsymbol{V} \mapsto P \otimes_{B_\tau} \boldsymbol{V} \cong \tilde{\mathcal{L}} \otimes_{\tilde{A}} \boldsymbol{V} \ .$$

Next, we extend P to a $\Pi^{\boldsymbol{\lambda}}(B_\tau)$-module by

$$\boldsymbol{P} := P \otimes_{B_\tau} \Pi^{\boldsymbol{\lambda}}(B_\tau) \cong \tilde{\mathcal{L}} \otimes_{\tilde{A}} \Pi^{\boldsymbol{\lambda}}(B_\tau) \ , \tag{4.6}$$

which is clearly a progenerator in the category of right $\Pi^{\boldsymbol{\lambda}}(B_\tau)$-modules. By part (b) of Lemma 3.1, the algebra B_τ is isomorphic to B: the isomorphism is given by

$$\tilde{\tau} : \ B \to B_\tau \ , \quad \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mapsto \begin{pmatrix} \tau(a) & \tau(b) \\ 0 & c \end{pmatrix} \ . \tag{4.7}$$

Since $\tilde{\tau}(\boldsymbol{\lambda}) = \boldsymbol{\lambda}$ for all $\boldsymbol{\lambda} \in S$, (4.7) canonically extends to an isomorphism of deformed preprojective algebras: $\Pi^{\boldsymbol{\lambda}}(B) \xrightarrow{\sim} \Pi^{\boldsymbol{\lambda}}(B_\tau)$, which we will also denote by $\tilde{\tau}$. Now, using this last isomorphism, we will regard $\boldsymbol{P}$ as a $\Pi^{\boldsymbol{\lambda}}(B)$-module and identify

$$\mathrm{End}_{\Pi^{\boldsymbol{\lambda}}(B)}(\boldsymbol{P}) = \tilde{\mathcal{L}} \otimes_{\tilde{A}} \Pi^{\boldsymbol{\lambda}}(B_\tau) \otimes_{\tilde{A}} \tilde{\mathcal{L}}^\vee \cong \tilde{\mathcal{F}} \otimes_{\tilde{A}} \Pi^{\boldsymbol{\lambda}}(B) \otimes_{\tilde{A}} \tilde{\mathcal{F}}^\vee \ , \tag{4.8}$$

where $\mathcal{F} := \mathcal{L}^\tau = \tau^{-1}(\mathcal{L})$ and $\tilde{\mathcal{F}} = \mathcal{F} \times \mathbb{C}$. With identification (4.8), we have the embedding

$$\tilde{\tau}^{-1} : \ A[\mathcal{J}] \cong \tilde{\mathcal{L}} \otimes_{\tilde{A}} B_\tau \otimes_{\tilde{A}} \tilde{\mathcal{L}}^\vee \ \hookrightarrow \ \mathrm{End}_{\Pi^{\boldsymbol{\lambda}}(B)}(\boldsymbol{P}) \ , \tag{4.9}$$

and, since $\mathrm{End}_{\mathcal{D}}(\mathcal{F}\mathcal{D}) = \mathcal{F} \otimes_A \mathcal{D} \otimes_A \mathcal{F}^\vee \cong \tilde{\mathcal{F}} \otimes_{\tilde{A}} \mathcal{D} \otimes_{\tilde{A}} \tilde{\mathcal{F}}^\vee$, the natural map

$$1 \otimes \boldsymbol{\theta} \otimes 1 : \ \mathrm{End}_{\Pi^{\boldsymbol{\lambda}}(B)}(\boldsymbol{P}) \to \mathrm{End}_{\mathcal{D}}(\mathcal{F}\mathcal{D}) \ , \tag{4.10}$$

where $\boldsymbol{\theta}$ is given by Lemma 4.1.

[5] The homomorphism $\mathrm{Pic}(\mathcal{D}) \to \mathrm{Pic}(X) \rtimes \mathrm{Aut}(X)$ defined in [CH1] differs from (4.4) by an involution. In this paper, we use the same definition as in [BW].

On the other hand, $\varphi(\mathcal{D}) = \mathrm{End}_{\mathcal{D}}(\mathcal{D}\mathcal{L}) = \mathcal{L}^{\vee}\mathcal{D}\mathcal{L}$ implies $\mathcal{D} = \mathcal{L}\,\varphi(\mathcal{D})\,\mathcal{L}^{\vee}$, so taking the inverse of φ defines an algebra isomorphism

$$\psi = \varphi^{-1} :\ \mathcal{D} \to \mathcal{F}\,\mathcal{D}\,\mathcal{F}^{\vee} = \mathrm{End}_{\mathcal{D}}(\mathcal{F}\mathcal{D})\ . \tag{4.11}$$

Combining (4.10) and (4.11) together, we get the diagram of algebra maps

$$\begin{array}{ccc} \Pi^{\boldsymbol{\lambda}}(A[\mathcal{J}]) & \dashrightarrow & \mathrm{End}_{\Pi^{\boldsymbol{\lambda}}(B)}(\boldsymbol{P}) \\ \Big\downarrow{\scriptstyle\boldsymbol{\theta}} & & \Big\downarrow{\scriptstyle 1\otimes\boldsymbol{\theta}\otimes 1} \\ \mathcal{D} & \xrightarrow{\ \psi\ } & \mathrm{End}_{\mathcal{D}}(\mathcal{F}\mathcal{D}) \end{array} \tag{4.12}$$

which obviously commutes when the dotted arrow is restricted to (4.9).

Proposition 4.3. *There is a unique algebra isomorphism* $\widetilde{\psi} :\ \Pi^{\boldsymbol{\lambda}}(A[\mathcal{J}]) \to \mathrm{End}_{\Pi^{\boldsymbol{\lambda}}(B)}(\boldsymbol{P})$, *extending* (4.9) *and making* (4.12) *a commutative diagram.*

We postpone the proof of Proposition 4.3 until Section 5.5. Meanwhile, we note that the isomorphism $\widetilde{\psi}$ makes $\boldsymbol{P}$ a left $\Pi^{\boldsymbol{\lambda}}(A[\mathcal{J}])$-module and thus a progenerator from $\Pi^{\boldsymbol{\lambda}}(A[\mathcal{I}])$ to $\Pi^{\boldsymbol{\lambda}}(A[\mathcal{J}])$. This assigns to $\mathcal{P} = (\mathcal{D}\mathcal{L})_{\varphi}$ the Morita equivalence:

$$\mathtt{Mod}\,\Pi^{\boldsymbol{\lambda}}(A[\mathcal{I}]) \to \mathtt{Mod}\,\Pi^{\boldsymbol{\lambda}}(A[\mathcal{J}])\ ,\quad \boldsymbol{V} \mapsto \boldsymbol{P} \otimes_{\Pi} \boldsymbol{V}\ ,$$

which, in turn, induces an isomorphism of representation varieties

$$f_{\mathcal{P}} :\ \mathcal{C}_n(X, \mathcal{I}) \xrightarrow{\sim} \mathcal{C}_n(X, \mathcal{J})\ . \tag{4.13}$$

Remark. We warn the reader that (4.13) *depends* on the choice of a specific representative in the class $[\mathcal{P}] \in \mathrm{Pic}(\mathcal{D})$, so, in general, we do not get an action of $\mathrm{Pic}(\mathcal{D})$ on $\bigsqcup_{[\mathcal{I}]\in \mathrm{Pic}(X)} \mathcal{C}_n(X, \mathcal{I})$. However, we will see below (Lemma 4.3) that $f_{\mathcal{P}}$ induces a well-defined action of $\mathrm{Pic}(\mathcal{D})$ on the union of *reduced* Calogero-Moser spaces $\overline{\mathcal{C}}_n(X, \mathcal{I})$.

Next, we describe an action of the canonical bundle $\Omega^1 X$ of X on the varieties $\mathcal{C}_n(X, \mathcal{I})$. We recall that, by [CH1], Theorem 1.15, the group homomorphism (4.4) is surjective and fits into the exact sequence

$$1 \to \Lambda \xrightarrow{\mathtt{dlog}} \Omega^1 X \xrightarrow{c} \mathrm{Pic}(\mathcal{D}) \xrightarrow{g} \mathrm{Pic}(X) \rtimes \mathrm{Aut}(X) \to 1\ , \tag{4.14}$$

where $\Lambda := A^{\times}/\mathbb{C}^{\times}$ is the multiplicative group of (nontrivial) units in A. The maps $\mathtt{dlog}$ and c in (4.14) are defined by

$$\mathtt{dlog} :\ \Lambda \to \Omega^1 X\ ,\quad u \mapsto u^{-1} du\ , \tag{4.15}$$

$$c :\ \Omega^1 X \to \mathrm{Pic}(\mathcal{D})\ ,\quad \omega \mapsto [\,\mathcal{D}_{\bar{\sigma}_\omega}]\ , \tag{4.16}$$

where $\bar{\sigma}_\omega \in \mathrm{Aut}(\mathcal{D})$ is the algebra automorphism acting identically on A and mapping $\partial \in \mathrm{Der}(A)$ to $\omega(\partial) + \partial \in \mathcal{D}_1$. Since the action of $\mathrm{Pic}(\mathcal{D})$ on $\mathrm{Pic}(X)$ factors through g, the image of $\Omega^1 X$ in $\mathrm{Pic}(\mathcal{D})$ under c stabilizes each point of $\mathrm{Pic}(X)$, and therefore, by equivariance of γ, preserves every fibre $\gamma^{-1}[\mathcal{I}] \subseteq \mathfrak{I}(\mathcal{D})$. Thus, letting $\Gamma := \Omega^1(X)/\Lambda$ and identifying Γ with $\mathrm{Im}(c)$, we get an action

$$\Gamma \times \gamma^{-1}[\mathcal{I}] \to \gamma^{-1}[\mathcal{I}]\ ,\quad [\mathcal{I}] \in \mathrm{Pic}(X)\ . \tag{4.17}$$

Now, let $\mathrm{DR}^1(B) := \Omega^1(B)/[B, \Omega^1 B]$ denote the space of Karoubi-de Rham differentials of the algebra $B = A[\mathcal{I}]$. Using the fact that B is smooth, we identify

$$\mathrm{DR}^1(B) \cong B \otimes_{B^{\mathrm{e}}} \Omega^1(B) \cong B \otimes_{B^{\mathrm{e}}} (\Omega^1 B)^{\star\star} \cong \mathrm{Hom}_{B^{\mathrm{e}}}((\Omega^1 B)^{\star},\, B)\ , \tag{4.18}$$

where $B^{\mathrm{e}} := B \otimes B^{\mathrm{o}}$, and $(\text{—})^\star$ stands for the duality over B^{e}. Explicitly, under the identification (4.18), $\bar{\omega} = \omega \,(\mathrm{mod}\,[B,\, \Omega^1 B]) \in \mathrm{DR}^1(B)$ corresponds to the homomorphism $\hat{\omega} : \Omega^1(B)^\star \to B$, $\delta \mapsto \mu[\delta(\omega)]$.

We let $\mathrm{DR}^1(B)$ act on $T_B\,(\Omega^1 B)^\star$ as follows: if $\bar{\omega} \in \mathrm{DR}^1(B)$, we define $\tilde{\sigma}_\omega$ to be the automorphism of $T_B\,(\Omega^1 B)^\star$ acting identically on B and mapping

$$(\Omega^1 B)^\star \to B \oplus (\Omega^1 B)^\star \;\hookrightarrow\; T_B\,(\Omega^1 B)^\star \ , \quad \delta \mapsto \hat{\omega}(\delta) + \delta \ .$$

By the universal property of tensor algebras, this uniquely determines $\tilde{\sigma}_\omega$, and it is clear that this map is bijective. Moreover, if $\Delta_B \in (\Omega^1 B)^\star$ is the canonical inclusion $\Omega^1 B \hookrightarrow B^{\mathrm{e}}$, then $\hat{\omega}(\Delta_B) = 0$, and hence $\tilde{\sigma}_\omega(\Delta_B) = \Delta_B$ for any $\bar{\omega} \in \mathrm{DR}^1(B)$. The assignment $\bar{\omega} \mapsto \tilde{\sigma}_\omega$ defines thus a homomorphism

$$\tilde{\sigma} : \mathrm{DR}^1(B) \to \mathrm{Aut}_B[T_B\,(\Omega^1 B)^\star] \tag{4.19}$$

from the additive group of $\mathrm{DR}^1(B)$ to the subgroup of B-linear automorphisms of $T_B(\Omega^1 B)^\star$ preserving the element Δ_B.

Identifying $\Omega^1 X$ with the module of Kähler differentials of A, we now construct an embedding $\Omega^1 X \hookrightarrow \mathrm{DR}^1(B)$. For this, we consider the exact sequence

$$0 \to \mathrm{HH}_1(\mathrm{B}) \xrightarrow{\alpha} \mathrm{DR}^1(\mathrm{B}) \to \mathrm{B} \to \mathrm{HH}_0(\mathrm{B}) \to 0 \ , \tag{4.20}$$

obtained by tensoring $0 \to \Omega^1(B) \to B^{\mathrm{e}} \to B \to 0$ with B, and compose the connecting map α in (4.20) with natural isomorphisms

$$\mathrm{HH}_1(\mathrm{B}) \cong \mathrm{HH}_1(\mathrm{A}) \cong \Omega^1\mathrm{X} \ . \tag{4.21}$$

The first isomorphism in (4.21) is induced by the algebra projection $\theta : B \to A$ (see [L], Theorem 1.2.15), while the second by the canonical map: $A^{\otimes 2} \to \Omega^1 X$, $f \otimes g \mapsto f\, dg$ (see [L], Prop. 1.1.10).

Finally, combining (4.19) with (4.20) and (4.21), we define

$$\sigma :\ \Omega^1 X \overset{\alpha}{\hookrightarrow} \mathrm{DR}^1(B) \xrightarrow{\tilde{\sigma}} \mathrm{Aut}_B[T_B\,(\Omega^1 B)^\star] \to \mathrm{Aut}_S[\Pi^{\boldsymbol{\lambda}}(B)] \ , \tag{4.22}$$

where the last map is induced by the algebra projection: $T_B\,(\Omega^1 B)^\star \twoheadrightarrow \Pi^{\boldsymbol{\lambda}}(B)$. An explicit description of $\tilde{\sigma}$ will be given in Section 5.4 (see Lemma 5.10).

Now, the group $\mathrm{Aut}_S[\Pi^{\boldsymbol{\lambda}}(B)]$ acts on $\mathrm{Rep}_S(\Pi^{\boldsymbol{\lambda}}(B), \boldsymbol{n})$ in the obvious way: if $\varrho : \Pi^{\boldsymbol{\lambda}}(B) \to \mathrm{End}(\boldsymbol{V})$ represents a point in $\mathrm{Rep}_S(\Pi^{\boldsymbol{\lambda}}(B), \boldsymbol{n})$, then $\sigma.\varrho = \varrho\,\sigma^{-1}$ for $\sigma \in \mathrm{Aut}_S[\Pi^{\boldsymbol{\lambda}}(B)]$. Clearly, this commutes with the $\mathrm{G}_S(\boldsymbol{n})$-action on $\mathrm{Rep}_S(\Pi^{\boldsymbol{\lambda}}(B), \boldsymbol{n})$, and hence induces an action of $\mathrm{Aut}_S[\Pi^{\boldsymbol{\lambda}}(B)]$ on $\mathcal{C}_n(X, \mathcal{I})$. Restricting this last action to $\Omega^1 X$ via (4.22), we define

$$\sigma^* :\ \Omega^1 X \to \mathrm{Aut}\,[\,\mathcal{C}_n(X, \mathcal{I})\,] \ , \quad \omega \mapsto [\,\sigma^*_\omega :\ \varrho \mapsto \varrho\,\sigma_\omega^{-1}\,] \ . \tag{4.23}$$

Equivalently, σ^*_ω is defined on $\mathcal{C}_n(X, \mathcal{I})$ by twisting the structure of $\Pi^{\boldsymbol{\lambda}}(B)$-modules by σ_ω^{-1}, i. e. $[\boldsymbol{V}] \mapsto [\boldsymbol{V}^{\sigma_\omega^{-1}}]$.

Restricting (4.23) further to Λ , via (4.15), we define the quotient varieties

$$\overline{\mathcal{C}}_n(X, \mathcal{I}) := \mathcal{C}_n(X, \mathcal{I})/\Lambda \ . \tag{4.24}$$

These varieties come equipped with the induced action of the group $\Gamma = \Omega^1(X)/\Lambda$.

Lemma 4.3. (1) *The action* (4.23) *agrees with* (4.13): *if* $\mathcal{P} = \mathcal{D}_{\bar{\sigma}_\omega}$, *then*

$$f_{\mathcal{P}} = \sigma^*_\omega \ , \ \forall\, \omega \in \Omega^1 X \ .$$

(2) *The map* (4.13) *induces an isomorphism of quotient varieties*

$$\bar{f}_{\mathcal{P}} :\ \overline{\mathcal{C}}_n(X, \mathcal{I}) \xrightarrow{\sim} \overline{\mathcal{C}}_n(X, \mathcal{J}) \ ,$$

which depends only on the class of $\mathcal{P}$ *in* $\mathrm{Pic}(\mathcal{D})$.

We will prove Lemma 4.3 together with Proposition 4.3 in Section 5.5. Here, we make only a few remarks. It follows from Lemma 4.3 that the action of Λ on $\mathcal{C}_n(X,\mathcal{I})$ defined above coincides with the natural action of $\mathrm{Aut}(\mathcal{I}) = A^{\times}$, so $\overline{\mathcal{C}}_n(X,\mathcal{I})$ depends only on the class of $\mathcal{I}$ in $\mathrm{Pic}(X)$ and the definition (4.24) agrees with the one given in the introduction.

For each $n \geq 0$, let $\overline{\mathcal{C}}_n(X)$ denote the union of reduced Calogero-Moser spaces over the entire $\mathrm{Pic}(X)$:

$$\overline{\mathcal{C}}_n(X) := \bigsqcup_{[\mathcal{I}]\in \mathrm{Pic}(X)} \overline{\mathcal{C}}_n(X,\mathcal{I}) \ .$$

By part (2) of Lemma 4.3, the assignment $[\mathcal{P}] \mapsto \bar{f}_{\mathcal{P}}$ defines then an action of $\mathrm{Pic}(\mathcal{D})$ on $\overline{\mathcal{C}}_n(X)$, and part (1) says that this action restricts to the action of Γ on each individual fibre $\overline{\mathcal{C}}_n(X,\mathcal{I})$, i. e. $\bar{f}_{c(\omega)} = \bar{\sigma}^*_{\omega}$ for all $\omega \in \Gamma$.

4.3. **The main theorem.** We may now put pieces together and state the main result of the present paper. We recall the functor $L\boldsymbol{\theta}^* = \mathrm{Tor}_1^{\Pi}(\mathcal{D}, \text{—}) : \mathtt{Mod}(\Pi) \to \mathtt{Mod}(\mathcal{D})$ associated to the algebra homomorphism $\boldsymbol{\theta} : \Pi \to \mathcal{D}$: when restricted to finite-dimensional representations, this functor is given by (4.3).

Theorem 4.2. *Let* X *be a smooth affine irreducible curve over* $\mathbb{C}$.

(a) *For each* $n \geq 0$ *and* $[\mathcal{I}] \in \mathrm{Pic}(X)$, *the functor* (4.3) *induces an injective map*

$$\omega_n : \overline{\mathcal{C}}_n(X,\mathcal{I}) \to \gamma^{-1}[\mathcal{I}] \ ,$$

which is equivariant under the action of the group Γ.

(b) *Amalgamating the maps* ω_n *for all* $n \geq 0$ *gives a bijective correspondence*

$$\omega : \bigsqcup_{n\geq 0} \overline{\mathcal{C}}_n(X,\mathcal{I}) \xrightarrow{\sim} \gamma^{-1}[\mathcal{I}] \ .$$

(c) *For any* $[\mathcal{I}]$ *and* $[\mathcal{J}]$ *in* $\mathrm{Pic}(X)$ *and for any* $[\mathcal{P}] \in \mathrm{Pic}(\mathcal{D})$, *such that* $[\mathcal{P}]\cdot[\mathcal{I}] = [\mathcal{J}]$, *there is a commutative diagram:*

$$\begin{array}{ccc} \overline{\mathcal{C}}_n(X,\mathcal{I}) & \xrightarrow{\bar{f}_{\mathcal{P}}} & \overline{\mathcal{C}}_n(X,\mathcal{J}) \\ {\scriptstyle \omega_n}\downarrow & & \downarrow{\scriptstyle \omega_n} \\ \gamma^{-1}[\mathcal{I}] & \xrightarrow{[\mathcal{P}]} & \gamma^{-1}[\mathcal{J}] \end{array} \tag{4.25}$$

where $\bar{f}_{\mathcal{P}}$ *is an isomorphism induced by* (4.13).

Remark. For technical reasons, we assumed above that $X \neq \mathbb{A}^1$. Theorem 4.2 holds true, however, in general: if $X = \mathbb{A}^1$, the map ω induced by $L\boldsymbol{\theta}^*$ agrees with the Calogero-Moser map constructed in [BW1, BW2] (see [BCE], Theorem 1). In this case, the ring $\mathcal{D}$ is isomorphic to the Weyl algebra $A_1(\mathbb{C})$, $\mathrm{Pic}(\mathcal{D})$ is isomorphic to the automorphism group $\mathrm{Aut}(A_1)$ of A_1 (see [St]) and Γ corresponds to the subgroup of KP flows in $\mathrm{Aut}(A_1)$ (see [BW1]). Since $\mathrm{Pic}(\mathbb{A}^1)$ is trivial, the last part of Theorem 4.2 implies the equivariance of ω under the action of $\mathrm{Aut}(A_1)$.

5. Proof of the Main Theorem

We proceed in four steps. First, we show that the functor (4.3) induces well-defined maps $\tilde{\omega}_n : \mathcal{C}_n(X,\mathcal{I}) \to \gamma^{-1}[\mathcal{I}]$, one for each integer $n \geq 0$. Second, we prove that every class $[M] \in \gamma^{-1}[\mathcal{I}]$ is contained in the image of $\tilde{\omega}_n$ for some n (which is uniquely determined by $[M]$). Third, we check that $\tilde{\omega}_n$ factors through the action of Λ on $\mathcal{C}_n(X,\mathcal{I})$ and prove that the induced map $\omega_n : \overline{\mathcal{C}}_n(X,\mathcal{I}) \to \gamma^{-1}[\mathcal{I}]$ is injective and Γ-equivariant. Finally, we prove Proposition 4.3 and Lemma 4.3 of Section 4.2, and show that the diagram (4.25) in Theorem 4.2 is commutative.

We begin by describing the algebras $\Pi^{\boldsymbol{\lambda}}(B)$ in terms of generators and relations.

5.1. **The structure of $\Pi^{\boldsymbol{\lambda}}(B)$.** Recall that, for each $\boldsymbol{\lambda} \in S$, we defined these algebras by

$$\Pi^{\boldsymbol{\lambda}}(B) = T_B \mathrm{Der}(B, B^{\otimes 2})/\langle \Delta_B - \boldsymbol{\lambda} \rangle \ ,$$

where Δ_B is the distinguished derivation in $\mathrm{Der}(B, B^{\otimes 2})$ mapping $x \mapsto x \otimes 1 - 1 \otimes x$. Now, $\mathrm{Der}(B, B^{\otimes 2})$ contains a canonical sub-bimodule $\mathrm{Der}_S(B, B^{\otimes 2})$, consisting of S-linear derivations. We write $\Delta_{B,S} : B \to B \otimes B$ for the inner derivation $x \mapsto \mathrm{ad}_{\mathbf{e}}(x)$, with $\mathbf{e} := e \otimes e + e_\infty \otimes e_\infty \in B \otimes B$. It is easy to see that $\Delta_{B,S}(x) = 0$ for all $x \in S$, so $\Delta_{B,S} \in \mathrm{Der}_S(B, B^{\otimes 2})$.

Lemma 5.1. *For any $\boldsymbol{\lambda} \in S$, there is a canonical algebra isomorphism*

$$\Pi^{\boldsymbol{\lambda}}(B) \cong T_B \mathrm{Der}_S(B, B^{\otimes 2})/\langle \Delta_{B,S} - \boldsymbol{\lambda} \rangle \ .$$

Proof. By universal property, the natural inclusion of bimodules

$$\mathrm{Der}_S(B, B^{\otimes 2}) \hookrightarrow \mathrm{Der}(B, B^{\otimes 2})$$

extends to their tensor algebras. Combined with canonical projection, this yields the algebra map

$$\phi : T_B \mathrm{Der}_S(B, B^{\otimes 2}) \hookrightarrow T_B \mathrm{Der}(B, B^{\otimes 2}) \twoheadrightarrow \Pi^{\boldsymbol{\lambda}}(B) \ .$$

An easy calculation shows that $\Delta_{B,S} = e\,\Delta_B\, e + e_\infty\, \Delta_B\, e_\infty$ in $\mathrm{Der}(B, B^{\otimes 2})$. So $\Delta_{B,S} - \boldsymbol{\lambda} = e\,(\Delta_B - \boldsymbol{\lambda})\,e + e_\infty(\Delta_B - \boldsymbol{\lambda})\,e_\infty$ belongs to the ideal $\langle \Delta_B - \boldsymbol{\lambda} \rangle \subseteq T_B \mathrm{Der}(B, B^{\otimes 2})$, and hence ϕ vanishes on $\Delta_{B,S} - \boldsymbol{\lambda}$, inducing an algebra map

$$\overline{\phi} : T_B \mathrm{Der}_S(B, B^{\otimes 2})/\langle \Delta_{B,S} - \boldsymbol{\lambda} \rangle \to \Pi^{\boldsymbol{\lambda}}(B) \ . \tag{5.1}$$

We will show that $\overline{\phi}$ is an isomorphism by constructing the inverse map.

Consider the cotangent exact sequence for the algebra B (cf. [CQ], Prop. 2.9):

$$0 \to B \otimes_S \Omega^1 S \otimes_S B \to \Omega^1 B \to \Omega^1_S B \to 0 \ .$$

Since S is separable, this sequence splits: in fact, the canonical projection $B \otimes B \twoheadrightarrow B \otimes_S B$ has the distinguished section

$$B \otimes_S B \hookrightarrow B \otimes B \ , \quad b_1 \otimes_S b_2 \mapsto b_1 \cdot \mathbf{e} \cdot b_2 \ , \tag{5.2}$$

which restricts to the bimodules of differentials $\Omega^1_S B \hookrightarrow \Omega^1 B$. On the other hand, it is easy to see that the image of $B \otimes_S \Omega^1 S \otimes_S B$ in $\Omega^1 B$ is equal to $(Be \otimes e_\infty B) \oplus (Be_\infty \otimes eB)$. Thus, we can identify

$$\Omega^1 B \cong \Omega^1_S B \oplus (Be \otimes e_\infty B) \oplus (Be_\infty \otimes eB) \ .$$

Dualizing this by $B \otimes B$ and identifying terms, we get an isomorphism

$$\mathrm{Der}(B, B^{\otimes 2}) \cong \mathrm{Der}_S(B, B^{\otimes 2}) \oplus (Be_\infty \otimes eB) \oplus (Be \otimes e_\infty B) \ , \tag{5.3}$$

where the spaces of derivations are equipped, as usual, with the inner bimodule structures, while the last two summands have the obvious (outer) bimodule structure. Note that Δ_B corresponds under (5.3) to the element $(\Delta_{B,S},\, e_\infty \otimes e,\, e \otimes e_\infty)$.

Now, the isomorphism (5.3) extends to tensor algebras, identifying

$$T_B\mathrm{Der}(B, B^{\otimes 2}) \cong T_B\mathrm{Der}_S(B, B^{\otimes 2}) \oplus \langle\, e_\infty \otimes e,\, e \otimes e_\infty \,\rangle \tag{5.4}$$

as algebras over B. Under this identification, the ideal $\langle \Delta_B - \boldsymbol{\lambda} \rangle$ decomposes as

$$\langle \Delta_B - \boldsymbol{\lambda} \rangle \cong \langle \Delta_{B,S} - \boldsymbol{\lambda} \rangle \oplus \langle\, e_\infty \otimes e,\, e \otimes e_\infty \,\rangle \ , \tag{5.5}$$

where the first bracket in the right-hand side stands for an ideal in $T_B\mathrm{Der}_S(B, B^{\otimes 2})$. Indeed, $\langle \Delta_B - \boldsymbol{\lambda} \rangle$ is generated in $T_B\mathrm{Der}(B, B^{\otimes 2})$ by the elements $e(\Delta_B - \boldsymbol{\lambda})e + e_\infty(\Delta_B - \boldsymbol{\lambda})e_\infty$, $e_\infty(\Delta_B - \boldsymbol{\lambda})e$ and $e(\Delta_B - \boldsymbol{\lambda})e_\infty$, while under our identification, these elements correspond to $\Delta_{B,S} - \boldsymbol{\lambda}$, $e_\infty \otimes e$ and $e \otimes e_\infty$ respectively.

Now, in view of (5.5), projecting onto the first summand of (5.4) induces an algebra isomorphism

$$T_B\mathrm{Der}(B, B^{\otimes 2})/\langle \Delta_B - \boldsymbol{\lambda} \rangle \overset{\sim}{\to} T_B\mathrm{Der}_S(B, B^{\otimes 2})/\langle \Delta_{B,S} - \boldsymbol{\lambda} \rangle \ ,$$

which is obviously the inverse of (5.1). This finishes the proof of the lemma. □

By Lemma 5.1, the structure of the algebras $\Pi^{\boldsymbol{\lambda}}(B)$ is determined by the bimodule $\mathrm{Der}_S(B, B^{\otimes 2})$. We now describe this bimodule explicitly, in terms of A, $\mathcal{I}$ and the dual module $\mathcal{I}^\vee = \mathrm{Hom}_A(\mathcal{I}, A)$. To fix notation we begin with a few fairly obvious remarks on bimodules over one-point extensions.

A bimodule Ξ over B is characterized by the following data: an A-A-bimodule T, a left A-module U, a right A-module V and a $\mathbb{C}$-vector space W given together with three A-module homomorphisms $f_1 : \mathcal{I} \otimes V \to T$, $f_2 : \mathcal{I} \otimes W \to U$, $g_1 : T \otimes_A \mathcal{I} \to U$ and a $\mathbb{C}$-linear map $g_2 : V \otimes_A \mathcal{I} \to W$, which fit into the commutative diagram

$$\begin{array}{ccc} \mathcal{I} \otimes V \otimes_A \mathcal{I} & \xrightarrow{\mathrm{Id} \otimes g_2} & \mathcal{I} \otimes W \\ {\scriptstyle f_1 \otimes_A \mathrm{Id}} \downarrow & & \downarrow {\scriptstyle f_2} \\ T \otimes_A \mathcal{I} & \xrightarrow{\quad g_1 \quad} & U \end{array} \tag{5.6}$$

These data can be conveniently organized by using the matrix notation

$$\Xi = \begin{pmatrix} T & U \\ V & W \end{pmatrix} \ , \tag{5.7}$$

with understanding that B acts on Λ by the usual matrix multiplication, via the maps f_1, f_2, g_1 and g_2. Note that the components of T are determined by

$$T = e\,\Xi\,e \ , \quad U = e\,\Xi\,e_\infty \ , \quad V = e_\infty \Xi\,e \ , \quad W = e_\infty \Xi\,e_\infty \ , \tag{5.8}$$

and the commutativity of (5.6) ensures the associativity of the action of B.

With above notation, the bimodule $\mathrm{Der}_S(B, B^{\otimes 2})$ can be described as follows.

Lemma 5.2. *There is an isomorphism of B-bimodules*

$$\mathrm{Der}_S(B, B^{\otimes 2}) \cong \begin{pmatrix} \mathrm{Der}(A, A^{\otimes 2}) & \mathrm{Der}(A, \mathcal{I} \otimes A) \\ 0 & 0 \end{pmatrix} \bigoplus \begin{pmatrix} \mathcal{I} \otimes \mathcal{I}^\vee & \mathcal{I} \otimes A \\ \mathcal{I}^\vee & A \end{pmatrix} \ ,$$

with $\Delta_{B,S}$ *corresponding to the element*

$$\left[\begin{pmatrix} \Delta_A & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -\sum_i \mathrm{v}_i \otimes \mathrm{w}_i & 0 \\ 0 & 1 \end{pmatrix}\right], \tag{5.9}$$

where $\{\mathrm{v}_i\}$ *and* $\{\mathrm{w}_i\}$ *are dual bases for the projective* A*-modules* $\mathcal{I}$ *and* $\mathcal{I}^\vee$*.*

Proof. We begin by describing the free bimodule $B \otimes B$: this can be decomposed as a direct sum of four bimodules $Be \otimes eB$, $Be \otimes e_\infty B$, $Be_\infty \otimes eB$ and $Be_\infty \otimes e_\infty B$, each of which is easy to identify using (5.8):

$$Be \otimes eB \cong \begin{pmatrix} A \otimes A & A \otimes \mathcal{I} \\ 0 & 0 \end{pmatrix}, \quad Be \otimes e_\infty B \cong \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix},$$

$$Be_\infty \otimes eB \cong \begin{pmatrix} \mathcal{I} \otimes A & \mathcal{I} \otimes \mathcal{I} \\ A & \mathcal{I} \end{pmatrix}, \quad Be_\infty \otimes e_\infty B \cong \begin{pmatrix} 0 & \mathcal{I} \\ 0 & \mathbb{C} \end{pmatrix}.$$

The action of B on each of these bimodules is determined by the natural maps (e. g., for $Be \otimes eB$, f_1, f_2 and g_2 are zero, while $g_1 : A \otimes A \otimes_A \mathcal{I} \xrightarrow{\sim} A \otimes \mathcal{I}$ is the canonical isomorphism).

Next, we describe $\Omega^1_S B$, the kernel of the multiplication map $\mu_B : B \otimes_S B \to B$. If we identify

$$B \otimes_S B \cong (Be \otimes eB) \oplus (Be_\infty \otimes e_\infty B) \cong \begin{pmatrix} A \otimes A & A \otimes \mathcal{I} \\ 0 & 0 \end{pmatrix} \bigoplus \begin{pmatrix} 0 & \mathcal{I} \\ 0 & \mathbb{C} \end{pmatrix}, \tag{5.10}$$

then μ_B is given by

$$\left[\begin{pmatrix} a_1 \otimes a_2 & a \otimes b \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & m \\ 0 & c \end{pmatrix}\right] \mapsto \begin{pmatrix} a_1 a_2 & ab + m \\ 0 & c \end{pmatrix},$$

and the elements of $\Omega^1_S B$ correspond to pairs of matrices

$$\left[\begin{pmatrix} \omega & a \otimes b \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -ab \\ 0 & 0 \end{pmatrix}\right] \tag{5.11}$$

with $\omega \in \Omega^1 A$, $a \in A$ and $b \in \mathcal{I}$. Now, the elements (5.11) with second component zero form a sub-B-bimodule in $\Omega^1_S B$, which is the image of the canonical map

$$i : \begin{pmatrix} \Omega^1 A & \Omega^1 A \otimes_A \mathcal{I} \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} A \otimes A & A \otimes \mathcal{I} \\ 0 & 0 \end{pmatrix} \bigoplus \begin{pmatrix} 0 & \mathcal{I} \\ 0 & \mathbb{C} \end{pmatrix}.$$

Since $\mathcal{I}$ is projective, $\mathrm{Ker}(i) = 0$, and we have the short exact sequence[6]

$$0 \to \begin{pmatrix} \Omega^1 A & \Omega^1 A \otimes_A \mathcal{I} \\ 0 & 0 \end{pmatrix} \xrightarrow{i} \Omega^1_S B \xrightarrow{\pi} \begin{pmatrix} 0 & \mathcal{I} \\ 0 & 0 \end{pmatrix} \to 0, \tag{5.12}$$

where π is the projection onto the second summand in (5.10).

Now, it is easy to see that the sequence (5.12) splits, and its splitting is determined by a choice of dual bases for the projective modules $\mathcal{I}$ and $\mathcal{I}^\vee$. In fact, if $\{\mathrm{v}_i\}_{i=1}^N \subset \mathcal{I}$ and $\{\mathrm{w}_i\}_{i=1}^N \subset \mathcal{I}^\vee$ are such bases (so that $b = \sum_{i=1}^N \mathrm{w}_i(b)\, \mathrm{v}_i$ for all $b \in \mathcal{I}$), then

$$\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mapsto \left[\begin{pmatrix} 0 & -\sum_i \mathrm{w}_i(b) \otimes \mathrm{v}_i \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\right]$$

[6] This sequence is isomorphic to the cotangent exact sequence (2.20) constructed in Proposition 2.2.

defines a B-bimodule homomorphism

$$s: \begin{pmatrix} 0 & \mathcal{I} \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} A\otimes A & A\otimes\mathcal{I} \\ 0 & 0 \end{pmatrix} \bigoplus \begin{pmatrix} 0 & \mathcal{I} \\ 0 & \mathbb{C} \end{pmatrix},$$

which is obviously a section of π with image in $\Omega^1_S B$. Thus, with identification (5.10), we have a decomposition

$$\Omega^1_S B \cong \begin{pmatrix} \Omega^1 A & \Omega^1 A\otimes_A \mathcal{I} \\ 0 & 0 \end{pmatrix} \bigoplus \begin{pmatrix} 0 & \mathcal{I} \\ 0 & 0 \end{pmatrix}, \tag{5.13}$$

the canonical inclusion $\Omega^1_S B \hookrightarrow B\otimes_S B$ corresponding to the map (i, s).

Now, to describe the bimodule $\mathrm{Der}_S(B, B^{\otimes 2})$ we dualize (5.13) over B^{e} and use the canonical isomorphism

$$\mathrm{Der}_S(B, B^{\otimes 2}) \cong \mathrm{Hom}_{B^{\mathrm{e}}}(\Omega^1_S B, B^{\otimes 2}) \,. \tag{5.14}$$

After some trivial calculations this yields

$$\begin{pmatrix} \mathrm{Der}(A, A^{\otimes 2}) & \mathrm{Der}(A, \mathcal{I}\otimes A) \\ 0 & 0 \end{pmatrix} \bigoplus \begin{pmatrix} \mathcal{I}\otimes\mathcal{I}^\vee & \mathcal{I}\otimes \mathrm{End}_A(\mathcal{I}) \\ \mathcal{I}^\vee & \mathrm{End}_A(\mathcal{I}) \end{pmatrix}. \tag{5.15}$$

Since A is commutative and $\mathcal{I}$ is a rank 1 projective, the action of A on $\mathcal{I}$ induces an isomorphism $A \xrightarrow{\sim} \mathrm{End}_A(\mathcal{I})$, so (5.15) is a required decomposition for $\mathrm{Der}_S(B, B^{\otimes 2})$.

The element (5.9) in (5.15) corresponds to the embedding

$$\Omega^1_S B \xrightarrow{(i,s)} B\otimes_S B \cong (Be\otimes eB)\oplus(Be_\infty\otimes e_\infty B) \hookrightarrow B\otimes B\,,$$

which, in turn, corresponds under (5.14) to the element $\Delta_{B,S} \in \mathrm{Der}_S(B, B^{\otimes 2})$. This completes the proof of the lemma. ☐

Now, we fix dual bases $\{\mathrm{v}_i\}_{i=1}^N$ and $\{\mathrm{w}_i\}_{i=1}^N$ for $\mathcal{I}$ and $\mathcal{I}^\vee$ and, combining the isomorphisms of Lemmas 5.1 and 5.2, identify $\Pi^{\boldsymbol\lambda}(B)$ with homomorphic image of the tensor algebra of the bimodule

$$\mathrm{Der}_S(B, B^{\otimes 2}) = \begin{pmatrix} \mathrm{Der}(A, A^{\otimes 2})\oplus(\mathcal{I}\otimes\mathcal{I}^\vee) & \mathrm{Der}(A, \mathcal{I}\otimes A)\oplus(\mathcal{I}\otimes A) \\ \mathcal{I}^\vee & A \end{pmatrix}.$$

With this identification, we have

Proposition 5.1. *The algebra $\Pi^{\boldsymbol\lambda}(B)$ is generated by (the images of) the following elements*

$$\hat{a} := \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix},\ \hat{\mathrm{v}}_i := \begin{pmatrix} 0 & \mathrm{v}_i \\ 0 & 0 \end{pmatrix},\ \hat{d} := \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix},\ \hat{\mathrm{w}}_i := \begin{pmatrix} 0 & 0 \\ \mathrm{w}_i & 0 \end{pmatrix},$$

where $\hat{a}$, $\hat{\mathrm{v}}_i \in B$ and $\hat{d}$, $\hat{\mathrm{w}}_i \in \mathrm{Der}_S(B, B^{\otimes 2})$ with $d \in \mathrm{Der}(A, A^{\otimes 2})$. Apart from the obvious relations induced by matrix multiplication, these elements satisfy

$$\hat{\Delta}_A - \sum_{i=1}^N \hat{\mathrm{v}}_i\cdot\hat{\mathrm{w}}_i = \lambda\, e\,,\quad \sum_{i=1}^N \hat{\mathrm{w}}_i\cdot\hat{\mathrm{v}}_i = \lambda_\infty\, e_\infty\,, \tag{5.16}$$

where "$\cdot$" denotes the action of B on the bimodule $\mathrm{Der}_S(B, B^{\otimes 2})$.

Proof. By Lemma 2.3, the matrices $\{\hat{a}\}$ and $\{\hat{\mathrm{v}}_i\}$ generate the algebra B, while $\{\hat{d}\}$ and $\{\hat{\mathrm{w}}_i\}$ generate the first and the second bimodule summand of (5.15) respectively. All together they thus generate the tensor algebra. Now, observe that the ideal $\langle \Delta_{B,S} - \boldsymbol{\lambda} \rangle$ in $\Pi^{\boldsymbol{\lambda}}(B)$ is generated by $e\,(\Delta_{B,S} - \boldsymbol{\lambda})\,e = e\,\Delta_{B,S}\,e - \lambda\,e$ and $e_\infty\,(\Delta_{B,S} - \boldsymbol{\lambda})\,e_\infty = e_\infty\,\Delta_{B,S}\,e_\infty - \lambda_\infty\,e_\infty$, since the sum of these elements is equal to $\Delta_{B,S} - \boldsymbol{\lambda}$. With identification of Lemma 5.2, we then have

$$e\,\Delta_{B,S}\,e = \begin{pmatrix} \Delta_A & 0 \\ 0 & 0 \end{pmatrix} - \sum_{i=1}^{N} \begin{pmatrix} \mathrm{v}_i \otimes \mathrm{w}_i & 0 \\ 0 & 0 \end{pmatrix} = \hat{\Delta}_A - \sum_{i=1}^{N} \hat{\mathrm{v}}_i \cdot \hat{\mathrm{w}}_i$$

and

$$e_\infty\,\Delta_{B,S}\,e_\infty = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \sum_{i=1}^{N} \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{w}_i \otimes_A \mathrm{v}_i \end{pmatrix} = \sum_{i=1}^{N} \hat{\mathrm{w}}_i \cdot \hat{\mathrm{v}}_i \ ,$$

whence the relations (5.16). □

Remark. The presentation of $\Pi^{\boldsymbol{\lambda}}(B)$ given in Proposition 5.1 is essentially independent of the choice of dual bases $\{\mathrm{v}_i\}$ and $\{\mathrm{w}_i\}$ in $\mathcal{I}$ and $\mathcal{I}^\vee$. To be precise, changing these bases to $\{\mathrm{v}'_j\}$ and $\{\mathrm{w}'_j\}$ (say) results in the automorphism of $\Pi^{\boldsymbol{\lambda}}(B)$ that fixes B and $\mathcal{I}^\vee$ elementwise, while mapping $\hat{d} \mapsto \hat{d} - \widehat{d(\omega)}$, where $\omega := \sum_i \mathrm{v}_i \otimes \mathrm{w}_i - \sum_j \mathrm{v}'_j \otimes \mathrm{w}'_j$. Note that ω belongs to the kernel of the natural pairing $\mu : \mathcal{I} \otimes \mathcal{I}^\vee \to A$. When we apply d to ω above, we use the canonical identification $\mathrm{Hom}_{A^e}(\mathrm{Ker}\,\mu,\, \mathcal{I} \otimes \mathcal{I}^\vee) \cong \mathrm{Der}(A,\, A^{\otimes 2})$ and then replace the tensor product $\mathcal{I} \otimes \mathcal{I}^\vee$ by multiplication in $\Pi^\lambda(B)$. In particular, for $d = \hat{\Delta}_A$, we have $\hat{\Delta}_A \mapsto \hat{\Delta}_A - \omega$, which agrees with the relations (5.16).

As a simple consequence of Proposition 5.1, we get

Lemma 5.3. *If* $\lambda_\infty \neq 0$*, the algebra* $\Pi^{\boldsymbol{\lambda}}(B)$ *is Morita equivalent to* $e\,\Pi^{\boldsymbol{\lambda}}(B)\,e$ *.*

Proof. By standard Morita theory (see, e.g., [MR], Prop. 3.5.6), it suffices to show that $\Pi^{\boldsymbol{\lambda}}(B)\,e\,\Pi^{\boldsymbol{\lambda}}(B) = \Pi^{\boldsymbol{\lambda}}(B)$. This last identity holds in $\Pi^{\boldsymbol{\lambda}}(B)$ if (and only if) $1 \in \Pi^{\boldsymbol{\lambda}}(B)\,e\,\Pi^{\boldsymbol{\lambda}}(B)$, or equivalently, if $e_\infty \in \Pi^{\boldsymbol{\lambda}}(B)\,e\,\Pi^{\boldsymbol{\lambda}}(B)$, since $e + e_\infty = 1$. Now, if $\lambda_\infty \neq 0$, the second relation of (5.16) can be written as

$$e_\infty = \frac{1}{\lambda_\infty} \sum_{i=1}^{N} \hat{\mathrm{w}}_i \cdot \hat{\mathrm{v}}_i = \frac{1}{\lambda_\infty} \sum_{i=1}^{N} \hat{\mathrm{w}}_i \cdot e \cdot \hat{\mathrm{v}}_i \ , \tag{5.17}$$

whence the result. □

We may now also show that $\mathcal{D}$ has projective dimension 1 in the category of Π-modules (the fact used in Section 4.1).

Lemma 5.4. *The multiplication map* $\Pi\,e_\infty \otimes_U e_\infty\Pi \to \Pi$ *gives a projective resolution of* $\mathcal{D}$ *in the category of (left and right)* Π*-modules, see* (4.2).

Proof. Observe that the map $\boldsymbol{\theta}$ of Lemma 4.1 restricts to a surjective algebra homomorphism: $e\Pi e \to \mathcal{D}$, mapping $e \mapsto 1$. If we identify $A \cong eBe \subset e\Pi e$ via $a \mapsto \hat{a}$ and $\mathcal{I} \cong eBe_\infty \subset e\Pi e_\infty$ via $\mathrm{v} \mapsto \hat{\mathrm{v}}$, then tensoring $\boldsymbol{\theta}|_{e\Pi e}$ with $\mathcal{I}$ yields $e\Pi e \otimes_{eBe} eBe_\infty \to \mathcal{D} \otimes_A \mathcal{I} \cong \mathcal{D}\mathcal{I}$. Since eBe_∞ is a projective eBe-module, the multiplication map $e\Pi e \otimes_{eBe} eBe_\infty \to e\Pi e_\infty$ is an isomorphism onto its image $e\Pi eBe_\infty \subseteq e\Pi e_\infty$. On the other hand, $e\Pi e_\infty \subseteq e\Pi eBe_\infty$, since $e\,p\,e_\infty = -\frac{1}{n}\sum_i e\,p\,\hat{\mathrm{w}}_i\,e\,\hat{\mathrm{v}}_i\,e_\infty$ for any $p \in \Pi$, by (5.17). Thus, identifying $e\Pi e \otimes_{eBe} eBe_\infty \cong e\Pi e_\infty$, we get a surjective map of left A-modules: $e\Pi e_\infty \twoheadrightarrow \mathcal{DI}$. Since $\mathcal{DI}$ is projective over A, this last map has an A-linear section, which we denote by $s : \mathcal{DI} \hookrightarrow e\Pi e_\infty$. Now, consider the composite map:

$$\mathcal{DI} \otimes e_\infty \Pi e \xrightarrow{s\otimes 1} e\Pi e_\infty \otimes e_\infty \Pi e \twoheadrightarrow e\,\Pi\, e_\infty \otimes_U e_\infty \Pi e \ , \tag{5.18}$$

which is a homomorphism of right $e\Pi e$-modules. Since $e\Pi e_\infty = e\Pi e B e_\infty = \sum_i e\Pi e\, \hat{\mathrm{v}}_i$, the elements of $e\,\Pi\, e_\infty$ can be written as linear combinations of words

$$\underbrace{P_1(\hat{a},\hat{d})\,\hat{\mathrm{v}}_{i_1}}_{e\Pi e_\infty}\underbrace{\hat{\mathrm{w}}_{i_1}\,P_2(\hat{a},\hat{d})\,\hat{\mathrm{v}}_{i_2}\hat{\mathrm{w}}_{i_2}\,\ldots\,\hat{\mathrm{v}}_{i_{n-1}}\hat{\mathrm{w}}_{i_{n-1}}P_n(\hat{a},\hat{d})\,\hat{\mathrm{v}}_{i_n}}_{e_\infty\Pi\, e_\infty} \ , \tag{5.19}$$

with $P_k(\hat{a},\hat{d}) \in e\Pi e$ being (noncommutative) polynomials in $\hat{a}$ and $\hat{d}$ with $a \in A$ and $d \in \mathrm{Der}(A, A^{\otimes 2})$. So (5.18) is surjective. On the other hand, using natural filtrations, it is easy to check that the composition of (5.18) with multiplication map $e\Pi e_\infty \otimes_U e_\infty \Pi e \to e\Pi e$ is injective. Hence (5.18) is injective and therefore an isomorphism. This implies that $e\Pi e_\infty \otimes_U e_\infty \Pi e$ is a right projective $e\Pi e$-module (since obviously so is $\mathcal{DI} \otimes e_\infty \Pi e$), and $0 \to e\,\Pi\, e_\infty \otimes_U e_\infty \Pi e \to e\Pi e \xrightarrow{\boldsymbol{\theta}} \mathcal{D} \to 0$ is an exact sequence of $e\Pi e$-modules. By Morita equivalence of Lemma 5.3, the complex $0 \to \Pi\, e_\infty \otimes_U e_\infty \Pi \to \Pi \to 0$ is then a projective resolution of $\mathcal{D}$ in the category of right Π-modules. A similar argument shows that this complex is also a projective resolution of $\mathcal{D}$ as a left Π-module. □

5.2. **The map ω is well-defined.** We show that the functor (4.3) maps the Π-modules of dimension vector $\boldsymbol{n} = (n, 1)$ to rank 1 torsion-free $\mathcal{D}$-modules M with $\gamma[M] = [\mathcal{I}]$.

Let $\boldsymbol{V}$ be a Π-module of dimension vector $\boldsymbol{n}$, and let $\boldsymbol{L} := \Pi e_\infty \otimes_U e_\infty \boldsymbol{V}$. Write $V := e\boldsymbol{V}$, $V_\infty := e_\infty \boldsymbol{V}$, and similarly $L := e\boldsymbol{L}$, $L_\infty := e_\infty \boldsymbol{L}$ (so that $\dim V_\infty = \dim L_\infty = 1$). Fix a vector $\xi \neq 0$ in V_∞ and define a character $\varepsilon : U \to \mathbb{C}$ by $u.\xi = \varepsilon(u)\xi$ for all $u \in U$. Note that ε does not depend on the choice of ξ and uniquely determines $\boldsymbol{L}$ (and $\boldsymbol{V}$). In fact, we have the isomorphism of Π-modules

$$\Pi\, e_\infty \Big/ \sum_{u\in U} \Pi\, e_\infty(u - \varepsilon(u)) \xrightarrow{\sim} \boldsymbol{L} \quad , \quad [e_\infty] \mapsto e_\infty \otimes \xi \ . \tag{5.20}$$

Now, under the Morita equivalence of Lemma 5.3, the Π-module homomorphism $\boldsymbol{\mu} : \boldsymbol{L} \to \boldsymbol{V}$ transforms to a homomorphism of $e\Pi e$-modules $\mu : L \to V$, the kernel of $\boldsymbol{\mu}$ being isomorphic to the kernel of μ, since $e_\infty(\mathrm{Ker}\,\boldsymbol{\mu}) = 0$ implies $e\,(\mathrm{Ker}\,\boldsymbol{\mu}) = \mathrm{Ker}\,\boldsymbol{\mu}$. Thus

$$L\boldsymbol{\theta}^*(\boldsymbol{V}) \cong \mathrm{Ker}\,[\,\mu :\, L \to V\,] \ , \tag{5.21}$$

which is naturally an isomorphism of $\mathcal{D}$-modules via $\boldsymbol{\theta}|_{e\Pi e} : e\Pi e \to \mathcal{D}$.

Next, we set $R := T_A\mathrm{Der}(A, A^{\otimes 2})$ and define the algebra map

$$R \to e\Pi e \ , \quad a \mapsto \hat{a} \ , \ d \mapsto \hat{d} \ , \tag{5.22}$$

where $a \in A$ and $d \in \mathrm{Der}(A, A^{\otimes 2})$. Extending the notation of Proposition 5.1, we will write $\hat{r} \in e\Pi e$ for the image of any element $r \in R$ under (5.22). Note that the canonical projection $R \twoheadrightarrow \Pi^1(A) = \mathcal{D}$ factors through (5.22), and the corresponding quotient map is $\boldsymbol{\theta}|_{e\Pi e}$.

Lemma 5.5. *There is an isomorphism of R-modules*

$$L \cong R\mathcal{I} \Big/ \sum_{i=1}^{N} \sum_{r\in R} R\Big[(\Delta_A - 1)\, r \mathrm{v}_i - \sum_{j=1}^{N} \varepsilon(\hat{\mathrm{w}}_j\, \hat{r}\, \hat{\mathrm{v}}_i)\, \mathrm{v}_j\Big] \ , \tag{5.23}$$

where L is regarded as an R-module via (5.22)*, and $R\mathcal{I} := R \otimes_A \mathcal{I}$.*

Proof. As in Lemma 5.4, we identify $A \cong eBe \subset e\Pi e$, $\mathcal{I} \cong eBe_\infty \subset e\Pi e_\infty$, and consider the composite map

$$R\mathcal{I} \xrightarrow{\pi_1} e\Pi e \otimes_{eBe} eBe_\infty \xrightarrow{\pi_2} e\Pi e_\infty \xrightarrow{\pi_3} e\Pi e_\infty \Big/ \sum_{u\in U} e\Pi e_\infty\, (u - \varepsilon(u)) \ , \tag{5.24}$$

where π_1 is the tensor product of (5.22) with $\mathcal{I}$, π_2 is the multiplication map and π_3 is the canonical projection. Now, we have seen in (the proof of) Lemma 5.4 that π_2 is surjective (in fact, an isomorphism). On the other hand, (5.19) shows every element of $e\Pi e_\infty$ is congruent (modulo $\sum_{u\in U} e\Pi e_\infty(u - \varepsilon(u))$) to a linear combination of words $P(\hat{a}, \hat{d})\, \hat{\mathrm{v}}$, with $a \in A$, $d \in \mathrm{Der}(A, A^{\otimes 2})$ and $\mathrm{v} \in \mathcal{I}$, which obviously span the image of $\pi_2 \circ \pi_1$. This implies that (5.24) is a surjective map: by (5.20), its image is isomorphic to L, while its kernel can be easily calculated, using the relations (5.16). □

Now, the tensor algebra R has a natural filtration:

$$A \subseteq A \oplus \mathrm{Der}(A, A^{\otimes 2}) \subseteq A \oplus \mathrm{Der}(A, A^{\otimes 2}) \oplus \mathrm{Der}(A, A^{\otimes 2})^{\otimes 2} \subseteq \ldots$$

This filtration induces the usual (differential) filtration on $\mathcal{D}$ via the canonical projection and module filtrations on L and $M \subseteq L$ via the isomorphism of Lemma 5.5. Writing $\bar{\mathcal{D}}$, $\overline{L}$, etc. for the associated graded rings relative to these filtrations, we have

$$\overline{M} \subseteq \overline{L} \cong \overline{R}\mathcal{I}/\overline{R}\,\bar{\Delta}_A \overline{R}\mathcal{I} \cong (\overline{R}/\overline{R}\,\bar{\Delta}_A\, \overline{R}) \otimes_A \mathcal{I} \cong \bar{\mathcal{D}} \otimes_A \mathcal{I} \cong \bar{\mathcal{D}}\mathcal{I} \ . \tag{5.25}$$

It follows from (5.25) that M is a rank 1 torsion-free module (because so is $\overline{M}$). Moreover, since $\dim_{\mathbb{C}} \overline{L}/\overline{M} = \dim_{\mathbb{C}} L/M < \infty$, by part (a) of Theorem 3.1 of Section 3.2, we conclude $\gamma[M] = [\mathcal{I}]$. This completes Step 1.

5.3. **The map ω is surjective.** Given a rank 1 torsion-free $\mathcal{D}$-module M, we now construct a Π-module $\boldsymbol{L}$, together with a $\Pi^{\boldsymbol{\lambda}}(B)$-module embedding $M \hookrightarrow \boldsymbol{L}$, such that $\boldsymbol{V} := \boldsymbol{L}/M$ has dimension $(n, 1)$ and $L\boldsymbol{\theta}^*[\boldsymbol{V}] \cong M$.

We begin with some preparations. We let $\widetilde{\mathcal{D}} := \bigoplus_{k=0}^{\infty} \mathcal{D}_k t^k$ denote the homogenization (the Rees algebra) of the ring $\mathcal{D}$ with respect to its canonical filtration $\{\mathcal{D}_k\}$, and let $\mathsf{GrMod}(\widetilde{\mathcal{D}})$ be the category of graded $\widetilde{\mathcal{D}}$-modules. There is a natural homomorphism of graded rings $i : \widetilde{\mathcal{D}} \to \bar{\mathcal{D}}$, mapping $a\, t^k \in \widetilde{\mathcal{D}}_k$ to $a\, (\mathrm{mod}\, \mathcal{D}_{k-1}) \in \bar{\mathcal{D}}_k$. Using this homomorphism, we will regard graded $\bar{\mathcal{D}}$-modules as objects of $\mathsf{GrMod}(\widetilde{\mathcal{D}})$. Since $\mathrm{Ker}(i) = \langle\, t\, \rangle$, we may identify $\bar{\mathcal{D}} \cong \widetilde{\mathcal{D}}/\langle\, t\, \rangle$. This implies that $\widetilde{\mathcal{D}}$ is Noetherian, since so is $\bar{\mathcal{D}}$ (see [L], Prop. 3.5).

Next, following [AZ], we define $\mathsf{Tors}(\widetilde{\mathcal{D}})$ to be the full subcategory of $\mathsf{GrMod}(\widetilde{\mathcal{D}})$ consisting of torsion modules. By definition, $\widetilde{M} \in \mathsf{GrMod}(\widetilde{\mathcal{D}})$ is *torsion*, if for every $m \in \widetilde{M}$ there is $k_m \in \mathbb{N}$ such that $\widetilde{\mathcal{D}}_k\, m = 0$ for all $k \geq k_m$. By [AZ], Sect. 2, $\mathsf{Tors}(\widetilde{\mathcal{D}})$ is a dense localizing subcategory of $\mathsf{GrMod}(\widetilde{\mathcal{D}})$. So the inclusion functor

$\mathsf{Tors}(\widetilde{\mathcal{D}}) \hookrightarrow \mathsf{GrMod}(\widetilde{\mathcal{D}})$ has a right adjoint: the torsion functor $\tau : \mathsf{GrMod}(\widetilde{\mathcal{D}}) \to \mathsf{Tors}(\widetilde{\mathcal{D}})$, which assigns to a graded module $\widetilde{M}$ its largest torsion submodule

$$\tau(\widetilde{M}) = \{m \in \widetilde{M} \ : \ \widetilde{\mathcal{D}}_k\, m = 0 \text{ for all } k \gg 0\} \ .$$

The functor τ is left exact, and we write $R^k\tau : \mathsf{GrMod}(\widetilde{\mathcal{D}}) \to \mathsf{Tors}(\widetilde{\mathcal{D}})$ for its right derived functors.

We also introduce the quotient category $\mathsf{Qgr}(\widetilde{\mathcal{D}}) := \mathsf{GrMod}(\widetilde{\mathcal{D}})/\mathsf{Tors}(\widetilde{\mathcal{D}})$. This is an abelian category that comes equipped with two canonical functors: the (exact) quotient functor $\pi : \mathsf{GrMod}(\widetilde{\mathcal{D}}) \to \mathsf{Qgr}(\widetilde{\mathcal{D}})$ and its right adjoint (and hence left exact) functor $\omega : \mathsf{Qgr}(\widetilde{\mathcal{D}}) \to \mathsf{GrMod}(\widetilde{\mathcal{D}})$. The relationship between π, ω and τ is described by the following result which is part of standard torsion theory (see, e. g., [AZ], Prop. 7.2).

Theorem 5.1. (1) *The adjunction map $\eta_{\widetilde{M}} : \widetilde{M} \to \omega\,\pi(\widetilde{M})$ fits into the exact sequence*

$$0 \to \tau(\widetilde{M}) \to \widetilde{M} \xrightarrow{\eta_{\widetilde{M}}} \omega\pi(\widetilde{M}) \to R^1\tau(\widetilde{M}) \ \to 0 \ , \tag{5.26}$$

which is functorial in $\widetilde{M} \in \mathsf{GrMod}(\widetilde{\mathcal{D}})$.

(2) *For $k \geq 1$, there are natural isomorphisms*

$$R^k\omega(\pi\widetilde{M}) \cong R^{k+1}\tau(\widetilde{M}) \ . \tag{5.27}$$

In particular, if $k \geq 1$, the modules $R^k\omega(\pi\widetilde{M})$ are torsion.

Now, given a graded module $\widetilde{M} = \bigoplus_{k\in\mathbb{Z}} \widetilde{M}_k$ and $n \in \mathbb{Z}$, we write $\widetilde{M}[n] := \bigoplus_{k\in\mathbb{Z}} \widetilde{M}_{k+n}$ and $\widetilde{M}_{\geq n} := \bigoplus_{k\geq n} \widetilde{M}_k$. Both are graded modules, $\widetilde{M}_{\geq n}$ being a submodule of $\widetilde{M}$. With this notation, we compute $R^k\omega(\pi\bar{\mathcal{D}})$, regarding $\bar{\mathcal{D}}$ as a $\widetilde{\mathcal{D}}$-module via the algebra map $i : \widetilde{\mathcal{D}} \to \bar{\mathcal{D}}$.

Lemma 5.6. (1) *The canonical map $\eta_{\bar{\mathcal{D}}} : \bar{\mathcal{D}} \xrightarrow{\sim} \omega\pi(\bar{\mathcal{D}})_{\geq 0}$ is an isomorphism.*
(2) *$R^k\omega(\pi\bar{\mathcal{D}}) = 0$ for $k \geq 1$.*

Proof. For graded $\bar{\mathcal{D}}$-modules $\bar{M}$ and $\bar{N}$, we define (cf. [AZ], Sect. 3)

$$\underline{\mathrm{Hom}}_{\bar{\mathcal{D}}}(\bar{M},\, \bar{N}) := \bigoplus_{k\in Z} \mathrm{Hom}_{\mathrm{GrMod}(\bar{\mathcal{D}})}(\bar{M},\, \bar{N}[k]) \ ,$$

and write $\underline{\mathrm{Ext}}^n_{\bar{\mathcal{D}}}(\bar{M},\, \bar{N})$ for the corresponding Ext-groups. Combining [AZ], Theorem 8.3 and Proposition 7.2, we then identify

$$R^k\omega(\pi\bar{\mathcal{D}}) \cong \varinjlim\ \underline{\mathrm{Ext}}^k_{\bar{\mathcal{D}}}(\bar{\mathcal{D}}_{\geq n},\, \bar{\mathcal{D}}) \ , \quad \forall\ k \geq 0 \ . \tag{5.28}$$

To compute the Ext-groups in (5.28) we use the long cohomology sequence

$$\underline{\mathrm{Ext}}^k_{\bar{\mathcal{D}}}(\bar{\mathcal{D}}_n[-n], \bar{\mathcal{D}}) \to \underline{\mathrm{Ext}}^k_{\bar{\mathcal{D}}}(\bar{\mathcal{D}}_{\geq n}, \bar{\mathcal{D}}) \to \underline{\mathrm{Ext}}^k_{\bar{\mathcal{D}}}(\bar{\mathcal{D}}_{\geq n+1}, \bar{\mathcal{D}}) \to \underline{\mathrm{Ext}}^{k+1}_{\bar{\mathcal{D}}}(\bar{\mathcal{D}}_n[-n], \bar{\mathcal{D}}) \tag{5.29}$$

arising from the short exact sequence $0 \to \bar{\mathcal{D}}_{\geq n+1} \to \bar{\mathcal{D}}_{\geq n} \to \bar{\mathcal{D}}_n[-n] \to 0$, and the following isomorphisms (for $n \geq 0$)

$$\underline{\mathrm{Ext}}^k_{\bar{\mathcal{D}}}(\bar{\mathcal{D}}_n[-n], \bar{\mathcal{D}}) = 0 \quad \text{if} \quad k \neq 1 \ , \tag{5.30}$$

and

$$\underline{\mathrm{Ext}}^1_{\bar{\mathcal{D}}}(\bar{\mathcal{D}}_n[-n], \bar{\mathcal{D}})_m \cong \begin{cases} 0 & \text{if} \quad m \neq -n-1 \\ \mathrm{Sym}^{-m}(\Omega^1 X) & \text{if} \quad m = -n-1 \end{cases} \ , \tag{5.31}$$

where Sym^q stands for the q-th symmetric power over A. It is easy to see that (5.29), (5.30) and (5.31), together with (5.28), imply both statements of the lemma. (In addition, we have $\omega(\pi\bar{\mathcal{D}})_n \cong \mathrm{Sym}^{-n}(\Omega^1 X)$ if $n < 0$.) It remains to prove (5.30) and (5.31).

First, observe that $\underline{\mathrm{Ext}}^k_{\bar{\mathcal{D}}}(\bar{\mathcal{D}}_n[-n], \bar{\mathcal{D}}) \cong \underline{\mathrm{Ext}}^k_{\bar{\mathcal{D}}}(\bar{\mathcal{D}}_n, \bar{\mathcal{D}})[n]$ for all $n \geq 0$, where $\bar{\mathcal{D}}_n$ is a graded $\bar{\mathcal{D}}$-module with a single component in degree 0. Such modules arise from the A-modules by restricting scalars via the algebra projection $\bar{\mathcal{D}} \to A$. So we can compute $\underline{\mathrm{Ext}}^k_{\bar{\mathcal{D}}}(\bar{\mathcal{D}}_n, \bar{\mathcal{D}})$ using the spectral sequence

$$\mathrm{Ext}^p_A(\bar{\mathcal{D}}_n, \underline{\mathrm{Ext}}^q_{\bar{\mathcal{D}}}(A, \bar{\mathcal{D}})) \underset{p}{\Rightarrow} \underline{\mathrm{Ext}}^{p+q}_{\bar{\mathcal{D}}}(\bar{\mathcal{D}}_n, \bar{\mathcal{D}}) . \tag{5.32}$$

To find $\underline{\mathrm{Ext}}^q_{\bar{\mathcal{D}}}(A, \bar{\mathcal{D}})$ we identify $\bar{\mathcal{D}}$ with the symmetric algebra $\mathrm{Sym}_A(\mathrm{Der}\, A)$ and use the canonical resolution of A as a $\bar{\mathcal{D}}$-module

$$0 \to \bar{\mathcal{D}} \otimes_A \mathrm{Der}(A)[-1] \xrightarrow{m} \bar{\mathcal{D}} \to A \to 0 . \tag{5.33}$$

Here the differential m is defined by the multiplication map of $\bar{\mathcal{D}}$. It follows from (5.33) that $\underline{\mathrm{Ext}}^q_{\bar{\mathcal{D}}}(A, \bar{\mathcal{D}}) = 0$ for $q \neq 1$, so the spectral sequence (5.32) collapses on the line $q = 1$, giving isomorphisms

$$\underline{\mathrm{Ext}}^1_{\bar{\mathcal{D}}}(\bar{\mathcal{D}}_n, \bar{\mathcal{D}}) \cong \mathrm{Hom}_A(\bar{\mathcal{D}}_n, \underline{\mathrm{Ext}}^1_{\bar{\mathcal{D}}}(A, \bar{\mathcal{D}})) , \tag{5.34}$$

$$\underline{\mathrm{Ext}}^2_{\bar{\mathcal{D}}}(\bar{\mathcal{D}}_n, \bar{\mathcal{D}}) \cong \mathrm{Ext}^1_A(\bar{\mathcal{D}}_n, \underline{\mathrm{Ext}}^1_{\bar{\mathcal{D}}}(A, \bar{\mathcal{D}})) = 0 , \tag{5.35}$$

$$\underline{\mathrm{Ext}}^k_{\bar{\mathcal{D}}}(\bar{\mathcal{D}}_n, \bar{\mathcal{D}}) = 0 \quad \text{for } k \neq 1, 2 . \tag{5.36}$$

Note that the Ext^1_A in (5.35) vanishes, since $\bar{\mathcal{D}}_n$ is a projective A-module. Combined together, (5.35) and (5.36) imply (5.30).

On the other hand, dualizing (5.33), we get

$$\begin{aligned} \underline{\mathrm{Ext}}^1_{\bar{\mathcal{D}}}(A, \bar{\mathcal{D}}) &\cong \mathrm{Coker}\,[\,\bar{\mathcal{D}} \to \underline{\mathrm{Hom}}_{\bar{\mathcal{D}}}(\bar{\mathcal{D}} \otimes_A \mathrm{Der}(A)[-1], \bar{\mathcal{D}})\,] \\ &\cong \mathrm{Coker}\,[\,\bar{\mathcal{D}} \to (\bar{\mathcal{D}} \otimes_A \Omega^1 X)[1]\,] \\ &\cong (\Omega^1 X)[1] . \end{aligned}$$

Hence, it follows from (5.34) that

$$\underline{\mathrm{Ext}}^1_{\bar{\mathcal{D}}}(\bar{\mathcal{D}}_n[-n], \bar{\mathcal{D}}) \cong \mathrm{Hom}_A(\bar{\mathcal{D}}_n, \Omega^1 X)[n+1] \cong \mathrm{Sym}^{n+1}(\Omega^1 X)[n+1] , \tag{5.37}$$

which is equivalent to (5.31). This finishes the proof of the lemma. □

Lemma 5.7. *If $\mathcal{I}$ is a flat A-module, then*

$$R^k\omega\,\pi(\widetilde{M} \otimes_A \mathcal{I}) \cong R^k\omega\,(\pi\widetilde{M}) \otimes_A \mathcal{I} , \quad \forall\, k \geq 0 ,$$

for any graded $\widetilde{\mathcal{D}}$-A-bimodule $\widetilde{M}$.

Proof. By [AZ], Prop. 7.2(1), we have

$$R^k\omega\,\pi(\widetilde{M} \otimes_A \mathcal{I}) \cong \varinjlim\ \underline{\mathrm{Ext}}^k_{\widetilde{\mathcal{D}}}(\widetilde{\mathcal{D}}_{\geq n}, \widetilde{M} \otimes_A \mathcal{I}) .$$

Since $\varinjlim$ commutes with tensor products, it suffices to prove that

$$\underline{\mathrm{Ext}}^k_{\widetilde{\mathcal{D}}}(\widetilde{\mathcal{D}}_{\geq n}, \widetilde{M} \otimes_A \mathcal{I}) \cong \underline{\mathrm{Ext}}^k_{\widetilde{\mathcal{D}}}(\widetilde{\mathcal{D}}_{\geq n}, \widetilde{M}) \otimes_A \mathcal{I} \quad \text{for } n \gg 0 . \tag{5.38}$$

Furthermore, by functoriality, it suffices to prove (5.38) only for $k = 0$. Now, in this last case, we have the natural map

$$\underline{\mathrm{Hom}}_{\widetilde{\mathcal{D}}}(\widetilde{\mathcal{D}}_{\geq n}, \widetilde{M}) \otimes_A \mathcal{I} \ \to\ \underline{\mathrm{Hom}}_{\widetilde{\mathcal{D}}}(\widetilde{\mathcal{D}}_{\geq n}, \widetilde{M} \otimes_A \mathcal{I}) , \quad f \otimes b \mapsto [x \mapsto f(x) \otimes b] ,$$

which is known to be an isomorphism if $\widetilde{\mathcal{D}}_{\geq n}$ is a finitely presented $\widetilde{\mathcal{D}}$-module (see [Ro], Lemma 3.83). Since $\widetilde{\mathcal{D}}_{\geq n}$ is an ideal in $\widetilde{\mathcal{D}}$, and $\widetilde{\mathcal{D}}$ is Noetherian, the result follows. □

Now, we turn to our problem. As in Section 3.2, we choose a good filtration $\{M_k\}$ on M so that $\overline{M} := \bigoplus_{k\in\mathbb{Z}} M_k/M_{k-1}$ is a torsion-free $\bar{\mathcal{D}}$-module. Then, by Theorem 3.1, there is an ideal $\mathcal{I} \subseteq A$ (unique up to isomorphism) and a graded embedding

$$\overline{f} : \overline{M} \hookrightarrow \bar{\mathcal{D}}\mathcal{I} , \tag{5.39}$$

such that $\dim_{\mathbb{C}} \mathrm{Coker}(\overline{f}) < \infty$. The filtration $\{M_k\}$ is uniquely determined by M up to a shift of degree (cf. Lemma 5.12 below); we fix this shift by requiring the map $\overline{f}$ to be of degree 0. The dimension $n := \dim_{\mathbb{C}} \mathrm{Coker}(\overline{f})$ is then an invariant of M, independent of the choice of filtration.

Since $\eta : \mathrm{Id} \to \omega\pi$ is a natural transformation of functors, the map (5.39) fits into the commutative diagram

$$\begin{array}{ccc} \overline{M} & \xrightarrow{\overline{f}} & \bar{\mathcal{D}}\mathcal{I} \\ {\scriptstyle \eta_{\overline{M}}}\downarrow & & \downarrow{\scriptstyle \eta_{\bar{\mathcal{D}}\mathcal{I}}} \\ \omega\pi(\overline{M}) & \xrightarrow{\omega\pi(\overline{f})} & \omega\pi(\bar{\mathcal{D}}\mathcal{I}) \end{array} \tag{5.40}$$

As $\mathrm{Ker}(\overline{f}) = 0$ and $\mathrm{Coker}(\overline{f}) \in \mathsf{Tors}(\widetilde{\mathcal{D}})$, $\pi(\overline{f})$ and, hence, $\omega\pi(\overline{f})$ are isomorphisms. On the other hand, by Lemma 5.7, the map $\eta_{\bar{\mathcal{D}}\mathcal{I}}$ can be factored as

$$\bar{\mathcal{D}}\mathcal{I} \cong \bar{\mathcal{D}} \otimes_A \mathcal{I} \xrightarrow{\eta_{\bar{\mathcal{D}}}\otimes 1} \omega\pi(\bar{\mathcal{D}}) \otimes_A \mathcal{I} \cong \omega\pi(\bar{\mathcal{D}}\mathcal{I})$$

and hence, by Lemma 5.6(1), $\eta_{\bar{\mathcal{D}}\mathcal{I}} : \bar{\mathcal{D}}\mathcal{I} \xrightarrow{\sim} \omega\pi(\bar{\mathcal{D}}\mathcal{I})_{\geq 0}$ is an isomorphism. Using these two isomorphisms, we identify

$$\omega\pi(\overline{M})_{\geq 0} \cong \bar{\mathcal{D}}\mathcal{I} . \tag{5.41}$$

It follows then from (5.26) and the commutativity of (5.40) that $\tau(\overline{M}) = 0$ and

$$R^1\tau(\overline{M})_{\geq 0} \cong \mathrm{Coker}(\overline{f}) \cong \bar{\mathcal{D}}\mathcal{I}/\overline{M} .$$

In particular, we have

$$\dim_{\mathbb{C}} R^1\tau(\overline{M})_{\geq 0} = n . \tag{5.42}$$

Next, we set $\widetilde{N} := \bigoplus_{k\in\mathbb{Z}} M/M_k$ and make $\widetilde{N}$ a graded $\widetilde{\mathcal{D}}$-module in the natural way, with $t \in \widetilde{\mathcal{D}}$ acting by the canonical projections $M/M_k \twoheadrightarrow M/M_{k+1}$.

Proposition 5.2. *The module $\widetilde{N}$ has the following properties:*
(1) $\tau(\widetilde{N}) = 0$,
(2) $\dim_{\mathbb{C}} R^1\tau(\widetilde{N})_{-1} = n$, *and* $\dim_{\mathbb{C}} R^1\tau(\widetilde{N})_{\geq -1} < \infty$,
(3) *The maps* $\omega\,\pi(\widetilde{N})_{k-1} \xrightarrow{t} \omega\,\pi(\widetilde{N})_k$ *are surjective for* $k \geq 0$.

Proof. (1) Given $\widetilde{M} \in \mathsf{GrMod}(\widetilde{\mathcal{D}})$, we write $i^!(\widetilde{M})$ for the largest submodule of $\widetilde{M}$ annihilated by the action of t, i. e. $i^!(\widetilde{M}) = \mathrm{Ker}(\widetilde{M} \xrightarrow{t\,\cdot} \widetilde{M}[1])$. Then, if $\widetilde{M} \in \mathsf{Tors}(\widetilde{\mathcal{D}})$ and $\widetilde{M} \neq 0$, we have $i^!(\widetilde{M}) \neq 0$. (Indeed, if $\widetilde{M} \neq 0$, the submodule $\widetilde{\mathcal{D}}m$ generated by a nonzero element $m \in \widetilde{M}$ is bounded, so its top graded component is annihilated by t.) Thus the assumption that $\tau(\widetilde{N}) \neq 0$ implies $i^!(\tau\widetilde{N}) \neq 0$.

On the other hand, $i^!(\widetilde{N}) \cong \overline{M}[1]$ and $\tau(\overline{M}[1]) = \tau(\overline{M})[1] = 0$, so $\tau(i^!\widetilde{N}) = 0$. Since $i^!(\tau\widetilde{N}) = i^!(\widetilde{N}) \cap \tau(\widetilde{N}) = \tau(i^!\widetilde{N})$, we arrive at contradiction. It follows that $\tau(\widetilde{N}) = 0$.

(2) For all $k \in \mathbb{Z}$, we have the exact sequences $0 \to M_k/M_{k-1} \to M/M_{k-1} \to M/M_k \to 0$ defined by the filtration inclusions. Combining these together, we get the exact sequence of graded $\widetilde{\mathcal{D}}$-modules

$$0 \to \overline{M} \to \widetilde{N}[-1] \xrightarrow{t} \widetilde{N} \to 0 \ . \tag{5.43}$$

Since $\tau(\widetilde{N}) = 0$, applying the torsion functor τ to (5.43) yields

$$0 \to R^1\tau(\overline{M}) \to R^1\tau(\widetilde{N}[-1]) \to R^1\tau(\widetilde{N}) \to R^2\tau(\overline{M}) \to \ \dots \tag{5.44}$$

By Theorem 5.1(2) and Lemma 5.7, the last term of (5.44) can be identified as

$$R^2\tau(\overline{M}) \cong R^1\omega(\pi\overline{M}) \cong R^1\omega[\pi(\bar{\mathcal{D}}\mathcal{I})] \cong R^1\omega(\pi\bar{\mathcal{D}}) \otimes_A \mathcal{I} \ ,$$

so $R^2\tau(\overline{M}) = 0$ by Lemma 5.6(2). We get thus the exact sequence

$$0 \to R^1\tau(\overline{M}) \to R^1\tau(\widetilde{N}[-1]) \xrightarrow{t} R^1\tau(\widetilde{N}) \to 0 \ . \tag{5.45}$$

Now, (5.42) implies that $R^1\tau(\overline{M})_{\geq 0}$ is bounded: i. e. there is an integer $d \geq 0$, such that $R^1\tau(\overline{M})_d \neq 0$, while $R^1\tau(\overline{M})_k = 0$ for all $k > d$. It follows then from (5.45) that t acts as a unit on $R^1\tau(\widetilde{N})_{\geq d}$: in particular, we have $i^!(R^1\tau(\widetilde{N})_{\geq d}) = 0$. But $R^1\tau(\widetilde{N})_{\geq d}$ is a submodule of $R^1\tau(\widetilde{N})$, which, by definition, is torsion. Hence, $i^!(R^1\tau(\widetilde{N})_{\geq d}) = 0$ forces $R^1\tau(\widetilde{N})_{\geq d} = 0$. Now, in degree d, the exact sequence (5.45) reads

$$0 \to R^1\tau(\overline{M})_d \to R^1\tau(\widetilde{N})_{d-1} \to R^1\tau(\widetilde{N})_d = 0 \ .$$

In other words, $R^1\tau(\overline{M})_d \xrightarrow{\sim} R^1\tau(\widetilde{N})_{d-1}$ is an isomorphism. Hence

$$\dim_{\mathbb{C}} R^1\tau(\widetilde{N})_{d-1} = \dim_{\mathbb{C}} R^1\tau(\overline{M})_d < \infty \ .$$

Similarly, if $d \geq 1$, then, for $k = d-1$, we have the exact sequence

$$0 \to R^1\tau(\overline{M})_{d-1} \to R^1\tau(\widetilde{N})_{d-2} \to R^1\tau(\widetilde{N})_{d-1} \to 0 \ ,$$

with first and last terms being finite-dimensional. It follows that

$$\dim_{\mathbb{C}} R^1\tau(\widetilde{N})_{d-2} = \dim_{\mathbb{C}} R^1\tau(\overline{M})_{d-1} + \dim_{\mathbb{C}} R^1\tau(\overline{M})_d < \infty \ .$$

Continuing this way, we get

$$\dim_{\mathbb{C}} R^1\tau(\widetilde{N})_{k-1} = \sum_{j=k}^{d} \dim_{\mathbb{C}} R^1\tau(\overline{M})_j \ , \quad k = 0, 1, \dots, d \ .$$

In particular, $\dim_{\mathbb{C}} R^1\tau(\widetilde{N})_{-1} = \dim_{\mathbb{C}} R^1\tau(\overline{M})_{\geq 0} = n$ by (5.42), and

$$\dim_{\mathbb{C}} R^1\tau(\widetilde{N})_{\geq -1} = \sum_{k=0}^{d-1} \dim_{\mathbb{C}} R^1\tau(\widetilde{N})_{k-1} < \infty \ .$$

(3) Applying $\omega\pi$ to (5.43) gives rise to the exact sequence

$$0 \to \omega\pi(\overline{M}) \to \omega\pi(\widetilde{N}[-1]) \xrightarrow{t} \omega\pi(\widetilde{N}) \to R^1\omega(\pi\overline{M}) \to \ \dots \tag{5.46}$$

Since $\pi(\overline{M}) \cong \pi(\bar{\mathcal{D}}\mathcal{I})$, we have $\omega\pi(\overline{M})_{\geq 0} \cong \omega\pi(\bar{\mathcal{D}}\mathcal{I})_{\geq 0} \cong \bar{\mathcal{D}}\mathcal{I}$, see (5.41), and $R^1\omega(\pi\overline{M}) \cong R^1\omega(\pi\bar{\mathcal{D}}\mathcal{I}) \cong R^1\omega(\pi\bar{\mathcal{D}})\otimes_A\mathcal{I} = 0$ by Lemma 5.6(2). Hence, truncating (5.46) at negative degrees, we get the exact sequence

$$0 \to \bar{\mathcal{D}}\mathcal{I} \to \omega\pi(\widetilde{N}[-1])_{\geq 0} \xrightarrow{t} \omega\pi(\widetilde{N})_{\geq 0} \to 0 \ . \tag{5.47}$$

The last statement of the proposition follows. □

Next, we consider the functorial exact sequence (5.26), with $\widetilde{M} = \widetilde{N}$. By Proposition 5.2(1), the first term of this sequence is zero, so we have

$$0 \to \widetilde{N} \xrightarrow{\eta_{\widetilde{N}}} \omega\pi(\widetilde{N}) \to R^1\tau(\widetilde{N}) \to 0 \ , \tag{5.48}$$

Since the canonical filtration on M is positive, $\widetilde{N}_k = M$ for all $k < 0$. Thus, setting $L := \omega\pi(\widetilde{N})_{-1}$ and $V := R^1\tau(\widetilde{N})_{-1}$, we get from (5.48) the exact sequence of A-modules

$$0 \to M \xrightarrow{\eta} L \to V \to 0 \ . \tag{5.49}$$

Now, replacing A by its one-point extension $B = A[\mathcal{I}]$, we lift (5.49) to an exact sequence of B-modules, as follows. First, we regard M as a B-module by restricting scalars via the algebra homomorphism $\theta : B \to A$, see (2.15). Next, we set $\boldsymbol{L} := L \oplus \mathbb{C}$ and make $\boldsymbol{L}$ a B-module by defining its structure map $\varphi : \mathcal{I}\otimes\mathbb{C} \cong \mathcal{I} \to L$ to be the degree 0 component of the canonical embedding $\bar{\mathcal{D}}\mathcal{I} \hookrightarrow \omega\pi(\widetilde{N}[-1])_{\geq 0}$ in (5.47). Every A-module homomorphism $M \to L$ extends then to a unique B-module homomorphism $M \to \boldsymbol{L}$, since $\mathrm{Hom}_A(M, L) \cong \mathrm{Hom}_B(M, \boldsymbol{L})$ via $f \mapsto (f, 0)$. In particular, the map η in (5.49) extends to an embedding $\boldsymbol{\eta} : M \hookrightarrow \boldsymbol{L}$, and we write $\boldsymbol{V} := \boldsymbol{L}/M$ for the cokernel of $\boldsymbol{\eta}$. Clearly, $\boldsymbol{V} \cong V \oplus \mathbb{C}$ as a vector space, and $\underline{\dim}(\boldsymbol{V}) = (n, 1)$, by Proposition 5.2(2). Summing up, we have constructed an exact sequence of B-modules

$$0 \to M \xrightarrow{\boldsymbol{\eta}} \boldsymbol{L} \to \boldsymbol{V} \to 0 \ , \tag{5.50}$$

with the quotient term being of dimension $(n, 1)$.

Now, recall that, by Lemma 4.1, θ extends to an algebra map $\boldsymbol{\theta} : \Pi^{\boldsymbol{\lambda}}(B) \to \Pi^1(A) \cong \mathcal{D}$, where $\boldsymbol{\lambda} = (1, -n)$. We will regard the $\mathcal{D}$-module M as a $\Pi^{\boldsymbol{\lambda}}(B)$-module by restricting scalars via $\boldsymbol{\theta}$.

Proposition 5.3. *The B-module structure on $\boldsymbol{L}$ defined above admits a* unique *extension to $\Pi^{\boldsymbol{\lambda}}(B)$, making $\boldsymbol{\eta} : M \to \boldsymbol{L}$ a homomorphism of $\Pi^{\boldsymbol{\lambda}}(B)$-modules.*

We will give a homological proof of this proposition, using Theorem 2.2 of Section 2.1. As explained in (the proof of) Theorem 2.2, a $\Pi^{\boldsymbol{\lambda}}(B)$-module structure on a B-module $\boldsymbol{M}$ is determined by an element of $\mathrm{End}(\boldsymbol{M})\otimes_{B^{\mathrm{e}}}\Omega^1 B$, lying in the fibre of $\boldsymbol{\lambda}\,(= \boldsymbol{\lambda}\cdot\mathrm{Id})$ under the evaluation map

$$\partial_{\boldsymbol{M}} : \mathrm{End}(\boldsymbol{M})\otimes_{B^{\mathrm{e}}}\Omega^1(B) \to \mathrm{End}(\boldsymbol{M}) \ , \qquad f\otimes d \mapsto f\Delta_B(d) \ . \tag{5.51}$$

In particular, the given $\Pi^{\boldsymbol{\lambda}}(B)$-module structure on M is determined by an element $\delta_M \in \mathrm{End}(M)\otimes_{B^{\mathrm{e}}}\Omega^1(B)$, such that $\partial_M(\delta_M) = \mathrm{Id}_M$. The B-module embedding $\boldsymbol{\eta}$ induces an embedding of B-bimodules: $\mathrm{End}(M) \hookrightarrow \mathrm{Hom}(M, \boldsymbol{L})$, and hence the natural map

$$\mathrm{End}(M)\otimes_{B^{\mathrm{e}}}\Omega^1(B) \hookrightarrow \mathrm{Hom}(M, \boldsymbol{L})\otimes_{B^{\mathrm{e}}}\Omega^1(B) \ . \tag{5.52}$$

Since $\Omega^1(B)$ is a projective bimodule, this last map is also an embedding, and we identify $\mathrm{End}(M)\otimes_{B^e}\Omega^1(B)$ with its image in $\mathrm{Hom}(M,\,\boldsymbol{L})\otimes_{B^e}\Omega^1(B)$ under (5.52).

Now, consider the commutative diagram

$$\begin{array}{ccc} \mathrm{End}(\boldsymbol{L})\otimes_{B^e}\Omega^1(B) & \xrightarrow{\tilde{\boldsymbol{\eta}}_*} & \mathrm{Hom}(M,\boldsymbol{L})\otimes_{B^e}\Omega^1(B) \\ \partial_{\boldsymbol{L}}\downarrow & & \partial_{M,\boldsymbol{L}}\downarrow \\ \mathrm{End}(\boldsymbol{L}) & \xrightarrow{\boldsymbol{\eta}_*} & \mathrm{Hom}(M,\boldsymbol{L}) \end{array} \tag{5.53}$$

where $\partial_{M,\boldsymbol{L}}$ is the evaluation map at Δ_B, $\boldsymbol{\eta}_*$ is the restriction (via $\boldsymbol{\eta}$), and $\tilde{\boldsymbol{\eta}}_*:=\boldsymbol{\eta}_*\otimes\mathrm{Id}$. Note that $\boldsymbol{\eta}_*$ and $\tilde{\boldsymbol{\eta}}_*$ are both surjective. With above identification, we have

$$\boldsymbol{\eta}_*(\boldsymbol{\lambda})=\partial_{M,\boldsymbol{L}}(\delta_M)=\boldsymbol{\eta}\ , \tag{5.54}$$

and our problem is to show that there is a unique element $\delta_{\boldsymbol{L}}\in\mathrm{End}(\boldsymbol{L})\otimes_{B^e}\Omega^1(B)$, such that

$$\partial_{\boldsymbol{L}}(\delta_{\boldsymbol{L}})=\boldsymbol{\lambda}\quad\text{and}\quad\tilde{\boldsymbol{\eta}}_*(\delta_{\boldsymbol{L}})=\delta_M\ . \tag{5.55}$$

To solve this problem homologically, we interpret the top and the bottom rows of (5.53) as 2-complexes of vector spaces $X^\bullet$ and $Y^\bullet$, with nonzero terms in degrees 0 and 1 and differentials $\tilde{\boldsymbol{\eta}}_*$ and $\boldsymbol{\eta}_*$, respectively. The pair of maps $(\partial_{\boldsymbol{L}},\,\partial_{M,\boldsymbol{L}})$ yields then a morphism of complexes $\partial^\bullet: X^\bullet\to Y^\bullet$ with mapping cone

$$C^\bullet(\partial):=\left[0\to X^0\xrightarrow{d^{-1}}X^1\oplus Y^0\xrightarrow{d^0}Y^1\to 0\right]\ . \tag{5.56}$$

By definition, the differentials in $C^\bullet(\partial)$ are given by $d^{-1}=(-\tilde{\boldsymbol{\eta}}_*,\,\partial_{\boldsymbol{L}})^T$ and $d^0=(\partial_{M,\boldsymbol{L}},\,\boldsymbol{\eta}_*)$. So (5.54) can be interpreted by saying that $(-\delta_M,\boldsymbol{\lambda})\in X^1\oplus Y^0$ is a 0-cocycle in $C^\bullet(\partial)$. Then, the cohomology class

$$c(\boldsymbol{\lambda},\delta_M):=[(-\delta_M,\boldsymbol{\lambda})] \tag{5.57}$$

represented by this cocycle, vanishes in $h^0(C^\bullet)$ if and only if there is $\delta_{\boldsymbol{L}}\in X^0$ such that $d^{-1}(\delta_{\boldsymbol{L}})=(-\delta_M,\boldsymbol{\lambda})$, i. e. (5.55) holds. Clearly, if it exists, such $\delta_{\boldsymbol{L}}$ is unique if and only if d^{-1} is injective, i. e. if and only if $h^{-1}(C^\bullet)=0$.

Now, the canonical exact sequence of complexes $0\to Y^\bullet\to C^\bullet\to X^\bullet[1]\to 0$ associated to the mapping cone yields the long cohomology sequence

$$0\to h^{-1}(C^\bullet)\to\mathrm{Ker}(\tilde{\boldsymbol{\eta}}_*)\to\mathrm{Ker}(\boldsymbol{\eta}_*)\to h^0(C^\bullet)\to 0\ . \tag{5.58}$$

Dualizing (5.50) by $\boldsymbol{L}$ and tensoring with $\Omega^1(B)$, we identify the two terms in the middle of (5.58) as

$$\mathrm{Ker}(\boldsymbol{\eta}_*)\cong\mathrm{Hom}(\boldsymbol{V},\boldsymbol{L})\quad\text{and}\quad\mathrm{Ker}(\tilde{\boldsymbol{\eta}}_*)\cong\mathrm{Hom}(\boldsymbol{V},\boldsymbol{L})\otimes_{B^e}\Omega^1(B)\ .$$

The map between these terms then becomes

$$\partial_{\boldsymbol{V},\boldsymbol{L}}:\mathrm{Hom}(\boldsymbol{V},\boldsymbol{L})\otimes_{B^e}\Omega^1(B)\to\mathrm{Hom}(\boldsymbol{V},\boldsymbol{L})\ ,\qquad f\otimes d\mapsto f\Delta_B(d)\ . \tag{5.59}$$

A simple calculation (see the proof of Theorem 2.2) shows that $\mathrm{Ker}(\partial_{\boldsymbol{V},\boldsymbol{L}})$ and $\mathrm{Coker}(\partial_{\boldsymbol{V},\boldsymbol{L}})$ are naturally isomorphic to the 1-st and 0-th Hochschild homology of the B-bimodule $\mathrm{Hom}(\boldsymbol{V},\boldsymbol{L})$. Hence, by exactness of (5.58), we have

$$h^0(C^\bullet)\cong\mathrm{H}_0(B,\,\mathrm{Hom}(\boldsymbol{V},\boldsymbol{L}))\quad\text{and}\quad h^{-1}(C^\bullet)\cong\mathrm{H}_1(B,\,\mathrm{Hom}(\boldsymbol{V},\boldsymbol{L}))\ .$$

Proposition 5.3 thus boils down to proving Lemma 5.8 and Lemma 5.9 below.

Lemma 5.8. $\mathrm{H}_1(B,\, \mathrm{Hom}(\boldsymbol{V}, \boldsymbol{L})) = 0$.

Proof. Recall that $L := \omega\,\pi(\widetilde{N})_{-1}$. Now, in addition, we set $L_0 := \omega\,\pi(\widetilde{N})_0$ and make this a B-module by restricting scalars via $\theta : B \to A$. Then, the A-module homomorphism $t_L : L \to L_0$ induced by the action of t extends to a unique B-module homomorphism $\boldsymbol{L} \to L_0$, which we denote by $\boldsymbol{t}$. (Indeed, the commutativity of the diagram (2.13) for $\boldsymbol{L} \to L_0$ amounts to $t_L \circ \varphi = 0$, which is true in view of our definition of φ.) By Proposition 5.2(3), the map t_L is surjective, and hence so is $\boldsymbol{t}$. It is easy to see that $\mathrm{Ker}(\boldsymbol{t}) \cong Be_\infty$ as a B-module, so we have a short exact sequence

$$0 \to Be_\infty \to \boldsymbol{L} \xrightarrow{\boldsymbol{t}} L_0 \to 0 \ . \tag{5.60}$$

Since Be_∞ is projective, tensoring (5.60) with $\boldsymbol{V}^* := \mathrm{Hom}(\boldsymbol{V},\, \mathbb{C})$ yields

$$0 \to \mathrm{Tor}_1^B(\boldsymbol{V}^*,\, \boldsymbol{L}) \to \mathrm{Tor}_1^B(\boldsymbol{V}^*,\, L_0) \to \boldsymbol{V}^* \otimes_B Be_\infty \to \boldsymbol{V}^* \otimes_B \boldsymbol{L} \to \boldsymbol{V}^* \otimes_B L_0 \to 0 \ .$$

On the other hand, if $\boldsymbol{V}$ is finite-dimensional, we have natural isomorphisms

$$\mathrm{H}_n(B,\, \mathrm{Hom}(\boldsymbol{V}, \boldsymbol{M})) \cong \mathrm{H}_n(B, \boldsymbol{M} \otimes \boldsymbol{V}^*) \cong \mathrm{Tor}_n^{B^{\mathrm{e}}}(B, \boldsymbol{M} \otimes \boldsymbol{V}^*) \cong \mathrm{Tor}_n^B(\boldsymbol{V}^*,\, \boldsymbol{M}) \ ,$$

for all $n \geq 0$ and an arbitrary B-module $\boldsymbol{M}$ (see [CE], Cor. 4.4, p.170). The above exact sequence can thus be identified with

$$\begin{aligned} &0 \to \mathrm{H}_1(B,\, \mathrm{Hom}(\boldsymbol{V}, \boldsymbol{L})) \to \mathrm{H}_1(B,\, \mathrm{Hom}(\boldsymbol{V}, L_0)) \to \\ &\quad \mathrm{H}_0(B,\, \mathrm{Hom}(\boldsymbol{V}, Be_\infty)) \to \mathrm{H}_0(B,\, \mathrm{Hom}(\boldsymbol{V}, \boldsymbol{L})) \to \mathrm{H}_0(B,\, \mathrm{Hom}(\boldsymbol{V}, L_0)) \to 0 \ . \end{aligned} \tag{5.61}$$

Hence, to prove the lemma it suffices to show that $\mathrm{H}_1(B,\, \mathrm{Hom}(\boldsymbol{V}, L_0)) = 0$, which is equivalent to

$$\mathrm{Tor}_1^B(\boldsymbol{V}^*,\, L_0) = 0 \ . \tag{5.62}$$

Since L_0 is an A-module, we can compute this last Tor, using the spectral sequence

$$\mathrm{Tor}_p^A(\mathrm{Tor}_q^B(\boldsymbol{V}^*,\, A),\, L_0) \underset{p}{\Rightarrow} \mathrm{Tor}_{p+q}^B(\boldsymbol{V}^*,\, L_0) \tag{5.63}$$

associated to the algebra map $\theta : B \to A$. By Lemma 2.3(4), this map is flat, so the spectral sequence (5.63) collapses at $q = 0$, giving an isomorphism

$$\mathrm{Tor}_1^B(\boldsymbol{V}^*,\, L_0) \cong \mathrm{Tor}_1^A(V^*,\, L_0) \ . \tag{5.64}$$

Now, for each $k \geq 0$, we set $L_k := \omega\pi(\widetilde{N})_k$ and write F_k for the kernel of the map $L_0 \xrightarrow{t^k} L_k$ induced by the action of $t^k \in \widetilde{\mathcal{D}}$ on $\omega\pi(\widetilde{N})$. By Proposition 5.2(3), the maps t^k are surjective for all $k \geq 0$, and thus $0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq \ldots$ is an A-module filtration on L_0. We claim that this filtration is exhaustive, so that

$$\varinjlim F_k \cong \bigcup_{k=0}^{\infty} F_k = L_0 \ . \tag{5.65}$$

Indeed, by Proposition 5.2, the kernel of the map $\eta : \widetilde{N}_{\geq 0} \to \omega\pi(\widetilde{N})_{\geq 0}$ is zero, while its cokernel $R^1\tau(\widetilde{N})_{\geq 0}$ is bounded. Hence, for any $a \in L_0$, we can find $k \in \mathbb{N}$, such that $t^k a$ is contained in the image of η , i. e. $t^k a = \eta(\bar{m})$ for some $\bar{m} \in M/M_k$. Now, the filtration on M is exhaustive, hence there exists $n \geq k$, such that $\bar{m}$ is represented (mod M_k) by some element $m \in M_n$. It follows that $t^{n-k}\bar{m} = 0$ in $\widetilde{N}$, and therefore $t^n a = t^{n-k}\eta(\bar{m}) = \eta(t^{n-k}\bar{m}) = 0$. This means that $a \in F_n$, so $\{F_k\}$ is exhaustive.

Now, by (5.47), for all $k \geq 0$, we have exact sequences

$$0 \to F_k \to F_{k+1} \to \bar{\mathcal{D}}_{k+1}\mathcal{I} \to 0 \ . \tag{5.66}$$

Since $\bar{\mathcal{D}}_{k+1}\mathcal{I}$ are projective A-modules for all $k \geq 0$, it follows, by induction, that F_k are also projective (and hence, flat) A-modules for $k \geq 1$. The direct limits of families of flat modules are flat, so by (5.65), L_0 is a flat A-module. Thus $\mathrm{Tor}_1^A(V^*, L_0) = 0$, and (5.62) follows from (5.64), finishing the proof of the lemma. □

Lemma 5.9. *$c(\boldsymbol{\lambda}, \delta_M) = 0$ in $\mathrm{H}_0(B, \mathrm{Hom}(\boldsymbol{V}, \boldsymbol{L}))$.*

Proof. By (5.61) and (5.62), we have the exact sequence

$$0 \to \mathrm{H}_0(B, \mathrm{Hom}(\boldsymbol{V}, Be_\infty)) \xrightarrow{\boldsymbol{\iota}_*} \mathrm{H}_0(B, \mathrm{Hom}(\boldsymbol{V}, \boldsymbol{L})) \xrightarrow{\boldsymbol{t}_*} \mathrm{H}_0(B, \mathrm{Hom}(\boldsymbol{V}, L_0)) \to 0, \tag{5.67}$$

where $\boldsymbol{\iota}_*$ is induced by the inclusion $\boldsymbol{\iota} : Be_\infty \hookrightarrow \boldsymbol{L}$ and $\boldsymbol{t}_*$ by the projection $\boldsymbol{t} : \boldsymbol{L} \twoheadrightarrow L_0$ in (5.60).

We show first that $\boldsymbol{t}_*(c(\boldsymbol{\lambda}, \delta_M)) = 0$. For this, we consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(\boldsymbol{L}, L_0) \otimes_{B^{\mathrm{e}}} \Omega^1(B) & \xrightarrow{\tilde{\boldsymbol{\eta}}_*} & \mathrm{Hom}(M, L_0) \otimes_{B^{\mathrm{e}}} \Omega^1(B) \\ \downarrow{\scriptstyle \partial_{\boldsymbol{L}, L_0}} & & \downarrow{\scriptstyle \partial_{M, L_0}} \\ \mathrm{Hom}(\boldsymbol{L}, L_0) & \xrightarrow{\boldsymbol{\eta}_*} & \mathrm{Hom}(M, L_0) \end{array} \tag{5.68}$$

which is the image of (5.53) under the natural projection by $\boldsymbol{t}$. Under this projection, the equations (5.54) become

$$\boldsymbol{\eta}_*(\boldsymbol{t}) = \partial_{M, L_0}(\tilde{\boldsymbol{t}}_*(\delta_M)) = \boldsymbol{t} \circ \boldsymbol{\eta} \ , \tag{5.69}$$

where $\tilde{\boldsymbol{t}}_* : \mathrm{Hom}(M, \boldsymbol{L}) \otimes_{B^{\mathrm{e}}} \Omega^1(B) \to \mathrm{Hom}(M, L_0) \otimes_{B^{\mathrm{e}}} \Omega^1(B)$ is defined by $f \otimes \omega \mapsto (\boldsymbol{t} \circ f) \otimes \omega$. Now, $\boldsymbol{t}$ induces a morphism of mapping cones associated to the diagrams (5.53) and (5.68), which, in turn, induces a map $\boldsymbol{t}_*$ on their cohomology. The class $\boldsymbol{t}_*(c(\boldsymbol{\lambda}, \delta_M)) \in \mathrm{H}_0(B, \mathrm{Hom}(\boldsymbol{V}, L_0))$ can thus be interpreted as the obstruction for the existence of an element $\delta_{\boldsymbol{L}, L_0} \in \mathrm{Hom}(\boldsymbol{L}, L_0) \otimes_{B^{\mathrm{e}}} \Omega^1(B)$, satisfying

$$\partial_{\boldsymbol{L}, L_0}(\delta_{\boldsymbol{L}, L_0}) = \boldsymbol{t} \quad \text{and} \quad \tilde{\boldsymbol{\eta}}_*(\delta_{\boldsymbol{L}, L_0}) = \tilde{\boldsymbol{t}}_*(\delta_M) \ . \tag{5.70}$$

We will show that $\boldsymbol{t}_*(c(\boldsymbol{\lambda}, \delta_M)) = 0$ by constructing such an element explicitly.

Recall that we have identified $\mathcal{D} \cong \Pi^1(A)$ as filtered A-rings. Combined with the canonical projection $R := T_A\mathrm{Der}(A, A^{\otimes 2}) \twoheadrightarrow \Pi^1(A)$, this identification yields a filtered homomorphism $R \to \mathcal{D}$, which, by the universal property of tensor algebras, lifts to a *graded algebra* homomorphism

$$R \to \widetilde{\mathcal{D}} \ . \tag{5.71}$$

In degree zero, (5.71) equals Id_A, while in degree one, it is the restriction of $R \to \mathcal{D}$ to $\mathrm{Der}(A, A^{\otimes 2})$, which is a surjective bimodule homomorphism onto $\mathcal{D}_1$ (see [CB], Lemma 4.6). Since $\widetilde{\mathcal{D}}$ is generated by elements of degree zero and one, (5.71) is surjective (in particular, $t \in \widetilde{\mathcal{D}}$ is the image of Δ_A).

We will regard graded $\widetilde{\mathcal{D}}$-modules as graded R-modules by restricting scalars via (5.71). Any graded R-module $\widetilde{M} = \bigoplus_{k \in \mathbb{Z}} \widetilde{M}_k$ determines (and is determined by) a family of A-bimodule homomorphisms

$$\mathrm{Der}(A, A^{\otimes 2}) \to \mathrm{Hom}(\widetilde{M}_k, \widetilde{M}_{k+1}) \ , \quad k \in \mathbb{Z} \ .$$

In particular, $\widetilde{M} = \omega\pi(\widetilde{N})$ yields

$$\mathrm{Der}(A, A^{\otimes 2}) \to \mathrm{Hom}(L,\, L_0)\ , \quad \Delta_A \mapsto t_L\ . \tag{5.72}$$

We will regard (5.72) as a homomorphism of B-bimodules by restricting the action of A (on both sides) via $\theta : B \to A$. Since θ is a pseudo-flat ring epimorphism, there is a canonical map

$$\mathrm{Der}(B, B^{\otimes 2}) \to \mathrm{Der}(A, A^{\otimes 2})\ , \quad \Delta_B \mapsto \Delta_A\ , \tag{5.73}$$

which is also a B-bimodule homomorphism (cf. [CB], the proof of Lemma 9.2). Combining (5.72) and (5.73), we get

$$\mathrm{Der}(B, B^{\otimes 2}) \to \mathrm{Hom}(L,\, L_0)\ , \quad \Delta_B \mapsto t_L\ ,$$

and hence an element

$$\delta_{L,L_0} \in \mathrm{Hom}_{B^{\mathrm{e}}}(\mathrm{Der}(B, B^{\otimes 2}),\, \mathrm{Hom}(L,\, L_0)) \cong \mathrm{Hom}(L,\, L_0) \otimes_{B^{\mathrm{e}}} \Omega^1(B)\ . \tag{5.74}$$

Now, consider the exact sequence of B-modules

$$0 \to L \xrightarrow{\boldsymbol{\alpha}} \boldsymbol{L} \to \boldsymbol{L}/L \to 0\ , \tag{5.75}$$

where L is viewed as a B-module via θ, and $\boldsymbol{\alpha}$ is the natural inclusion with cokernel $\boldsymbol{L}/L \cong \mathbb{C}$. Dualizing $\boldsymbol{\alpha}$ by L_0 and tensoring with $\Omega^1(B)$, we get the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(\boldsymbol{L}, L_0) \otimes_{B^{\mathrm{e}}} \Omega^1(B) & \xrightarrow{\tilde{\boldsymbol{\alpha}}_*} & \mathrm{Hom}(L, L_0) \otimes_{B^{\mathrm{e}}} \Omega^1(B) \\ \partial_{\boldsymbol{L},L_0} \big\downarrow & & \big\downarrow \partial_{L,L_0} \\ \mathrm{Hom}(\boldsymbol{L}, L_0) & \xrightarrow{\quad \boldsymbol{\alpha}_* \quad} & \mathrm{Hom}(L, L_0) \end{array} \tag{5.76}$$

with $\partial_{L,L_0}(\delta_{L,L_0}) = \boldsymbol{\alpha}_*(\boldsymbol{t}) = t_L$. Since $e \in B$ acts as identity on L_0 and as zero on $\boldsymbol{L}/L$, we have

$$\mathrm{H}_0(B,\, \mathrm{Hom}(\boldsymbol{L}/L, L_0)) \cong (\boldsymbol{L}/L)^* \otimes_B L_0 = 0\ .$$

Hence, there is an element $\delta_{\boldsymbol{L},L_0} \in \mathrm{Hom}(\boldsymbol{L}, L_0) \otimes_{B^{\mathrm{e}}} \Omega^1(B)$, such that $\partial_{\boldsymbol{L},L_0}(\delta_{\boldsymbol{L},L_0}) = \boldsymbol{t}$ and $\tilde{\boldsymbol{\alpha}}_*(\delta_{\boldsymbol{L},L_0}) = \delta_{L,L_0}$. We claim that this element satisfies (5.70).

In fact, since $\boldsymbol{\eta} = \boldsymbol{\alpha} \circ \eta$, we have

$$\tilde{\boldsymbol{\eta}}_*(\delta_{\boldsymbol{L},L_0}) = [\tilde{\eta}_* \circ \tilde{\boldsymbol{\alpha}}_*](\delta_{\boldsymbol{L},L_0}) = \tilde{\eta}_*(\delta_{L,L_0})\ ,$$

where $\tilde{\eta}_* : \mathrm{Hom}(L,\, L_0) \otimes_{B^{\mathrm{e}}} \Omega^1(B) \to \mathrm{Hom}(M,\, L_0) \otimes_{B^{\mathrm{e}}} \Omega^1(B)$ is induced by the inclusion $M \hookrightarrow L$. On the other hand, $\tilde{\boldsymbol{t}}_*(\delta_M) = \tilde{t}_*(\delta_M)$, where $\tilde{t}_* : \mathrm{Hom}(M,\, L) \otimes_{B^{\mathrm{e}}} \Omega^1(B) \to \mathrm{Hom}(M,\, L_0) \otimes_{B^{\mathrm{e}}} \Omega^1(B)$ is induced by the projection $L \twoheadrightarrow L_0$. Now, for any $d \in \mathrm{Der}(B, B^{\otimes 2})$, we compare

$$[\tilde{\eta}_*(\delta_{L,L_0})](d) = d_L \circ \eta = \eta \circ t_M \circ d_M = t_L \circ \eta \circ d_M = [\tilde{t}_*(\delta_M)](d)\ , \tag{5.77}$$

where $d_L \in \mathrm{Hom}(L, L_0)$ and $d_M \in \mathrm{End}(M)$ are the maps induced by d via (5.73), and $t_L \in \mathrm{Hom}(L, L_0)$, $t_M \in \mathrm{Hom}(M, M/M_0)$ are induced by the action of $t \in \widetilde{\mathcal{D}}$ on $\omega\,\pi(\widetilde{N})$ and $\widetilde{N}$, respectively. The second and the third equalities in (5.77) follow from the fact that $\widetilde{N} \hookrightarrow \omega\pi(\widetilde{N})$ is a homomorphism of graded $\widetilde{\mathcal{D}}$-modules.

Thus, the existence of $\delta_{\boldsymbol{L},L_0}$ implies that $\boldsymbol{t}_*(c(\boldsymbol{\lambda}, \delta_M)) = 0$. Returning to (5.67), we see then that $c(\boldsymbol{\lambda}, \delta_M)$ is contained in the image of $\boldsymbol{\iota}_*$, i. e. $c(\boldsymbol{\lambda}, \delta_M) = \boldsymbol{\iota}_*(\tilde{c})$ for some $\tilde{c} \in \mathrm{H}_0(B, \mathrm{Hom}(\boldsymbol{V}, Be_\infty))$. Now, to show that $\tilde{c} = 0$ we define the trace map

$$\mathrm{Tr} : \mathrm{Hom}(\boldsymbol{V},\, Be_\infty) \to \mathrm{Hom}(\boldsymbol{V}, \boldsymbol{L}) \to \mathrm{End}(\boldsymbol{V}) \to \mathbb{C}\ , \quad f \mapsto \mathrm{tr}_{\boldsymbol{V}}[\,\boldsymbol{\pi} \circ \boldsymbol{\iota} \circ f\,]\ ,$$

where $\boldsymbol{\iota} : Be_\infty \hookrightarrow \boldsymbol{L}$ is the embedding defined in (5.60), $\boldsymbol{\pi} : \boldsymbol{L} \twoheadrightarrow \boldsymbol{V}$ is the canonical projection in (5.50), and $\mathrm{tr}_{\boldsymbol{V}}$ is the usual trace on $\boldsymbol{V}$. Since $\boldsymbol{\iota}$ and $\boldsymbol{\pi}$ are homomorphisms of B-modules, this induces a linear map

$$\mathrm{Tr}_* :\ \mathrm{H}_0(B, \mathrm{Hom}(\boldsymbol{V}, Be_\infty)) \xrightarrow{\boldsymbol{\iota}_*} \mathrm{H}_0(B, \mathrm{Hom}(\boldsymbol{V}, \boldsymbol{L})) \xrightarrow{\boldsymbol{\pi}_*} \mathrm{H}_0(B, \mathrm{End}(\boldsymbol{V})) \xrightarrow{\mathrm{tr}_{\boldsymbol{V}}} \mathbb{C}\ .$$

We claim that Tr_* is an isomorphism. Indeed, it is easy to see that $\mathrm{Tr}_* \neq 0$, while

$$\mathrm{H}_0(B, \mathrm{Hom}(\boldsymbol{V}, Be_\infty)) \cong \boldsymbol{V}^* \otimes_B Be_\infty \cong \boldsymbol{V}^* e_\infty \cong (e_\infty \boldsymbol{V})^* \cong \mathbb{C}\ .$$

Now, since $\boldsymbol{\pi} \circ \boldsymbol{\eta} = 0$, we have $\boldsymbol{\pi}_*(c(\boldsymbol{\lambda}, \delta_M)) = [\,\boldsymbol{\lambda} \cdot \mathrm{Id}_{\boldsymbol{V}}\,]$, and hence

$$\mathrm{Tr}_*(\tilde{c}) := \mathrm{tr}_{\boldsymbol{V}}[\,\boldsymbol{\pi}_*\, \boldsymbol{\iota}_*(\tilde{c})\,] = \mathrm{tr}_{\boldsymbol{V}}[\,\boldsymbol{\pi}_*(c)\,] = \mathrm{tr}_{\boldsymbol{V}}[\,\boldsymbol{\lambda} \cdot \mathrm{Id}_{\boldsymbol{V}}\,] = 0\ .$$

It follows that $\tilde{c} = 0$ and $c(\boldsymbol{\lambda},\, \delta_M) = 0$, finishing the proof of the lemma and Proposition 5.3. □

Now, by Proposition 5.3, the given B-module structure on $\boldsymbol{V} = \boldsymbol{L}/M$ extends to a (unique) $\Pi^{\boldsymbol{\lambda}}(B)$-module structure, making (5.50) an exact sequence of $\Pi^{\boldsymbol{\lambda}}(B)$-modules. Tensoring this exact sequence with $e_\infty \Pi$, we get an isomorphism of U-modules $e_\infty \boldsymbol{L} \cong e_\infty \boldsymbol{V}$, and thence an isomorphism of Π-modules $\Pi e_\infty \otimes_U e_\infty \boldsymbol{L} \cong \Pi e_\infty \otimes_U e_\infty \boldsymbol{V}$, where $\Pi = \Pi^{\boldsymbol{\lambda}}(B)$ and $U = e_\infty \Pi e_\infty$. The last isomorphism fits into the commutative diagram

(5.78)
$$\begin{array}{ccc} \Pi e_\infty \otimes_U e_\infty \boldsymbol{L} & \cong & \Pi e_\infty \otimes_U e_\infty \boldsymbol{V} \\ {\scriptstyle \boldsymbol{\mu}_{\boldsymbol{L}}} \downarrow & & \downarrow {\scriptstyle \boldsymbol{\mu}_{\boldsymbol{V}}} \\ \boldsymbol{L} & \xrightarrow{\ \boldsymbol{\pi}\ } & \boldsymbol{V} \end{array}$$

where $\boldsymbol{\mu}_{\boldsymbol{L}}$ and $\boldsymbol{\mu}_{\boldsymbol{V}}$ are the natural multiplication-action maps. We claim that $\boldsymbol{\mu}_{\boldsymbol{L}}$ is an isomorphism. Indeed, by Morita equivalence (see Lemma 5.3), it suffices to show that $e\boldsymbol{\mu}_{\boldsymbol{L}} : e\Pi e_\infty \otimes_U e_\infty \boldsymbol{L} \to L$ is an isomorphism of $e\Pi e$-modules. Filtering $e\Pi e_\infty \otimes_U e_\infty \boldsymbol{L}$ as in Section 5.2 and L as in (the proof of) Lemma 5.8, it is easy to see that $e\boldsymbol{\mu}_{\boldsymbol{L}}$ preserves filtrations inducing an isomorphism on the associated graded modules (these last modules are both isomorphic to $\bar{\mathcal{D}}\mathcal{I}$, cf. (5.25) and (5.66)). Hence $e\boldsymbol{\mu}_{\boldsymbol{L}}$ is an isomorphism. It follows now from (5.78) that $\mathrm{Ker}(\boldsymbol{\pi}) \cong \mathrm{Ker}(\boldsymbol{\mu}_{\boldsymbol{V}})$, and thus $M \cong L\boldsymbol{\theta}^*(\boldsymbol{V})$. This completes Step 2.

5.4. **The map ω is injective and Γ-equivariant.** For two Π-modules $\boldsymbol{V}$ and $\boldsymbol{V}'$ of dimension vector $\boldsymbol{n} = (n, 1)$, we will show that

(5.79) $$L\boldsymbol{\theta}^*(\boldsymbol{V}) \cong L\boldsymbol{\theta}^*(\boldsymbol{V}') \quad \Longleftrightarrow \quad \boldsymbol{V}' \cong \boldsymbol{V}^{\sigma_\omega} \quad \text{for some } \omega = u^{-1}du \in \Omega^1 X\ ,$$

where $\boldsymbol{V}^\sigma$ denotes the Π-module $\boldsymbol{V}$ twisted by an automorphism $\sigma \in \mathrm{Aut}_S\, \Pi^{\boldsymbol{\lambda}}(B)$.

We begin by describing the action (4.22) in terms of generators of $\Pi^{\boldsymbol{\lambda}}(B)$ (see Proposition 5.1).

Lemma 5.10. *The homomorphism* $\sigma : \Omega^1 X \to \mathrm{Aut}_S\, \Pi$ *is given by*

(5.80) $$\sigma_\omega(\hat{a}) = \hat{a}\ , \quad \sigma_\omega(\hat{\mathrm{v}}_i) = \hat{\mathrm{v}}_i\ , \quad \sigma_\omega(\hat{\mathrm{w}}_i) = \hat{\mathrm{w}}_i\ , \quad \sigma_\omega(\hat{d}) = \hat{d} + \widehat{\omega(d)}\ ,$$

where $\omega \in \Omega^1 X$ *acts on* $d \in \mathrm{Der}(A, A^{\otimes 2})$ *via the natural identification*

$$\Omega^1 X = \mathrm{DR}^1(A) \cong \mathrm{Hom}_{A^{\mathrm{e}}}((\Omega^1 A)^\star,\, A) \cong \mathrm{Hom}_{A^{\mathrm{e}}}(\mathrm{Der}(A,\, A^{\otimes 2}),\, A)\ ,\ \ cf.\ (4.18)\,.$$

Proof. By Lemma 5.1, we can define (4.22) in terms of relative differentials

$$\sigma : \Omega^1 X \xrightarrow{\alpha} \mathrm{DR}^1_S(B) \xrightarrow{\tilde{\sigma}} \mathrm{Aut}_B[T_B(\Omega^1_S B)^\star] \to \mathrm{Aut}_S\, \Pi^{\boldsymbol{\lambda}}(B)\ . \tag{5.81}$$

Tensoring then $0 \to \Omega^1_S B \to B \otimes_S B \to B \to 0$ with B and using the identification (4.21), we get the exact sequence

$$0 \to \Omega^1 X \xrightarrow{\alpha} (\Omega^1_S B)_\natural \to \mathrm{H}_0(B, B \otimes_S B) \to \mathrm{H}_0(B,B) \to 0\ .$$

It is easy to see from (5.10) that $B \otimes_S B \to B$ induces an isomorphism $\mathrm{H}_0(B, B\otimes_S B) \xrightarrow{\sim} \mathrm{H}_0(B,B)$. Hence, by exactness of the above sequence, α is an isomorphism as well. Explicitly, with identification (5.13), the elements of $(\Omega^1_S B)_\natural = \Omega^1_S B/[B, \Omega^1_S B]$ can be represented by matrices $\hat{\omega} = \begin{pmatrix} \omega & 0 \\ 0 & 0 \end{pmatrix}$ with $\omega \in (\Omega^1 A)_\natural$, and α is given by $\omega \mapsto \hat{\omega} \bmod [B, \Omega^1_S B]$. It follows then that σ_ω acts on $\Pi^{\boldsymbol{\lambda}}(B)$ as described in (5.80). □

We may also describe the algebra map $\boldsymbol{\theta} : \Pi^{\boldsymbol{\lambda}}(B) \to \mathcal{D}$ of Lemma 4.1 in terms of generators of $\Pi^{\boldsymbol{\lambda}}(B)$:

$$\boldsymbol{\theta}(\hat{a}) = \overline{a}\ , \quad \boldsymbol{\theta}(\hat{d}) = \overline{d}\ , \quad \boldsymbol{\theta}(\hat{\mathrm{v}}_i) = \boldsymbol{\theta}(\hat{\mathrm{w}}_i) = 0\ , \tag{5.82}$$

where $\overline{a}$ and $\overline{d}$ denote the classes of $a \in A$ and $d \in \mathrm{Der}(A, A^{\otimes 2})$ in $T_A\mathrm{Der}(A, A^{\otimes 2})$ modulo the ideal $\langle \Delta_A - 1 \rangle$. The first and the second equations in (5.82) are immediate from (2.3), while the last two follow from the relations $\hat{\mathrm{v}}_i e_\infty = \hat{\mathrm{v}}_i$, $e_\infty \hat{\mathrm{w}}_i = \hat{\mathrm{w}}_i$.

Comparing now (5.80) and (5.82), we get

Lemma 5.11. *The group homomorphism* $\bar{\sigma} : \Omega^1 X \xrightarrow{\sigma} \mathrm{Aut}_S\, \Pi \to \mathrm{Aut}_{\mathbb{C}}\, \mathcal{D}$ *induced by* σ *is given by*

$$\bar{\sigma}_\omega(a) = a\ , \quad \bar{\sigma}_\omega(\partial) = \partial + \omega(\partial)\ , \tag{5.83}$$

where $a \in A$, $\partial \in \mathrm{Der}(A)$ *and* $\omega \in \Omega^1 X$.

In particular, if $\omega = \mathtt{dlog}(u) = u^{-1}du \in \Omega^1 X$ for some $u \in \Lambda$, then $\bar{\sigma}_\omega(a) = a = u\,a\,u^{-1}$ (since A is commutative), and $\bar{\sigma}_\omega(\partial) = u\,\partial\,u^{-1}$. Thus, the induced action of $\Lambda \subset \Omega^1 X$ on $\mathcal{D}$ is given by *inner* automorphisms. (In contrast, Λ does not act by inner automorphisms on the whole of $\Pi^{\boldsymbol{\lambda}}(B)$.)

Now, by functoriality, $\boldsymbol{V}' \cong \boldsymbol{V}^{\sigma_\omega}$ implies $\boldsymbol{L}' \cong \boldsymbol{L}^{\sigma_\omega}$ and $L\boldsymbol{\theta}^*(\boldsymbol{V}') \cong L\boldsymbol{\theta}^*(\boldsymbol{V})^{\sigma_\omega}$ for any $\omega \in \Omega^1 X$. So the map $\mathcal{C}_n(X, \mathcal{I}) \to \mathfrak{I}(\mathcal{D})$ induced by $L\boldsymbol{\theta}^*$ is equivariant under the action of $\Omega^1 X$. On the other hand, $L\boldsymbol{\theta}^*(\boldsymbol{V})$ is a $\mathcal{D}$-module, on which the twisting by ω acts via (5.83), i. e. $L\boldsymbol{\theta}^*(\boldsymbol{V})^{\sigma_\omega} = L\boldsymbol{\theta}^*(\boldsymbol{V})^{\bar{\sigma}_\omega}$. It is well-known (and easy to check) that the inner automorphisms induce trivial auto-equivalences of module categories, so if $\omega = u^{-1}du$ with $u \in \Lambda$, then $L\boldsymbol{\theta}^*(\boldsymbol{V})^{\sigma_\omega} \cong L\boldsymbol{\theta}^*(\boldsymbol{V})$. This proves the implication "$\Leftarrow$" in (5.79) and, in combination with Step 1, yields a Γ-equivariant map

$$\omega_n : \overline{\mathcal{C}}_n(X, \mathcal{I}) \to \gamma^{-1}[\mathcal{I}]\ . \tag{5.84}$$

It remains to show that ω_n is injective. We will need the following lemma, which is a mild generalization of [BW2], Lemma 10.1, and [NS], Lemma 3.2. Our proof below uses the arguments of [NS].

Lemma 5.12. *Let M be a (nonzero) ideal of $\mathcal{D}$ equipped with two good filtrations $\{M_k\}$ and $\{M'_k\}$, such that the associated graded modules $\overline{M}$ and $\overline{M}'$ are both torsion-free. Then, there is $k_0 \in \mathbb{Z}$, such that $M_k = M'_{k-k_0}$ for all $k \in \mathbb{Z}$.*

Proof. Since $\{M_k\}$ is a good filtration, $\overline{M}$ is finitely generated and, hence, bounded from the left. Fix a nonzero element $m_0 \in M$, and let $k_0 := \deg(m_0) - \deg'(m_0)$, where deg and deg′ denote the dergee functions for the given filtrations on M. Now, since $\mathcal{D}$ is an Ore domain, for any nonzero $m \in M$, we can find nonzero $a \in \mathcal{D}$ and $b \in \mathcal{D}$ such that $a\, m_0 = b\, m$. This implies $\mathsf{gr}(a)\,\mathsf{gr}(m_0) = \mathsf{gr}(b)\,\mathsf{gr}(m) \neq 0$, since $\{M_k\}$ is torsion-free, and therefore

$$\deg(a) + \deg(m_0) = \deg(a\, m_0) = \deg(b\, m) = \deg(b) + \deg(m)\ . \tag{5.85}$$

Similarly, replacing $\{M_k\}$ by $\{M'_k\}$ in the above argument, we get

$$\deg(a) + \deg'(m_0) = \deg(b) + \deg'(m)\ . \tag{5.86}$$

Comparing (5.85) and (5.86) yields $\deg(m) - \deg'(m) = \deg(m_0) - \deg'(m_0)$. Thus, for any nonzero $m \in M$, we have $\deg(m) - \deg'(m) = k_0$, the integer k_0 being independent of m. This implies

$$m \in M_k \ \Leftrightarrow\ \deg(m) \le k \ \Leftrightarrow\ \deg'(m) \le k - k_0 \ \Leftrightarrow\ m \in M'_{k-k_0}\ ,$$

so that $M_k = M'_{k-k_0}$ for all $k \in \mathbb{Z}$, as claimed. □

Now, given two Π-modules $\boldsymbol{V}$ and $\boldsymbol{V}'$ of dimension $\boldsymbol{n}$, we set $\boldsymbol{L} := \Pi e_\infty \otimes_U e_\infty \boldsymbol{V}$, $L := e\boldsymbol{L}$, $M := L\boldsymbol{\theta}^*(\boldsymbol{V})$, and similarly for $\boldsymbol{V}'$. In addition, we denote by $\eta : M \hookrightarrow L$ and $\boldsymbol{\eta} : M \hookrightarrow \boldsymbol{L}$ the natural inclusions (and similarly for M').

Proposition 5.4. *If $M \cong M'$ as $\mathcal{D}$-modules, then $\boldsymbol{L} \cong \boldsymbol{L}'$ as B-modules.*

Proof. First, we show that every $\mathcal{D}$-module isomorphism $f : M \to M'$ lifts to an A-module isomorphism $f_L : L \to L'$, making commutative the diagram

$$\begin{array}{ccc} L & \xrightarrow{f_L} & L' \\ \eta \big\uparrow & & \big\uparrow \eta' \\ M & \xrightarrow{f} & M' \end{array} \tag{5.87}$$

To this end, we identify the module L as in Lemma 5.5 and filter it as in (the end of) Section 5.2: thus, $L = R\mathcal{I}/\Sigma$, see (5.23), with filtration $F_0L := \mathcal{I}\ (\mathrm{mod}\,\Sigma)$, $F_1L := \mathcal{I} + \mathrm{Der}(A, A^{\otimes 2})\mathcal{I}\ (\mathrm{mod}\,\Sigma)$, etc. Now, we set $\widetilde{L} := \bigoplus_{k\in\mathbb{Z}} L/F_kL$ and define the structure of a graded $R[t]$-module on $\widetilde{L}$ by letting t act by the canonical projections $L/F_kL \twoheadrightarrow L/F_{k+1}L$, and $d \in \mathrm{Der}(A, A^{\otimes 2})$ by $[x]_k \mapsto [d\cdot x]_{k+1}$, where $[x]_k \in \widetilde{L}_k$ is the class of $x \in L$ mod F_kL. By (5.23), we have $\Delta_A \cdot x \equiv x\ (\mathrm{mod}\, F_0L)$ for all $x \in L$, so $\Delta_A\,[x]_k = [x]_{k+1} = t\,[x]_k$ for all $k \ge -1$. Since $R[t]/\langle \Delta_A - t\rangle \cong \widetilde{\mathcal{D}}$, we may (and will) regard $\widetilde{L}_{\ge -1}$ as a graded $\widetilde{\mathcal{D}}$-module.

Next, we equip M with the induced filtration $M_k := M \cap F_kL$ via the inclusion $\eta : M \hookrightarrow L$, and set $\widetilde{N} := \bigoplus_{k\in\mathbb{Z}} M/M_k$. The map η naturally extends to a graded embedding $\tilde{\eta} : \widetilde{N} \hookrightarrow \widetilde{L}$, and $\widetilde{N}$ becomes a graded $\widetilde{\mathcal{D}}$-module via the induced action of $R[t]$ on $\widetilde{L}$. It follows from (5.25) that $\overline{M} := \bigoplus_{k\in\mathbb{Z}} M_k/M_{k+1}$ is a torsion-free $\bar{\mathcal{D}}$-module, and hence $\tau(\widetilde{N}) = 0$ by Proposition 5.2(1).

Now, let $\eta_{\widetilde{N}} : \widetilde{N} \hookrightarrow \omega\pi(\widetilde{N})$ be the completion of $\widetilde{N}$ relative to $\mathsf{Tors}(\widetilde{\mathcal{D}})$, see (5.26). Ring-theoretically, this can be characterized as the largest essential extension of $\widetilde{N}$ in $\mathsf{GrMod}(M)$ with cokernel in $\mathsf{Tors}(\widetilde{\mathcal{D}})$. Since $\mathrm{Coker}[\,\tilde{\eta} : \widetilde{N}_{\geq -1} \hookrightarrow \widetilde{L}_{\geq -1}\,]$ is finite-dimensional, $\eta_{\widetilde{N}}$ extends through $\tilde{\eta}$, giving an embedding: $\widetilde{L}_{\geq -1} \hookrightarrow \omega\pi(\widetilde{N})_{\geq -1}$. We claim that this embedding is an isomorphism. Indeed, in degree (-1), we have

$$\omega\pi(\widetilde{N})_{-1}/\widetilde{L}_{-1} = \omega\pi(\widetilde{N})_{-1}/L \cong (\omega\pi(\widetilde{N})_{-1}/M)/(L/M) \ ,$$

while, by Proposition 5.2(2), $\dim_{\mathbb{C}}[\,\omega\pi(\widetilde{N})_{-1}/M\,] = n = \dim_{\mathbb{C}}[\,L/M\,]$. Hence $\widetilde{L}_{-1} \cong \omega\pi(\widetilde{N})_{-1}$. On the other hand, by (5.47), for each $k \geq 0$, the kernel of the projection $\omega\pi(\widetilde{N})_{k-1}/\widetilde{L}_{k-1} \overset{t}{\twoheadrightarrow} \omega\pi(\widetilde{N})_k/\widetilde{L}_k$ is isomorphic to $\bar{\mathcal{D}}_k\mathcal{I}/\overline{L}_k$, while, by (5.25), this last quotient is zero. It follows, by induction, that $\widetilde{L}_k \cong \omega\pi(\widetilde{N})_k$ for all $k \geq -1$, and thus $\widetilde{L}_{\geq -1} \cong \omega\pi(\widetilde{N})_{\geq -1}$.

Now, replacing L by L', we repeat the above construction. The $\mathcal{D}$-module M' comes then equipped with two filtrations: one is induced from L' via the inclusion $\eta' : M' \hookrightarrow L'$, i. e. $M'_k := M' \cap F_k L'$, and the other is transferred from M via the given isomorphism $f : M \to M'$. Both these filtrations satisfy the assumptions of Lemma 5.12 and, hence, coincide up to a degree shift. This last shift is equal to 0, since both $\overline{M}'$ and $f(\overline{M})$ have finite codimension in $\overline{L}'$, so $M'_k = f(M_k)$ for all $k \in \mathbb{Z}$. The map f extends then to an isomorphism of graded $\widetilde{\mathcal{D}}$-modules $\tilde{f} : \widetilde{N} \to \widetilde{N}'$ and further, by functoriality, to $\omega\pi(\tilde{f}) : \omega\pi(\widetilde{N}) \to \omega\pi(\widetilde{N}')$. As a result, we get the commutative diagram

$$\begin{array}{ccccccc} \widetilde{L}_{\geq -1} & \longrightarrow & \omega\pi(\widetilde{N})_{\geq -1} & \xrightarrow{\omega\pi(\tilde{f})} & \omega\pi(\widetilde{N}')_{\geq -1} & \longleftarrow & \widetilde{L}'_{\geq -1} \\ \uparrow{\scriptstyle \tilde{\eta}} & & \uparrow & & \uparrow & & \uparrow{\scriptstyle \tilde{\eta}'} \\ \widetilde{N}_{\geq -1} & = & \widetilde{N}_{\geq -1} & \xrightarrow{\tilde{f}} & \widetilde{N}'_{\geq -1} & = & \widetilde{N}'_{\geq -1} \end{array} \tag{5.88}$$

with all horizontal arrows being isomorphisms. The graded components of $\tilde{f}$, $\tilde{\eta}$ and $\tilde{\eta}'$ in degree (-1) are precisely the maps f, η and η'. So restricting to degree (-1) and composing the first two maps at the top of (5.88) with the inverse of the third yields a required extension $f_L : L \to L'$ in (5.87).

Now, with our identifications of L and L', the B-module structures on $\boldsymbol{L}$ and $\boldsymbol{L}'$ are determined (up to isomorphism) by the triples $(L, \mathbb{C}, \varphi)$ and $(L', \mathbb{C}, \varphi')$, where $\varphi : \mathcal{I} \hookrightarrow L$ and $\varphi' : \mathcal{I} \hookrightarrow L'$ are the canonical embeddings with images F_0L and F_0L' respectively. Since F_0L is the kernel of the projection $\widetilde{L}_{-1} \twoheadrightarrow \widetilde{L}_0$ induced by the action of t on $\widetilde{L}$, the map f_L restricts to F_0L, giving an isomorphism $f_L|_0 : F_0L \to F_0L'$. Letting $u := (\varphi')^{-1} f_L|_0\, \varphi \in \mathrm{Aut}_A(\mathcal{I})$ and identifying $\mathrm{Aut}_A(\mathcal{I}) = \mathrm{End}_A(\mathcal{I})^{\times} \cong A^{\times}$ via the action map, we have $u\,\varphi' = \varphi' u = f_L\,\varphi$. Hence

$$\boldsymbol{g} := (u^{-1} f_L,\, \mathrm{Id}) : \ L \oplus \mathbb{C} \to L' \oplus \mathbb{C} \tag{5.89}$$

makes (2.13) a commutative diagram and thus defines an isomorphism of B-modules $\boldsymbol{L} \xrightarrow{\sim} \boldsymbol{L}'$. Summing up, if $f : M \xrightarrow{\sim} M'$ is an isomorphism of $\mathcal{D}$-modules, then $\boldsymbol{g} : \boldsymbol{L} \xrightarrow{\sim} \boldsymbol{L}'$ is an isomorphism of B-modules. Note, however, that $\boldsymbol{g}$ does not restrict to f through the inclusion $\boldsymbol{\eta} : M \hookrightarrow \boldsymbol{L}$. $\square$

Now, keeping the notation of Proposition 5.4, consider two Π-modules $\boldsymbol{V}$ and $\boldsymbol{V}'$ of dimension $\boldsymbol{n}$, with $M \cong M'$. Fix an isomorphism $f : M \to M'$ and define $\boldsymbol{g}$ as in (5.89). Taking $\omega = u^{-1}du \in \Omega^1 X$ and twisting $\boldsymbol{\eta}$ by $\sigma = \sigma_\omega \in \mathrm{Aut}_S \Pi$, consider the diagram

$$\begin{array}{ccc} \boldsymbol{L}^\sigma & \xrightarrow{\ \boldsymbol{g}\ } & \boldsymbol{L}' \\ \boldsymbol{\eta} \big\uparrow & & \big\uparrow \boldsymbol{\eta}' \\ M^\sigma & \xrightarrow{\ fu^{-1}\ } & M' \end{array} \tag{5.90}$$

We claim that (5.90) is a commutative diagram of Π-*module* homomorphisms, with horizontal arrows being isomorphisms.

To see this, we recall that σ induces on $\mathcal{D}$ the inner automorphism: $D \mapsto u\,D\,u^{-1}$. The multiplication map $u^{-1} : M^\sigma \to M$ is then a $\mathcal{D}$-module (and *a fortiori* Π-module) isomorphism. The bottom arrow in (5.90) is thus the composition of two Π-module isomorphisms, and hence a Π-module isomorphism as well. On the other hand, by definition, σ acts as the identity on $B \subset \Pi$. So, if we restrict to B-modules, the fact that (5.90) is a commutative diagram in $\mathtt{Mod}(B)$ follows immediately from commutativity of (5.87) and (5.89).

Now, identifying $M^\sigma \cong M'$ via fu^{-1} and $\boldsymbol{L}^\sigma \cong \boldsymbol{L}'$ via $\boldsymbol{g}$ in (5.90), we get two (*a priori* different) Π-module structures on $\boldsymbol{L}'$. Both of these are extensions of the given Π-module structure on M' via $\boldsymbol{\eta}'$. Hence, by Proposition 5.3, they coincide. It follows that $\boldsymbol{g} : \boldsymbol{L}^\sigma \to \boldsymbol{L}'$ is an isomorphism of Π-modules, which, by commutativity of (5.90), induces a required isomorphism $\boldsymbol{V}^\sigma \cong \boldsymbol{V}'$. This completes Step 3.

5.5. **The equivariance of ω under the action of** $\mathrm{Pic}(\mathcal{D})$**.** We begin with Proposition 4.3 and Lemma 4.3. As in Section 4, we will assume that $X \neq \mathbb{A}^1$. By [CH1], Prop. 1.4, the automorphism group of the algebra $\mathcal{D}$ is then isomorphic to the semi-direct product $\mathrm{Aut}(X) \ltimes \Omega^1 X$: the isomorphism is given by

$$\mathrm{Aut}(X) \ltimes \Omega^1 X \xrightarrow{\sim} \mathrm{Aut}(\mathcal{D}) \ , \quad (\nu,\, \omega) \mapsto \bar{\nu}\, \bar{\sigma}_\omega \ , \tag{5.91}$$

where $\bar{\nu} \in \mathrm{Aut}(\mathcal{D}) : D \mapsto \nu\, D\, \nu^{-1}$, and $\bar{\sigma}_\omega$ is defined by (5.83). Note that $\bar{\sigma}$ identifies $\Omega^1 X$ with the subgroup of automorphisms of $\mathcal{D}$ acting as identity on A.

Now, if $\mathcal{F}$ is a line bundle on X, the algebra $\mathrm{End}_{\mathcal{D}}(\mathcal{F}\mathcal{D})$ is canonically isomorphic to the ring of twisted differential operators on X with coefficients in $\mathcal{F}$. As X is an affine curve, this last ring is isomorphic to $\mathcal{D}$, so the set of all algebra isomorphisms: $\mathcal{D} \to \mathrm{End}_{\mathcal{D}}(\mathcal{F}\mathcal{D})$ is non-empty and equals $\psi_0\, \mathrm{Aut}(\mathcal{D})$, where ψ_0 is a fixed isomorphism. By [CH1], Theorem 1.8, the isomorphism ψ_0 can be chosen in such a way that $\psi_0\,|_A = \mathrm{Id}$: specifically, fixing dual bases for $\mathcal{F}$ and $\mathcal{F}^\vee$, say $\{\alpha_i\} \subset \mathcal{F}$ and $\{\beta_i\} \subset \mathcal{F}^\vee$, and identifying $\mathrm{End}_{\mathcal{D}}(\mathcal{F}\mathcal{D}) = \mathcal{F}\,\mathcal{D}\,\mathcal{F}^\vee$ as in Section 4.2, we define $\psi_0 : \mathcal{D} \xrightarrow{\sim} \mathrm{End}_{\mathcal{D}}(\mathcal{F}\mathcal{D})$ by

$$\psi_0(a) = a \ , \quad \psi_0(\partial) = \sum_i \alpha_i\, \partial\, \beta_i \ , \tag{5.92}$$

where $a \in A$ and $\partial \in \mathrm{Der}(A)$. With (5.91), every isomorphism $\psi : \mathcal{D} \to \mathrm{End}_{\mathcal{D}}(\mathcal{F}\mathcal{D})$ can then be decomposed as

$$\psi = \psi_0\, \bar{\nu}\, \bar{\sigma}_\omega \ , \tag{5.93}$$

where $\nu \in \mathrm{Aut}(X)$ and $\omega \in \Omega^1 X$ are uniquely determined by ψ.

Proof of Proposition 4.3. Given a line bundle $\mathcal{I}$ and an invertible bimodule $\mathcal{P} = (\mathcal{D}\mathcal{L})_\varphi$, with $\varphi : \mathcal{D} \xrightarrow{\sim} \mathrm{End}_{\mathcal{D}}(\mathcal{D}\mathcal{L})$, we set $\tau := \varphi|_A$, $\mathcal{J} := \mathcal{L}\tau(\mathcal{I})$, $\mathcal{F} := \mathcal{L}^\tau = \tau^{-1}(\mathcal{L})$, and $\psi = \varphi^{-1} : \mathcal{D} \to \mathrm{End}_{\mathcal{D}}(\mathcal{F}\mathcal{D})$, as in Section 4.2. To construct the isomorphism $\widetilde{\psi}$, satisfying the conditions of Lemma 4.3, we decompose ψ as in (5.93), and extend each decomposition factor through $\boldsymbol{\theta}$ individually. Since ψ_0 and $\bar{\sigma}_\omega$ act on A as identity, we have $\nu = \psi|_A = \tau^{-1}$, so $\bar{\nu} = \bar{\tau}^{-1}$ in (5.93). Thus, we set

$$\widetilde{\psi} := \widetilde{\psi}_0\, \tilde{\tau}^{-1}\, \sigma_\omega\ ,$$

where $\sigma_\omega \in \mathrm{Aut}_S[\Pi^{\boldsymbol{\lambda}}(A[\mathcal{J}])]$ is defined in Section 4.2 (see (4.22), with B replaced by $A[\mathcal{J}]$) and $\tilde{\tau}^{-1} : \Pi^{\boldsymbol{\lambda}}(A[\mathcal{J}]) \xrightarrow{\sim} \Pi^{\boldsymbol{\lambda}}(A[\mathcal{J}^\tau])$ is the isomorphism induced by $\tilde{\tau}^{-1} : A[\mathcal{J}] \to A[\mathcal{J}^\tau]$, see (4.7). The commutativity of the middle square of the diagram

$$\begin{array}{ccccccc}
\Pi^{\boldsymbol{\lambda}}(A[\mathcal{J}]) & \xrightarrow{\sigma_\omega} & \Pi^{\boldsymbol{\lambda}}(A[\mathcal{J}]) & \xrightarrow{\tilde{\tau}^{-1}} & \Pi^{\boldsymbol{\lambda}}(A[\mathcal{J}^\tau]) & \xrightarrow{\widetilde{\psi}_0} & \mathrm{End}_{\Pi^{\boldsymbol{\lambda}}(B)}(\boldsymbol{P}) \\
\downarrow{\scriptstyle\boldsymbol{\theta}} & & \downarrow{\scriptstyle\boldsymbol{\theta}} & & \downarrow{\scriptstyle\boldsymbol{\theta}} & & \downarrow{\scriptstyle\boldsymbol{\theta}} \\
\mathcal{D} & \xrightarrow{\bar{\sigma}_\omega} & \mathcal{D} & \xrightarrow{\bar{\tau}^{-1}} & \mathcal{D} & \xrightarrow{\psi_0} & \mathrm{End}_{\mathcal{D}}(\mathcal{F}\mathcal{D})
\end{array} \tag{5.94}$$

is immediate, while the leftmost square commutes by Lemma 5.11. It remains to define the isomorphism $\widetilde{\psi}_0$. To this end, we identify

$$\mathrm{End}_{\Pi^{\boldsymbol{\lambda}}(B)}(\boldsymbol{P}) \cong \tilde{\mathcal{F}} \otimes_{\tilde{A}} \Pi^{\boldsymbol{\lambda}}(B) \otimes_{\tilde{A}} \tilde{\mathcal{F}}^\vee \qquad \text{(see (4.8))}$$

Since $\mathcal{J}^\tau = \mathcal{L}^\tau \mathcal{I} = \mathcal{F}\mathcal{I}$, we have then the natural embedding

$$A[\mathcal{J}^\tau] \cong \tilde{\mathcal{F}} \otimes_{\tilde{A}} B \otimes_{\tilde{A}} \tilde{\mathcal{F}}^\vee \hookrightarrow \tilde{\mathcal{F}} \otimes_{\tilde{A}} \Pi^{\boldsymbol{\lambda}}(B) \otimes_{\tilde{A}} \tilde{\mathcal{F}}^\vee\ , \tag{5.95}$$

which we take as a definition of $\widetilde{\psi}_0$ on $A[\mathcal{J}^\tau]$. This does not depend on the choice of dual bases, and induces the identity map on A, as required.

Next, we construct a bimodule isomorphism:

$$\mathrm{Der}_S(A[\mathcal{F}\mathcal{I}],\, A[\mathcal{F}\mathcal{I}]^{\otimes 2}) \to \tilde{\mathcal{F}} \otimes_{\tilde{A}} \mathrm{Der}_S(B, B^{\otimes 2}) \otimes_{\tilde{A}} \tilde{\mathcal{F}}^\vee\ , \tag{5.96}$$

using the dual bases for $\mathcal{F}$ and $\mathcal{I}$. By Lemma 5.2, we first identify the domain of (5.96) with

$$\begin{pmatrix} \mathrm{Der}(A, A^{\otimes 2}) & \mathrm{Der}(A, \mathcal{F}\mathcal{I} \otimes A) \\ 0 & 0 \end{pmatrix} \bigoplus \begin{pmatrix} \mathcal{F}\mathcal{I} \otimes (\mathcal{F}\mathcal{I})^\vee & \mathcal{F}\mathcal{I} \otimes A \\ (\mathcal{F}\mathcal{I})^\vee & A \end{pmatrix} \tag{5.97}$$

and the codomain with

$$\begin{pmatrix} \mathcal{F} \otimes \mathrm{Der}(A, A^{\otimes 2}) \otimes \mathcal{F}^\vee & \mathrm{Der}(A, \mathcal{I} \otimes \mathcal{F}) \\ 0 & 0 \end{pmatrix} \bigoplus \begin{pmatrix} \mathcal{F}\mathcal{I} \otimes (\mathcal{F}\mathcal{I})^\vee & \mathcal{F}\mathcal{I} \otimes A \\ (\mathcal{F}\mathcal{I})^\vee & A \end{pmatrix}.$$

The first summand of (5.97) is generated (as a bimodule over $A[\mathcal{F}\mathcal{I}]$) by the elements $\hat{d} \in e\,\mathrm{Der}(A, A^{\otimes 2})\, e$ (see Proposition 5.1): with above identifications, we define the map (5.96) on this first summand by

$$\hat{d} = \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} \sum_i \alpha_i \otimes d \otimes \beta_i & 0 \\ 0 & 0 \end{pmatrix}, \tag{5.98}$$

while letting it be the identity on the second. Obviously, this yields an isomorphism of bimodules and, together with (5.95), induces the required algebra map $\widetilde{\psi}_0$. The commutativity of the last square of the diagram (5.94) is verified by a simple calculation, using formulas (5.82).

To complete the proof of Proposition 4.3, it remains to show the uniqueness of $\widetilde{\psi}$. For this, we will argue as in the proof of Proposition 5.3. Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{End}_{\Pi^{\boldsymbol\lambda}(B)}(\boldsymbol{P}) \otimes_{A[\mathcal{J}]^{\mathrm{e}}} \Omega^1(A[\mathcal{J}]) & \xrightarrow{\ \tilde{\boldsymbol\theta}\ } & \mathrm{End}_{\mathcal{D}}(\mathcal{F}\mathcal{D}) \otimes_{A[\mathcal{J}]^{\mathrm{e}}} \Omega^1(A[\mathcal{J}]) \\ \partial_{\boldsymbol P}\downarrow & & \downarrow\partial_{\mathcal{L}} \\ \mathrm{End}_{\Pi^{\boldsymbol\lambda}(B)}(\boldsymbol{P}) & \xrightarrow{\ 1\otimes\boldsymbol\theta\otimes 1\ } & \mathrm{End}_{\mathcal{D}}(\mathcal{F}\mathcal{D}) \end{array} \tag{5.99}$$

where $\mathrm{End}_{\Pi}(\boldsymbol{P})$ is regarded as a right module over $A[\mathcal{J}]^{\mathrm{e}}$ via (4.9). The maps $\partial_{\boldsymbol P}$ and $\partial_{\mathcal{L}}$ in (5.99) are given by $f\otimes d \mapsto f\Delta(d)$ and $\tilde{\boldsymbol\theta}$ is induced by $1\otimes\boldsymbol\theta\otimes 1$. Now, given ψ, the set of all algebra homomorphisms $\widetilde{\psi} : \Pi^{\boldsymbol\lambda}(A[\mathcal{J}]) \to \mathrm{End}_{\Pi^{\boldsymbol\lambda}(B)}(\boldsymbol{P})$, satisfying the conditions of Lemma 4.3, can be identified with a common fibre of $\partial_{\boldsymbol P}$ and $\tilde{\boldsymbol\theta}$: i. e., the subset of elements $\delta \in \mathrm{End}_{\Pi}(\boldsymbol{P}) \otimes_{A[\mathcal{J}]^{\mathrm{e}}} \Omega^1(A[\mathcal{J}])$, such that

$$\partial_{\boldsymbol P}(\delta) = \boldsymbol\lambda\cdot\mathrm{Id} \quad \text{ and } \quad \tilde{\boldsymbol\theta}(\delta) = \psi\,\boldsymbol\theta\ .$$

We have already seen that this subset is non-empty; on the other hand, by (the proof of) Proposition 5.3, it is homogeneous under the action of the Hochschild homology group $\mathrm{H}_1(A[\mathcal{J}],\, \mathrm{Ker}\,\boldsymbol\theta^{\otimes})$, where $\boldsymbol\theta^{\otimes} := 1\otimes\boldsymbol\theta\otimes 1$ is the bottom arrow in (5.99). Thus, it suffices to show that $\mathrm{H}_1(A[\mathcal{J}],\, \mathrm{Ker}\,\boldsymbol\theta^{\otimes}) = 0$. With identification (4.8), we have

$$\mathrm{Ker}\,\boldsymbol\theta^{\otimes} \cong \tilde{\mathcal{F}} \otimes_{\tilde A} (\mathrm{Ker}\,\boldsymbol\theta) \otimes_{\tilde A} \tilde{\mathcal{F}}^{\vee}\ .$$

Since $A[\mathcal{J}] \cong \tilde{\mathcal{F}} \otimes_{\tilde A} B \otimes_{\tilde A} \tilde{\mathcal{F}}^{\vee}$, see (4.9), we may identify $\mathrm{H}_1(A[\mathcal{J}],\, \mathrm{Ker}\,\boldsymbol\theta^{\otimes}) \cong \mathrm{H}_1(B,\, \mathrm{Ker}\,\boldsymbol\theta)$. On the other hand, by Lemma 5.4, $\mathrm{Ker}\,\boldsymbol\theta \cong \Pi^{\boldsymbol\lambda}(B)e_\infty \otimes_U e_\infty\Pi^{\boldsymbol\lambda}(B)$, which is easily seen to be a flat B-bimodule. Thus $\mathrm{H}_1(B,\, \mathrm{Ker}\,\boldsymbol\theta) = 0$, as required. This finishes the proof of Proposition 4.3. □

Proof of Lemma 4.3. (1) We will keep the notation of Proposition 4.3. For $\mathcal{P} = \mathcal{D}_{\bar\sigma_\omega}$, we have then $\mathcal{L} \cong A$, $\varphi = \bar\sigma_\omega$, $\tau = \mathrm{Id}_A$ and $\psi = \bar\sigma_\omega^{-1}$. Now, since $\mathcal{F} = \mathcal{L}^\tau \cong A$, we may choose $\psi_0 = \mathrm{Id}_{\mathcal{D}}$. Then $\widetilde{\psi} = \sigma_\omega^{-1} \in \mathrm{Aut}_S[\Pi^{\boldsymbol\lambda}(B)]$, so the bimodule $\boldsymbol{P}$ is isomorphic to $\Pi^{\boldsymbol\lambda}(B)$ with left multiplication twisted by σ_ω^{-1}. Hence, for $\mathcal{P} = \mathcal{D}_{\bar\sigma_\omega}$, the isomorphism (4.13) is given by

$$f_{\mathcal{P}} : \mathcal{C}_n(X,\mathcal{I}) \to \mathcal{C}_n(X,\mathcal{I})\ , \quad [\boldsymbol{V}] \mapsto [\boldsymbol{V}^{\sigma_\omega^{-1}}]\ ,$$

which agrees with our definition of σ_ω^*, see (4.23).

(2) For $\mathcal{P} = (\mathcal{D}\mathcal{L})_\varphi$, the isomorphism $f_{\mathcal{P}} : \mathcal{C}_n(X,\mathcal{I}) \to \mathcal{C}_n(X,\mathcal{J})$ is equivariant under Λ in the following sense

$$f_{\mathcal{P}} \circ \sigma_\omega^* = \sigma_{\omega_\tau}^* \circ f_{\mathcal{P}}\ , \quad \forall\, u \in \Lambda\ , \tag{5.100}$$

where $\omega = \mathtt{dlog}(u) \in \Omega^1 X$ and $\omega_\tau = \mathtt{dlog}[\tau(u)] \in \Omega^1 X$. Indeed, the composite map $f_{\mathcal{P}}\circ\sigma_\omega^*$ is induced by tensoring Π-modules with the bimodule ${}_{\psi}\boldsymbol{P}_{\sigma_\omega} = {}_{\psi}\boldsymbol{P}\otimes_\Pi \Pi_{\sigma_\omega}$, on which $\Pi^{\boldsymbol\lambda}(A[\mathcal{J}])$ acts on the left via the isomorphism $\widetilde{\psi}$. Since $\tilde\tau\,\sigma_\omega = \sigma_{\omega_\tau}\,\tilde\tau$, we have

$${}_{\psi}\boldsymbol{P}_{\sigma_\omega} \cong {}_{\sigma_\omega^{-1}\psi}\boldsymbol{P} \cong {}_{\psi\sigma_{\omega_\tau}^{-1}}\boldsymbol{P} \cong (\Pi'_{\sigma_{\omega_\tau}}) \otimes_{\Pi'} ({}_{\psi}\boldsymbol{P})\ ,$$

where $\Pi' := \Pi^{\boldsymbol\lambda}(A[\mathcal{J}])$. Obviously, this implies (5.100). Now, it follows from (5.100) that $f_{\mathcal{P}}$ induces a well-defined map $\bar f_{\mathcal{P}}$ on the quotient varieties. To see that this map depends only on the class $[\mathcal{P}] \in \mathrm{Pic}(\mathcal{D})$, we note that $[\mathcal{P}]$ determines φ (and hence, $\psi = \varphi^{-1}$) up to an *inner* automorphism of $\mathcal{D}$. By Proposition 4.3, this means that $\widetilde{\psi}$ (and hence, $f_{\mathcal{P}}$) are determined by $[\mathcal{P}]$ up to an automorphism $\sigma_\omega \in \mathrm{Aut}_S[\Pi']$ with $\omega = \mathtt{dlog}(u) \in \Omega^1 X$, $u \in \Lambda$. Since such automorphisms act trivially on $\overline{\mathcal{C}}_n(X, \mathcal{I})$, the induced map $\bar{f}_{\mathcal{P}}$ is uniquely determined by $[\mathcal{P}] \in \mathrm{Pic}(\mathcal{D})$. □

Finally, we prove the last part of Theorem 4.2.

Proof of Theorem 4.2(c). Let $\boldsymbol{V}$ be a $\Pi^{\boldsymbol{\lambda}}(B)$-module, representing a point of $\mathcal{C}_n(X, \mathcal{I})$. The corresponding class $\omega_n[\boldsymbol{V}] \in \gamma^{-1}[\mathcal{I}]$ can then be represented by an ideal M, fitting into the exact sequence

$$0 \to M \to \boldsymbol{L} \to \boldsymbol{V} \to 0 \ , \tag{5.101}$$

where $\boldsymbol{L} = \Pi e_\infty \otimes_U e_\infty \boldsymbol{V}$. Now, given an invertible bimodule $\mathcal{P} \in (\mathcal{DL})_\varphi$, we write $\Pi' = \Pi^{\boldsymbol{\lambda}}(A[\mathcal{J}])$, $U' = e_\infty \Pi' e_\infty$ and observe that

$$\boldsymbol{P} \otimes_\Pi (\Pi e_\infty \otimes_U e_\infty \Pi) \otimes_\Pi \boldsymbol{P}^* \cong \Pi' e_\infty \otimes_{U'} e_\infty \Pi' \ ,$$

where $\boldsymbol{P}$ is the progenerator from Π to Π' determined by $\mathcal{P}$. On the other hand, we have

$${}_\psi \boldsymbol{P} \otimes_\Pi \mathcal{D} \cong {}_\psi(\tilde{\mathcal{F}} \otimes_{\tilde{A}} \Pi \otimes_\Pi \mathcal{D}) \cong {}_\psi(\tilde{\mathcal{F}} \otimes_{\mathcal{D}} \mathcal{D}) \cong {}_\psi(\mathcal{FD}) \cong (\mathcal{DL})_\varphi = \mathcal{P} \ ,$$

where we use the notation of Proposition 4.3. Tensoring now (5.101) with $\boldsymbol{P}$ shows that the Π'-modules $\boldsymbol{V}' := \boldsymbol{P} \otimes_\Pi \boldsymbol{V}$ and $M' := \mathcal{P} \otimes_{\mathcal{D}} M$ fit into the exact sequence

$$0 \to M' \to \boldsymbol{L}' \to \boldsymbol{V}' \to 0 \ ,$$

with $\boldsymbol{L}' = \Pi' e_\infty \otimes_{U'} e_\infty \boldsymbol{V}'$. This means that $[M'] \in \gamma^{-1}[\mathcal{J}]$ corresponds under ω_n to $[\boldsymbol{V}'] \in \mathcal{C}_n(X, \mathcal{J})$, verifying the commutativity of (4.25) and completing the proof of Theorem 4.2 . □

6. Explicit Construction of Ideals. Examples

6.1. **Distinguished representatives.** Given a rank one torsion-free $\mathcal{D}$-module M, we choose an embedding $e : M \hookrightarrow Q$, where Q is the quotient field of the ring $\mathcal{D}$. As in [BC], we may think of this embedding in abstract terms: as an injective envelope of M in the category of $\mathcal{D}$-modules. Such an envelope exists and is uniquely determined by M up to *isomorphism* in $\mathtt{Mod}(\mathcal{D})$. We will fix this isomorphism at a later stage of our calculation.

Now, regarding M and Q as modules over $R = T_A\mathrm{Der}(A,\, A^{\otimes 2})$, we may try to extend e to L through the embedding $\eta : M \hookrightarrow L$. It is easy to see, however, that such an extension does not exist in $\mathtt{Mod}(R)$, since

$$\mathrm{Hom}_R(L,\, Q) \cong \mathrm{Hom}_{e\Pi e}(L,\, Q) \cong \mathrm{Hom}_\Pi(\boldsymbol{L},\, Q) \cong \mathrm{Hom}_U(e_\infty \boldsymbol{V}, e_\infty Q) = 0 \ ,$$

by Morita equivalence of Lemma 5.3. On the other hand, we have

Lemma 6.1. *There is a unique A-linear map $e_L : L \to Q$ extending e in $\mathtt{Mod}(A)$.*

Proof. Let $\eta_* : \mathrm{Hom}_A(L,\, Q) \to \mathrm{Hom}_A(M,\, Q)$ be the restriction map. We have $\mathrm{Ker}(\eta_*) \cong \mathrm{Hom}_A(V,\, Q) = 0$, since V is a torsion A-module, while Q is torsion-free. On the other hand, $\mathrm{Coker}(\eta_*)$ is isomorphic to a submodule of $\mathrm{Ext}^1_A(V,\, Q)$, while $\mathrm{Ext}^1_A(V,\, Q) = 0$, since Q is an injective A-module. It follows that η_* is an isomorphism, and the result follows. □

Our aim is to compute e_L explicitly, in terms of representation V. First, we consider the map

$$\mathrm{ad}:\ \mathrm{Hom}_A(L,\,Q) \to \mathrm{Der}_A(R,\,\mathrm{Hom}(L,\,Q))\ , \tag{6.1}$$

taking $f: L \to Q$ to the inner derivation $\mathrm{ad}_f(r)(x) := rf(x) - f(rx)$, where $r \in R$ and $x \in L$. Since $\mathrm{Ker}(\mathrm{ad}) \cong \mathrm{Hom}_R(L,\,Q) = 0$, the map (6.1) is injective, and every $f \in \mathrm{Hom}_A(L,\,Q)$ is uniquely determined by ad_f. In addition, if f restricts to an R-linear map $M \to Q$, then $\eta_*(\mathrm{ad}_f) = 0$ in $\mathrm{Der}_A(R,\,\mathrm{Hom}(M,\,Q))$, and ad_f is determined by a (unique) derivation in $\mathrm{Der}_A(R,\,\mathrm{Hom}(V,\,Q))$. This follows immediately from the exact sequence

$$0 \to \mathrm{Der}_A(R,\,\mathrm{Hom}(V,\,Q)) \to \mathrm{Der}_A(R,\,\mathrm{Hom}(L,\,Q)) \xrightarrow{\eta_*} \mathrm{Der}_A(R,\,\mathrm{Hom}(M,\,Q))\ .$$

Thus, e_L is uniquely determined by a derivation $\delta_V \in \mathrm{Der}_A(R,\,\mathrm{Hom}(V,\,Q))$, satisfying

$$e_L(rx) - re_L(x) = \delta_V(r)[\pi(x)]\ , \quad \forall\, r \in R\ , \quad \forall\, x \in L\ , \tag{6.2}$$

where $\pi: L \to V$ is the canonical projection. Furthermore, by the Leibniz rule, the restriction map

$$\mathrm{Der}_A(R,\,\mathrm{Hom}(V,\,Q)) \xrightarrow{\sim} \mathrm{Hom}_{A^{\mathrm{e}}}(\mathrm{Der}(A, A^{\otimes 2}),\,\mathrm{Hom}(V,\,Q))$$

is an isomorphism: every derivation in $\mathrm{Der}_A(R,\,\mathrm{Hom}(V,\,Q))$ is uniquely determined by its restriction to $\mathrm{Der}(A, A^{\otimes 2})$. We thus need to compute δ_V on $\mathrm{Der}(A, A^{\otimes 2})$ only.

Let $\mathbb{C}(X\times X)^{\mathrm{reg}}$ be the subring of rational functions on $X\times X$, regular outside the diagonal of $X\times X$. Clearly, $\mathbb{C}(X\times X)^{\mathrm{reg}}$ is closed under the natural left-right multiplication by A, and thus can be viewed as a bimodule over A. Let $\flat$ denote the involution on $\mathbb{C}(X\times X)^{\mathrm{reg}}$ induced by interchanging the factors in $X\times X$.

Lemma 6.2. (1) *The assignment*

$$\nu:\ d \mapsto [\,d(a)/(a\otimes 1 - 1\otimes a)\,]^{\flat}\ , \quad a \in A\setminus\mathbb{C}\ , \tag{6.3}$$

defines an injective bimodule homomorphism $\mathrm{Der}(A, A^{\otimes 2}) \longrightarrow \mathbb{C}(X\times X)^{\mathrm{reg}}$.

(2) *If* $a \in A$, $d \in \mathrm{Der}(A, A^{\otimes 2})$ *and* $d(a) = \sum_j f_j \otimes g_j$, *then*

$$[d, a] = \sum_j g_j\, \Delta_A\, f_j\ .$$

Proof. We identify $A^{\mathrm{e}} = A\otimes A^{o}$ with the ring $\mathcal{O}(X\times X) = A\otimes A$ of regular functions on $X\times X$. Recall that $\mathrm{Der}(A, A^{\otimes 2})$ is isomorphic to $\mathrm{Hom}_{A^{\mathrm{e}}}(\Omega^1 A\,, A^{\otimes 2})$, with a homomorphism θ giving rise to the derivation d with $d(a) = \theta(a\otimes 1 - 1\otimes a)$. Geometrically, we can think of $\Omega^1(A) \subset A^{\otimes 2}$ as the vanishing ideal of the diagonal in $X\times X$. Then $\Omega^1(A)^\star := \mathrm{Hom}_{A^{\otimes 2}}(\Omega^1 A\,, A^{\otimes 2})$ can be realized as the subspace of functions in $\mathbb{C}(X\times X)^{\mathrm{reg}}$ with (at most) simple poles along the diagonal, and the canonical pairing between $\Omega^1 A$ and $\Omega^1(A)^\star$ is given by multiplication in $\mathcal{O}(X\times X)$. With this identification, the above isomorphism between $\mathrm{Der}(A, A^{\otimes 2})$ and $\mathrm{Hom}_{A^{\mathrm{e}}}(\Omega^1 A\,, A^{\otimes 2})$ becomes the map $d \mapsto d(a)/(a\otimes 1 - 1\otimes a)$. Under this map the left (respectively, right) action of A on the double derivations becomes the right (respectively, left) action of A on functions on $X\times X$. Thus, twisting this map by $\flat$ produces the bimodule homomorphism ν.

For the second part, we note that $[d, a]$ lies in the kernel of the homomorphism $R \twoheadrightarrow \Pi^0(A) = R/\langle \Delta_A\rangle$, therefore, $[d, a] = \sum_i x_i\,\Delta_A\, y_i$ with some $x_i\,, y_i \in A$.

Also, since A is commutative, it follows that $[d,a](a) = d(a)\cdot(a\otimes 1 - 1\otimes a)$. This implies that $\sum_i y_i \otimes x_i = \sum_j f_j \otimes g_j$, as needed. ☐

To compute $\delta_V(d) \in \mathrm{Hom}(V,Q)$ we identify $\mathrm{Hom}(V,Q) \cong Q\otimes V^*$. Since V is an R-module, there is a natural action of $R^{\mathrm{e}} := R\otimes R^\circ$ on this space:

$$R^{\mathrm{e}} \to Q\otimes \mathrm{End}(V^*)\ , \tag{6.4}$$

which is the tensor product of the dual representation $\varrho^* : R^\circ \to \mathrm{End}(V^*)$ with composition of the natural maps $R \twoheadrightarrow \Pi^1(A) \cong \mathcal{D} \hookrightarrow Q$. With an abuse of notation, we will write $a\otimes b^*$ for the image of an element $a\otimes b^\circ \in R^{\mathrm{e}}$ under (6.4).

Now, restricting (6.4) to $A\otimes A \subset R^{\mathrm{e}}$, we get a ring homomorphism

$$A\otimes A \longrightarrow Q\otimes \mathrm{End}(V^*)\,. \tag{6.5}$$

Since V is finite-dimensional, the elements $a\otimes 1 - 1\otimes a$ with $a\in A\setminus\mathbb{C}$ are mapped under (6.5) to invertible elements in $\mathbb{C}(X)\otimes\mathrm{End}(V^*) \subset Q\otimes\mathrm{End}(V^*)$. Hence (6.5) extends canonically to a ring homomorphism

$$\mathbb{C}(X\times X)^{\mathrm{reg}} \longrightarrow Q\otimes\mathrm{End}(V^*)\,.$$

Combining this with (6.3), we get a well-defined bimodule map

$$\nu_V : \mathrm{Der}(A, A^{\otimes 2}) \longrightarrow Q\otimes\mathrm{End}(V^*)\ , \tag{6.6}$$

which takes Δ_A to $1\otimes \mathrm{Id}_{V^*}$.

We can now express the derivation δ_V in terms of ν_V. To this end, we choose dual bases $\{\mathrm{v}_i\}$ and $\{\mathrm{w}_i\}$ for $\mathcal{I}$ and $\mathcal{I}^\vee$; by Proposition 5.1, this gives generators $\hat{a}$, $\hat{d}$, $\hat{\mathrm{v}}_i$ and $\hat{\mathrm{w}}_i$ for the algebra Π. Identifying $L_\infty \cong V_\infty \cong \mathbb{C}$, we think of $\hat{\mathrm{v}}_i$ and $\hat{\mathrm{w}}_i$ acting on $\boldsymbol{L}$ as linear maps $\mathrm{v}_i : \mathbb{C}\to L$ and $\mathrm{w}_i : L\to\mathbb{C}$, i. e. as elements of L and L^*, respectively. Similarly, when acting on $\boldsymbol{V}$, the elements $\hat{\mathrm{v}}_i$ and $\hat{\mathrm{w}}_i$ give rise to vectors $\bar{\mathrm{v}}_i \in V$ and covectors $\bar{\mathrm{w}}_i \in V^*$. Note that $\bar{\mathrm{v}}_i = \pi\mathrm{v}_i$ and $\bar{\mathrm{w}}_i\pi = \mathrm{w}_i$, where $\pi : L \twoheadrightarrow V$ is the canonical projection. Further, we fix an embedding $\mathcal{I} \hookrightarrow A$ and identify L as in Lemma 5.5. Then we twist $e : M \hookrightarrow Q$ by an automorphism of Q in such a way that $e_L(\mathrm{v}) = \mathrm{v}$ for all $\mathrm{v}\in\mathcal{I}\subset A\subset Q$. This is possible, since $e_L : L\to Q$ is an A-linear extension of e, see Lemma 6.1.

With above notation and conventions, we have

Proposition 6.1. *The derivation* $\delta_V : \mathrm{Der}(A, A^{\otimes 2}) \to Q\otimes V^*$ *is given by*

$$\delta_V(d) = \sum_i \nu_V(d)[\mathrm{v}_i\,\bar{\mathrm{w}}_i]\ . \tag{6.7}$$

Proof. First, we compute $\delta_V(d)$ for $d = \Delta_A$. By definition, we have

$$\delta_V(\Delta_A)(\pi(x)) = e_L(\Delta_A\, x) - \Delta_A\, e_L(x) = \sum_i e_L(\mathrm{v}_i\,\mathrm{w}_i\, x) = \sum_i \mathrm{v}_i\mathrm{w}_i(x)\ ,$$

where we used the fact that Δ_A acts as $1+\sum_i \mathrm{v}_i\,\mathrm{w}_i$ on L and as the identity on Q. Replacing $\mathrm{w}_i(x)$ by $\bar{\mathrm{w}}_i(\pi(x))$ and thinking of $\delta_V(\Delta_A)$ as an element of $Q\otimes V^*$, we get

$$\delta_V(\Delta_A) = \sum_i \mathrm{v}_i\bar{\mathrm{w}}_i\,.$$

Next, we observe that if $r = [d,a]$ in $\mathrm{Der}(A, A^{\otimes 2})$, then $\delta_V(r) = [\delta_V(d), a]$, since $\delta_V(a) = 0$. On the other hand, $[d,a] = \sum_j g_j\,\Delta_A\, f_j$, so $\delta_V(r) = \sum_j g_j\,\delta_V(\Delta_A)\, f_j$.

Thus, $[\delta_V(d), a] = \sum_j g_j\, \delta_V(\Delta_A)\, f_j$, or, if we think of $\delta_V(d)$ as an element of $Q\otimes V^*$,

$$(1\otimes a^* - a\otimes 1)\, \delta_V(d) = \left(\sum_j g_j \otimes f_j^*\right)\, \delta_V(\Delta_A)\,.$$

By Lemma 6.2, this implies that $\delta_V(d) = \nu_V(d)\, [\delta_V(\Delta_A)]$, as needed. □

Now, we are in position to state the main result of this section. For $\mathrm{v} \in \mathcal{I}$ and $d \in \mathrm{Der}(A, A^{\otimes 2})$, we define the following element of Q:

$$\kappa(d, \mathrm{v}) := \mathrm{v} - (1\otimes d^* - d \otimes 1)^{-1}\, \delta_V(d)\, [1 \otimes \bar{\mathrm{v}}]\ , \tag{6.8}$$

where $\bar{\mathrm{v}} = \pi(\mathrm{v}) \in V$ and $(1\otimes d^* - d\otimes 1)^{-1} \in Q \otimes \mathrm{End}(V^*)$.

Theorem 6.1. *Let $\boldsymbol{V}$ be a $\Pi^{\boldsymbol{\lambda}}(B)$-module of dimension $\boldsymbol{n} = (n, 1)$ representing a point in $\mathcal{C}_n(X, \mathcal{I})$. Then the corresponding ideal class $\omega[\boldsymbol{V}] \in \mathfrak{I}(\mathcal{D})$ can be represented by the (fractional) ideal M generated by the elements*

$$\det{}_{V^*}(1\otimes a^* - a\otimes 1)\, \mathrm{v}\,, \qquad \det{}_{V^*}(1\otimes d^* - d\otimes 1)\, \kappa(d, \mathrm{v})\ ,$$

where $a \in A$, $d \in \mathrm{Der}(A, A^{\otimes 2})$ and $\mathrm{v} \in \mathcal{I}$.

Theorem 6.1 needs some explanations.

1. Formally, the element $\kappa(d, \mathrm{v})$ is defined by (6.8) only for those $d \in \mathrm{Der}(A, A^{\otimes 2})$, for which $1 \otimes d^* - d\otimes 1$ is invertible in $Q\otimes \mathrm{End}(V^*)$. It is easy to see, however, that the product $\det_{V^*}(1\otimes d^* - d\otimes 1)\, \kappa(d, \mathrm{v}) \in M$ makes sense for all d's (cf. [BC], Remark 2, p. 83).

2. To generate M as an ideal it suffices to take a, d and v from some (finite) generating sets of the ring A, the bimodule $\mathrm{Der}(A, A^{\otimes 2})$ and the ideal $\mathcal{I}$.

Proof. To simplify the notation, we will denote the elements of $\mathcal{I}$ (resp., $\mathcal{I}^\vee$) and the corresponding elements of V (resp., V^*) by the same letter.

By formula (5.20), the class $\omega(\boldsymbol{V})$ can be represented by $\widetilde{M} = \mathrm{Ker}[\pi : L \to V]$. Our goal is to show that the two kinds of determinants given in the proposition generate $M := e_L(\widetilde{M})$.

Using the Leibniz rule, for any $r \in R$ and $m \geq 1$, we have

$$\delta_V(r^m) = \left(\sum_{s=0}^{m-1} r^s \otimes (r^*)^{m-s-1}\right) \delta_V(r) = \frac{1\otimes (r^*)^m - r^m \otimes 1}{1\otimes r^* - r\otimes 1}\, \delta_V(r)\ , \tag{6.9}$$

provided $1\otimes r^* - r\otimes 1 \in Q\otimes \mathrm{End}(V^*)$ is invertible.

Consider the characteristic polynomial $p(t) = \chi_r(t) := \det_\rho(r - t\,\mathrm{Id}_V)$ of $r \in R$ in the representation $\varrho : R \to \mathrm{End}(V)$. It is obvious that, for any $x \in L$, the element $p(r)x$ lies in the kernel of $\pi : L \twoheadrightarrow V$, thus $p(r)x \in \widetilde{M}$. To compute its image under e_L, we write

$$e_L(p(r)x) = p(r)e_L(x) + \delta_V(p(r))[1\otimes \overline{x}]\,, \qquad \overline{x} = \pi(x)\,.$$

Using (6.9), we obtain

$$\delta_V(p(r)) = \frac{1\otimes p(r^*) - p(r)\otimes 1}{1\otimes r^* - r\otimes 1}\delta_V(r) = -(p(r)\otimes 1)(1\otimes r^* - r\otimes 1)^{-1}\delta_V(r)\ ,$$

where we used that $p(t) = \chi_r(t)$ annihilates $r^* \in \mathrm{End}(V^*)$. As a result, taking $x = \mathrm{v} \in \mathcal{I}$, we get

$$e_L(\chi_r(r)\mathrm{v}) = \chi_r(r)\left(\mathrm{v} - (1\otimes r^* - r\otimes 1)^{-1}\delta_V(r)[1\otimes \bar{\mathrm{v}}]\right) \in M\,. \tag{6.10}$$

Choosing different $r \in R$, we obtain in this way various elements in M. In particular, taking $r = a \in A$, we have $\delta_V(a) = 0$, so the formula (6.10) produces the elements of the first kind $\chi_a(a)\mathrm{v} \in M$. Taking $r = d \in \mathrm{Der}(A, A^{\otimes 2})$, on the other hand, produces the elements $\chi_d(d)\kappa(d, \mathrm{v})$, which are the elements of the second kind in M.

Finally, a simple filtration argument shows that the elements $\chi_a(a)\mathrm{v}$ and $\chi_d(d)\mathrm{v}$, with a, d and v running over the generating sets of A, $\mathrm{Der}(A, A^{\otimes 2})$ and $\mathcal{I}$, generate over Π a submodule $\widetilde{N} \subset \widetilde{M}$ of finite codimension in L, therefore, $\widetilde{N} = \widetilde{M}$. Thus, the images of these elements under e_L generate $M = e_L(\widetilde{M})$. □

6.2. **Examples.**

6.2.1. *The affine line.* Let $X = \mathbb{A}^1$. Choosing a global coordinate on X, we identify $A = \mathcal{O}(X) \cong \mathbb{C}[x]$. In this case, $\mathrm{Der}(A, A^{\otimes 2})$ is a free bimodule of rank 1; as a generator of $\mathrm{Der}(A, A^{\otimes 2})$, we may take a derivation y defined by $y(x) = 1 \otimes 1$. It is easy to check that $\Delta_A = yx - xy$ in $\mathrm{Der}(A, A^{\otimes 2})$. The tensor algebra $R = T_A\mathrm{Der}(A, A^{\otimes 2})$ is isomorphic to the free algebra $\mathbb{C}\langle x, y\rangle$, and $\Pi^1(A) \cong \mathbb{C}\langle x, y\rangle/\langle xy - yx + 1\rangle$ is just the Weyl algebra $A_1(\mathbb{C})$. The map ν of Lemma 6.2 is given by

$$\nu(y) = (1 \otimes x - x \otimes 1)^{-1}\,, \qquad \nu(\Delta) = 1\,. \tag{6.11}$$

Any line bundle $\mathcal{I}$ on X is trivial, so we only need to consider $B = A[\mathcal{I}]$ with $\mathcal{I} = A$. The n-th Calogero-Moser variety $\mathcal{C}_n := \mathcal{C}_n(X, A)$ can be described as the space of equivalence classes of matrices

$$\{(\bar{X}, \bar{Y}, \bar{\mathrm{v}}, \bar{\mathrm{w}})\,:\, \bar{X} \in \mathrm{End}(\mathbb{C}^n),\, \bar{Y} \in \mathrm{End}(\mathbb{C}^n),\, \bar{\mathrm{v}} \in \mathrm{Hom}(\mathbb{C}, \mathbb{C}^n),\, \bar{\mathrm{w}} \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C})\}$$

satisfying the relation

$$\bar{Y}\bar{X} - \bar{X}\bar{Y} = \mathrm{Id}_n + \bar{\mathrm{v}}\bar{\mathrm{w}} \tag{6.12}$$

modulo the natural action of $\mathrm{GL}_n(\mathbb{C})$:

$$(\bar{X},\, \bar{Y},\, \bar{\mathrm{v}},\, \bar{\mathrm{w}}) \mapsto (g\bar{X}g^{-1},\, g\bar{Y}g^{-1},\, g\bar{\mathrm{v}},\, \bar{\mathrm{w}}g^{-1}),\ g \in \mathrm{GL}_n(\mathbb{C})\ .$$

If we choose $\mathrm{v} = 1$ as a generator of $\mathcal{I} = A$, then the ideal M of $\mathcal{D} \cong \Pi^1(A)$ corresponding to a point $(\bar{X}, \bar{Y}, \bar{\mathrm{v}}, \bar{\mathrm{w}})$ will be given by

$$M = \mathcal{D} \cdot \det(\bar{X} - x\,\mathrm{Id}_n) \ + \ \mathcal{D} \cdot \det(\bar{Y} - y\,\mathrm{Id}_n)\,\kappa\ , \tag{6.13}$$

where

$$\kappa = 1 - \bar{\mathrm{v}}^t(\bar{Y}^t - y\,\mathrm{Id}_n)^{-1}(\bar{X}^t - x\,\mathrm{Id}_n)^{-1}\bar{\mathrm{w}}^t\,. \tag{6.14}$$

This agrees with the description of ideals of $A_1(\mathbb{C})$ given in [BC].

6.2.2. *The complex torus.* Let $X = \mathbb{C}^*$. We identify $A = \mathcal{O}(X)$ with $\mathbb{C}[x, x^{-1}]$, the ring of Laurent polynomials. As in the affine line case, the bimodule $\mathrm{Der}(A, A^{\otimes 2})$ is freely generated by the derivation y defined by $y(x) = 1 \otimes 1$. The tensor algebra $R = T_A\mathrm{Der}(A, A^{\otimes 2})$ is isomorphic to the free product $\mathbb{C}\langle x^{\pm 1}, y\rangle := \mathbb{C}[x, x^{-1}] \star \mathbb{C}[y]$, and $\Delta_A = yx - xy$ in R. The matrix description of the Calogero-Moser spaces $\mathcal{C}_n$ and the formulas for the corresponding fractional ideals of $\mathcal{D} \cong \Pi^1(A) = \mathbb{C}\langle x^{\pm 1}, y\rangle/\langle xy - yx + 1\rangle$ are the same as above, except for the fact that x and $\bar{X}$ are now invertible. A new feature is that A has now nontrivial units x^r, $r \in \mathbb{Z}$. The corresponding group Λ can be identified with $\mathbb{Z}$ and its action on $\mathcal{C}_n$ is given by

$$r.(\bar{X},\, \bar{Y},\, \bar{\mathrm{v}},\, \bar{\mathrm{w}}) = (\bar{X},\, \bar{Y} + r\bar{X}^{-1},\, \bar{\mathrm{v}},\, \bar{\mathrm{w}})\ , \quad r \in \mathbb{Z}\ .$$

Thus, by Theorem 4.2, the classes of ideals of $\mathcal{D} \cong \Pi^1(A)$ are parameterized by the points of the quotient variety $\overline{\mathcal{C}}_n = \mathcal{C}_n/\mathbb{Z}$.

It is worth mentioning that one may choose a different generator for the bimodule $\mathrm{Der}(A, A^{\otimes 2})$, taking, for example, $z = yx$, instead of y. Then $\Delta_A = z - xzx^{-1}$, which leads to an alternative matrix description of $\mathcal{C}_n$ and the corresponding ideals.

6.2.3. *A general plane curve.* Let X be a smooth curve in $\mathbb{C}^2$ defined by the equation $F(x,y) = 0$, with $F(x,y) := \sum_{r,s} a_{rs}x^r y^s \in \mathbb{C}[x,y]$. In this case, the algebra $A \cong \mathbb{C}[x,y]/\langle F(x,y)\rangle$ is generated by x and y and the module $\mathrm{Der}(A)$ is (freely) generated by the derivation ∂ defined by

$$\partial(x) = F'_y(x,y)\,, \qquad \partial(y) = -F'_x(x,y)\,.$$

The bimodule $\mathrm{Der}(A, A^{\otimes 2})$ is generated by the distinguished derivation $\Delta = \Delta_A$ and the element z defined by

$$z(x) = \sum_{r,s} a_{rs}\frac{x^r y^s \otimes 1 - x^r \otimes y^s}{y \otimes 1 - 1 \otimes y} = -\frac{\sum_{r,s} a_{rs}x^r \otimes y^s}{y \otimes 1 - 1 \otimes y}\,,$$

$$z(y) = -\sum_{r,s} a_{rs}\frac{x^r \otimes y^s - 1 \otimes x^r y^s}{x \otimes 1 - 1 \otimes x} = -\frac{\sum_{r,s} a_{rs}x^r \otimes y^s}{x \otimes 1 - 1 \otimes x}\,.$$

These generators satisfy the following commutation relations

$$[z,\, x] = \sum_{r,s} a_{rs} \sum_{k=0}^{s-1} y^{s-k-1}\Delta y^k x^r\,, \tag{6.15}$$

$$[z,\, y] = -\sum_{r,s} a_{rs} \sum_{l=0}^{r-1} y^s x^{r-l-1}\Delta x^l\,. \tag{6.16}$$

By Proposition 5.1, the algebra $\Pi^{\boldsymbol{\lambda}}(B)$ is then generated by the elements $\hat{x},\ \hat{y},\ \hat{z}$, $\hat{\mathrm{v}}_i,\ \hat{\mathrm{w}}_i$ and $\hat{\Delta}$, subject to the relations (6.15), (6.16) and (5.16). The assignment $x \mapsto x,\ y \mapsto y,\ z \mapsto \partial,\ \Delta \mapsto 1$ extends to an isomorphism between $\Pi^1(A)$ and the ring $\mathcal{D}$ of differential operators on X.

The bimodule map ν of Lemma 6.2 is given by

$$\nu(z) = -\frac{\sum_{r,s} a_{rs}y^s \otimes x^r}{(1 \otimes x - x \otimes 1)(1 \otimes y - y \otimes 1)}\,, \qquad \nu(\Delta) = 1\,. \tag{6.17}$$

Let us now describe generic points of the representation varieties $\mathcal{C}_n(X, \mathcal{I})$. First, consider the case when $\mathcal{I}$ is trivial. Choose n distinct points $p_i = (x_i,\, y_i) \in X$, $i = 1, \ldots, n$, and define

(6.18)
$$(\bar{X},\, \bar{Y},\, \bar{Z},\, \bar{\mathrm{v}},\, \bar{\mathrm{w}}) \in \mathrm{End}(\mathbb{C}^n) \times \mathrm{End}(\mathbb{C}^n) \times \mathrm{End}(\mathbb{C}^n) \times \mathrm{Hom}(\mathbb{C}, \mathbb{C}^n) \times \mathrm{Hom}(\mathbb{C}^n, \mathbb{C})$$

by the following formulas

$$\bar{X} = \mathrm{diag}(x_1, \ldots, x_n)\ ,\ \bar{Y} = \mathrm{diag}(y_1, \ldots, y_n)\ ,\ \bar{\mathrm{v}}^t = -\bar{\mathrm{w}} = (1, \ldots, 1)\ , \tag{6.19}$$

$$\bar{Z}_{ii} = \alpha_i \quad \text{and} \quad \bar{Z}_{ij} = \frac{F(x_j,\, y_i)}{(x_i - x_j)(y_i - y_j)} \quad (\text{for } i \neq j)\ ,$$

where $\alpha_1, \ldots, \alpha_n$ are arbitrary scalars. Then, a straightforward calculation, using the relations (6.15) and (6.16), shows that the assignment

$$\hat{x} \mapsto \bar{X}\ , \quad \hat{y} \mapsto \bar{Y}\ , \quad \hat{z} \mapsto \bar{Z}\ , \quad \hat{\mathrm{v}} \mapsto \bar{\mathrm{v}}\ , \quad \hat{\mathrm{w}} \mapsto \bar{\mathrm{w}}\ , \quad \hat{\Delta} \mapsto \mathrm{Id}_n + \bar{\mathrm{v}}\,\bar{\mathrm{w}}$$

extends to a representation of $\Pi^{\boldsymbol{\lambda}}(B)$, with $B = A[A]$ and $\boldsymbol{\lambda} = (1, -n)$, on the vector space $\boldsymbol{V} = \mathbb{C}^n \oplus \mathbb{C}$.

Remark. The matrix $\bar{Z}$ defined above is a generalization of the classical *Moser matrix* in the theory of integrable systems (see [KKS]).

Now, let $\mathcal{I}$ be an arbitrary line bundle on X. As before, we identify $\mathcal{I}$ with an ideal in A and assume that the zero set $\mathcal{V}(\mathcal{I})$ of $\mathcal{I}$ does not include the points p_i (this can always be achieved by changing the embedding of $\mathcal{I}$ in A, if necessary). Then, $\mathcal{I}^\vee$ can be identified with a fractional ideal of A generated by rational functions with poles in $\mathcal{V}(\mathcal{I})$, and the pairing $\mathcal{I} \times \mathcal{I}^\vee \to A$ is given by multiplication in $\mathbb{C}(X)$. The evaluation of $\mathrm{v} \in \mathcal{I}$ at $p_1, \ldots, p_n$ defines a vector $\bar{\mathrm{v}} \in \mathbb{C}^n$; in a similar fashion, any $\mathrm{w} \in \mathcal{I}^\vee$ defines a row vector $\bar{\mathrm{w}} = (\bar{\mathrm{w}}_1, \ldots, \bar{\mathrm{w}}_n) \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C})$, with $\bar{\mathrm{w}}_j = -\mathrm{w}(p_j)$. If $\{\mathrm{v}_i\}$, $\{\mathrm{w}_i\}$ are dual bases for $\mathcal{I}$ and $\mathcal{I}^\vee$, then $\sum_i \mathrm{v}_i \otimes \mathrm{w}_i$ gives a rational function on $X \times X$, which we denote by ϕ; the fact that the bases are dual implies $\phi(p, p) = 1$ for all $p \in X$. As a result, the $n \times n$ matrix $\sum_i \bar{\mathrm{v}}_i \bar{\mathrm{w}}_i$ equals $\| -\phi(p_i, p_j)\|$, with all the diagonal entries being equal to -1.

Now, let $\bar{X}$ and $\bar{Y}$ be the diagonal matrices as in (6.19), and let $\bar{Z}$ be given by

$$\bar{Z}_{ii} = \alpha_i \quad \text{and} \quad \bar{Z}_{ij} = \frac{F(x_j, y_i)\,\phi(p_i, p_j)}{(x_i - x_j)(y_i - y_j)} \quad (\text{for } i \neq j)\ .$$

It is straightforward to check that the assignment

$$\hat{x} \mapsto \bar{X}\ , \quad \hat{y} \mapsto \bar{Y}\ , \quad \hat{z} \mapsto \bar{Z}\ , \quad \hat{\mathrm{v}}_i \mapsto \bar{\mathrm{v}}_i\ , \quad \hat{\mathrm{w}}_i \mapsto \bar{\mathrm{w}}_i\ , \quad \hat{\Delta} \mapsto \mathrm{Id}_n + \sum_i \bar{\mathrm{v}}_i\, \bar{\mathrm{w}}_i$$

extends to a representation of $\Pi^{\boldsymbol{\lambda}}(B)$ with $B = A[\mathcal{I}]$.

Remark. In a similar way, one can describe generic points of $\mathcal{C}_n(X,\mathcal{I})$ for an arbitrary (not necessarily plane) curve X.

To illustrate Theorem 6.1 we now describe the fractional ideal representing the class $\omega[\boldsymbol{V}]$ for an arbitrary $[\boldsymbol{V}] \in \mathcal{C}_n(X,\mathcal{I})$. Again, we consider first the case when $\mathcal{I}$ is trivial. In that case, we identify $\mathcal{I} = \mathcal{I}^\vee = A$ and choose $\mathrm{v} = \mathrm{w} = 1$ as the generators of $\mathcal{I}$ and $\mathcal{I}^\vee$. A Π-module $\boldsymbol{V} = \mathbb{C}^n \oplus \mathbb{C}$ may then be described by the matrices (6.18), which, apart from (6.15) and (6.16), satisfy the following relations

$$F(\bar{X}, \bar{Y}) = 0\ , \quad [\bar{X}, \bar{Y}] = 0 \quad \text{and} \quad \bar{\Delta} = \mathrm{Id}_n + \bar{\mathrm{v}}\,\bar{\mathrm{w}}\ .$$

The dual representation $\varrho^* : \Pi^\circ \to \mathrm{End}(\boldsymbol{V}^*)$ is given by the transposed matrices.

Now, formula (6.17) yields

$$\nu_V(z) = -(\bar{X}^t - x\,\mathrm{Id}_n)^{-1}(\bar{Y}^t - y\,\mathrm{Id}_n)^{-1}F(\bar{X}^t, y\,\mathrm{Id}_n) \in \mathbb{C}(X) \otimes \mathrm{End}(V^*)\ ,$$

and the element $\kappa = \kappa(z, 1) \in Q$, see (6.8), is given by

$$\kappa = 1 + \bar{\mathrm{v}}^t\,(\bar{Z}^t - z\,\mathrm{Id}_n)^{-1}(\bar{X}^t - x\,\mathrm{Id}_n)^{-1}(\bar{Y}^t - y\,\mathrm{Id}_n)^{-1}F(\bar{X}^t, y\,\mathrm{Id}_n)\,\bar{\mathrm{w}}^t\ . \tag{6.20}$$

Thus, if $[\boldsymbol{V}] \in \mathcal{C}_n(X, A)$ is determined by the data $(\bar{X}, \bar{Y}, \bar{Z}, \bar{\mathrm{v}}, \bar{\mathrm{w}})$, the corresponding class $\omega[\boldsymbol{V}]$ is represented by the (fractional) ideal

$$M = \mathcal{D}\cdot\det(\bar{X} - x\,\mathrm{Id}_n) + \mathcal{D}\cdot\det(\bar{Y} - y\,\mathrm{Id}_n) + \mathcal{D}\cdot\det(\bar{Z} - z\,\mathrm{Id}_n)\,\kappa\ , \tag{6.21}$$

where κ is defined by (6.20).

In the general case, when $\mathcal{I}$ is arbitrary, κ is replaced by

$$\kappa(\mathrm{v}) = \mathrm{v} + \sum_i \left(\bar{\mathrm{v}}^t(\bar{Z}^t - z\,\mathrm{Id}_n)^{-1}(\bar{X}^t - x\,\mathrm{Id}_n)^{-1}(\bar{Y}^t - y\,\mathrm{Id}_n)^{-1}F(\bar{X}^t, y\,\mathrm{Id}_n)\,\bar{\mathrm{w}}_i^t\right)\mathrm{v}_i\ ,$$

and the corresponding class $\omega[\boldsymbol{V}] \in \gamma^{-1}[\mathcal{I}]$ is represented by

$$(6.22)\quad M = \sum_i [\,\mathcal{D}\cdot\det(\bar{X}-x\,\mathrm{Id}_n)\,\mathrm{v}_i + \mathcal{D}\cdot\det(\bar{Y}-y\,\mathrm{Id}_n)\,\mathrm{v}_i + \mathcal{D}\cdot\det(\bar{Z}-z\,\mathrm{Id}_n)\,\kappa(\mathrm{v}_i)\,]\,.$$

6.2.4. *A hyperelliptic curve.* This is a special case of the plane curve X described by the equation $y^2 = P(x)$, where $P(x) = \sum_s a_s x^s$ is an arbitrary polynomial with simple roots. Some of the above formulas simplify in this case, so we write them down below.

We have $A = \mathcal{O}(X) \cong \mathbb{C}[x, y]/\langle y^2 - P(x)\rangle$, and the module $\mathrm{Der}(A)$ is freely generated by the derivation ∂ with $\partial(x) = 2y$ and $\partial(y) = P'(x)$.

The bimodule $\mathrm{Der}(A, A^{\otimes 2})$ is generated by Δ and the element z defined by

$$z(x) = y \otimes 1 + 1 \otimes y\,, \qquad z(y) = (P(x) \otimes 1 - 1 \otimes P(x))/(x \otimes 1 - 1 \otimes x)\,.$$

These generators satisfy the following relations

$$(6.23)\qquad [z,\, x] = y\Delta + \Delta y\,, \qquad [z,\, y] = \sum_s a_s \sum_{l=0}^{s-1} x^{s-l-1}\Delta x^l\,.$$

The assignment $x \mapsto x$, $y \mapsto y$, $z \mapsto \partial$, $\Delta \mapsto 1$ extends to an algebra isomorphism: $\Pi^1(A) \xrightarrow{\sim} \mathcal{D}$.

Given an ideal $\mathcal{I} \subset A$, let us fix dual bases $\{\mathrm{v}_i\}$ and $\{\mathrm{w}_i\}$ for $\mathcal{I}$ and $\mathcal{I}^\vee$, respectively. A point of the Calogero-Moser variety $\mathcal{C}_n(X, \mathcal{I})$ is determined by the following data:

1. A representation of A on the vector space $V = \mathbb{C}^n$, i.e. a pair of matrices $(\bar{X}, \bar{Y}) \in \mathrm{End}(\mathbb{C}^n) \times \mathrm{End}(\mathbb{C}^n)$, satisfying $[\bar{X}, \bar{Y}] = 0$ and $\bar{Y}^2 = P(\bar{X})$.
2. A pair of A-module maps $\mathcal{I} \to V$ and $\mathcal{I}^\vee \to V^*$. We write $\bar{\mathrm{v}}_i \in V$ and $\bar{\mathrm{w}}_i \in V^*$ for the images of the basis elements under these maps.
3. A matrix $\bar{Z} \in \mathrm{End}(\mathbb{C}^n)$, such that $\bar{X}$, $\bar{Y}$, $\bar{Z}$ and $\bar{\Delta} := \mathrm{Id}_n + \sum_i \bar{\mathrm{v}}_i\bar{\mathrm{w}}_i$ satisfy the relations (6.23). (In the simplest case $\mathcal{I} = A$, the second piece of the data is simply a pair $\bar{\mathrm{v}} \in V$, $\bar{\mathrm{w}} \in V^*$, and we have $\bar{\Delta} = \mathrm{Id}_n + \bar{\mathrm{v}}\bar{\mathrm{w}}$.)

In this case, the bimodule map ν of Lemma 6.2 is given by

$$\nu(z) = (1 \otimes y + y \otimes 1)/(1 \otimes x - x \otimes 1)\ , \quad \nu_V(z) = (\bar{X}^t - x\,\mathrm{Id})^{-1}(\bar{Y}^t + y\,\mathrm{Id})\,,$$

so that (6.20) becomes

$$\kappa = 1 - \bar{\mathrm{v}}^t(\bar{Z}^t - z\,\mathrm{Id})^{-1}(\bar{X}^t - x\,\mathrm{Id})^{-1}(\bar{Y}^t + y\,\mathrm{Id})\,\mathrm{w}^t\ ,$$

while for the general $\mathcal{I}$ we have

$$\kappa(\mathrm{v}) = \mathrm{v} - \sum_i \big(\,\bar{\mathrm{v}}^t(\bar{Z}^t - z\,\mathrm{Id})^{-1}(\bar{X}^t - x\,\mathrm{Id})^{-1}(\bar{Y}^t + y\,\mathrm{Id})\,\bar{\mathrm{w}}_i^t\,\big)\ \mathrm{v}_i\,.$$

The corresponding fractional ideals are given by the same formulas (6.21) and (6.22).

Appendix A. (by G. Wilson) Half-Forms on Riemann Surfaces

In this note I provide a proof for one of the key facts (Proposition A.1 below) needed to understand the relationship between deformed preprojective algebras and rings of differential operators. The note owes a great deal to conversations with Graeme Segal.

Statement of problem. Let X be a compact Riemann surface, and let Δ be the diagonal divisor in $X \times X$. We have the inclusion

$$\mathcal{O}_{X\times X}(-\Delta) \hookrightarrow \mathcal{O}_{X\times X}(\Delta)$$

of the sheaf of functions that vanish on Δ into the sheaf of functions that are allowed a simple pole on Δ. The quotient sheaf $\mathcal{O}_{X\times X}(\Delta)/\mathcal{O}_{X\times X}(-\Delta)$ is supported on the first infinitesimal neighbourhood Δ_1 of Δ. Similarly, if $\mathcal{L}$ is a line bundle on X, we have the sheaf $\mathcal{D}_1(\mathcal{L})$ of differential operators of order ≤ 1 on $\mathcal{L}$. This is usually regarded as a sheaf on X, but since we can compose a differential operator with a function either on the left or on the right, it has two commuting structures of $\mathcal{O}_X$-module, so it too can be regarded as a sheaf on $X \times X$, again supported on Δ_1.

Fix a square root $\Omega^{1/2}$ of the canonical bundle Ω_X; the choice of square root will be immaterial, because the corresponding sheaves of differential operators $\mathcal{D}(\Omega^{1/2})$ are canonically isomorphic to each other. Our aim is to understand the following fact stated in [G].

Proposition A.1. *There is a canonical isomorphism (of sheaves over $X \times X$)*

$$\chi : \mathcal{O}_{X\times X}(\Delta)/\mathcal{O}_{X\times X}(-\Delta) \to \mathcal{D}_1(\Omega^{1/2}) .$$

A consequence is that the sheaf of deformed preprojective algebras formed from $\mathcal{O}_X$ is canonically isomorphic to the the sheaf $\mathcal{D}(\Omega^{1/2})$ of differential operators on $\Omega^{1/2}$. This is explained in [G], Section 13.

The isomorphism in Proposition A.1 does not seem to be a well-known fact, and at first sight looks puzzling, because there are no half-forms in the left hand side. The proof sketched in the current version of [G] is not very convincing, so it seems worth recording the following simple explanation shown to me by Segal: although Proposition A.1 itself does not look familiar, it can be obtained by combining two familiar facts, of a slightly different nature. While we are about it, we shall deal also with a slight generalization, twisting by an arbitrary line bundle $\mathcal{L}$ on X.

We use the following notation: Δ_n is the nth infinitesimal neighbourhood of the diagonal in $X \times X$, so that we have a canonical identification

$$\mathcal{L} \,/\, \mathcal{L}(-(n+1)\Delta) \;\simeq\; \mathcal{L} \,|\, \Delta_n \,. \tag{A.1}$$

The two projections $X \times X \to X$ are denoted by p_1 and p_2. If U is a simply-connected coordinate patch on X and z is a parameter on U, we write (z_1, z_2) for the induced parameters on $U \times U \subset X \times X$. The parameter z determines a trivialization (non-vanishing section) dz of $\Omega_X \,|\, U$. Fixing also an isomorphism[7] $\kappa : (\Omega^{1/2})^{\otimes 2} \simeq \Omega_X$, we may choose a trivialization $dz^{1/2}$ of $\Omega^{1/2} \,|\, U$ such that $\kappa(dz^{1/2} \otimes dz^{1/2}) = dz$ (there are only two choices, differing by a sign).

A proof of Proposition A.1. Let $\mathcal{L}$ be a line bundle on X.

Proposition A.2. *There is a canonical identification (of sheaves over $X \times X$)*

$$p_1^*(\mathcal{L}) \otimes p_2^*(\mathcal{L}^* \otimes \Omega_X)\,((n+1)\Delta) \,|\, \Delta_n \;\simeq\; \mathcal{D}_n(\mathcal{L}) \,. \tag{A.2}$$

[7]Of course κ is uniquely determined up to a constant multiple. The isomorphism χ in Proposition A.1 does not depend on this multiple, but some of the intermediate steps below do.

Proof. The action of a (local) section of the sheaf on the left of (A.2) on a section of $\mathcal{L}$ is given by contracting with the factor $p_2^*(\mathcal{L}^*)$ and then taking the residue on the diagonal of the resulting differential. Let us spell that out in more detail in the case where $\mathcal{L}$ is the trivial bundle and $n = 1$. The sheaf on the left of (A.2) is then just $p_2^*(\Omega_X)(2\Delta) \,|\, \Delta_1 \;=\; p_2^*(\Omega_X)(2\Delta) \,/\, p_2^*(\Omega_X)$. In terms of a parameter z, a local section of this sheaf has the form

$$\frac{\varphi(z_1, z_2)\, dz_2}{(z_2 - z_1)^2} \quad \text{modulo regular terms}$$

(where φ is regular). To see how this acts on a function $f(z)$, we have to calculate the residue

$$\mathrm{res}_{z_2 = z_1} \frac{f(z_2)\varphi(z_1, z_2) dz_2}{(z_2 - z_1)^2}$$

(z_1 is held fixed during the calculation). Expanding

$$f(z_2) = f(z_1) + f'(z_1)(z_2 - z_1) + \ \dots\ ,$$

and

$$\frac{\varphi(z_1, z_2)}{(z_2 - z_1)^2} = \frac{a(z_1)}{(z_2 - z_1)^2} + \frac{b(z_1)}{z_2 - z_1} + \ \dots\ ,$$

we find that the residue is

$$a(z)\frac{df}{dz} + b(z) f \Big|_{z = z_1} .$$

The proposition is now clear. □

Now let U be a coordinate patch on X. We consider the classical[8] differential γ given in terms of a parameter z by

$$\text{(A.3)} \qquad \gamma \ := \ \frac{dz_1^{1/2} dz_2^{1/2}}{z_1 - z_2} .$$

It is a non-vanishing section (over $U \times U$) of the line bundle

$$p_1^*(\Omega^{1/2}) \otimes p_2^*(\Omega^{1/2})(\Delta) .$$

It depends on the parameter z; however, its restriction to Δ does not. Indeed, when we identify $\mathcal{O}_{X \times X}(-\Delta) \,|\, \Delta$ with the canonical bundle on the diagonal, $z_1 - z_2$ corresponds to dz, so $\gamma \,|\, \Delta$ becomes the constant section $1 \in \mathcal{O}(U)$. Furthermore, because γ is skew in the two variables, its restriction to Δ_1 is also independent of the choice of z. Thus for any sheaf $\mathcal{M}$ over $X \times X$, multiplication by γ gives a well-defined global isomorphism

$$\mathcal{M} \,|\, \Delta_1 \ \simeq\ \mathcal{M} \otimes p_1^*(\Omega^{1/2}) \otimes p_2^*(\Omega^{1/2})(\Delta) \,|\, \Delta_1 .$$

In particular, for any line bundle $\mathcal{L}$ over X, we get an isomorphism

$$\text{(A.4)} \qquad p_1^*(\mathcal{L}) \otimes p_2^*(\mathcal{L}^*)(\Delta) \,|\, \Delta_1 \ \simeq\ p_1^*(\mathcal{L} \otimes \Omega^{1/2}) \otimes p_2^*(\mathcal{L}^* \otimes \Omega^{1/2})(2\Delta) \,|\, \Delta_1 .$$

Tensoring our chosen isomorphism $\kappa : (\Omega^{1/2})^{\otimes 2} \simeq \Omega_X$ with $(\Omega^{1/2})^*$, we get an isomorphism $\Omega^{1/2} \ \simeq\ (\Omega^{1/2})^* \otimes \Omega_X$, and hence for any $\mathcal{L}$ an isomorphism

$$\mathcal{L}^* \otimes \Omega^{1/2} \ \simeq\ (\mathcal{L} \otimes \Omega^{1/2})^* \otimes \Omega_X .$$

---

[8]it is the principal part of the *Szegö kernel* on $X \times X$.

Inserting this into (A.4) and taking account of (A.1) and (A.2) gives us an isomorphism (now independent of κ)

$$p_1^*(\mathcal{L}) \otimes p_2^*(\mathcal{L}^*)(\Delta) \,/\, p_1^*(\mathcal{L}) \otimes p_2^*(\mathcal{L}^*)(-\Delta) \;\simeq\; \mathcal{D}_1(\mathcal{L} \otimes \Omega^{1/2})\,. \tag{A.5}$$

Taking $\mathcal{L} = \mathcal{O}_X$, we get Proposition A.1.

Remarks. 1. Reversing the arguments in [G], we easily get from (A.5) a construction of any $\mathcal{D}(\mathcal{L})$ as a sheaf of "twisted deformed preprojective algebras".

2. The differential γ in (A.3) is invariant under a *linear fractional* change of parameter. Thus if we fix a projective structure on X (thought of as an altas with linear fractional transition functions), then γ is well-defined globally on some analytic neighbourhood of Δ, not merely on Δ_1. This remark is the starting point for the papers [BR].

3. The considerations above give an explicit formula for the isomorphism χ in Proposition A.1: an element of $\mathcal{O}_{X\times X}(\Delta)/\mathcal{O}_{X\times X}(-\Delta)$ has a unique local representative of the form

$$a(z_1)(z_2 - z_1)^{-1} + b(z_1) \;+\; \ldots,$$

and χ maps this to the operator

$$f\,dz^{1/2} \;\mapsto\; \Big(a(z)\frac{df}{dz} \;+\; b(z)f\Big)\,dz^{1/2}\ .$$

In an earlier version of this note I verified Proposition A.1 by checking directly that the map χ defined by this formula is independent of the chosen parameter z; however, the calculation is surprisingly complicated (and unilluminating).

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