ALGEBRO-GEOMETRIC SCHRÖDINGER OPERATORS IN MANY DIMENSIONS

OLEG CHALYKH

ABSTRACT. We collect known results about the Schrödinger operators $L = -\Delta + u(x)$ generalizing to higher dimension those algebro-geometric operators $L = -\frac{d^2}{dx^2} + u(x)$ with rational, trigonometric and elliptic potential which appear in the finite-gap theory.

1. INTRODUCTION

The present paper is a survey of the results of [1]–[5] on higher-dimensional analogues of the so-called *algebro-geometric* operators of the form

$$L = -\frac{d^2}{dx^2} + u(x) \tag{1.1}$$

which appeared in the framework of the finite-gap theory in 70-s. A great achievement of that theory was constructing explicit solutions for various nonlinear integrable equations, for instance KdV, and the spectral theory of the operator (1.1) was an important tool of the theory, see [6] and references therein.

Apart from their relation to the KdV equation, the algebro-geometric Sturm–Liouville operators (1.1) are interesting for a lot of other reasons (commutative rings of differential operators, exact solutions, finite-gapness, interesting moduli spaces). And while there is no completely satisfactory analogue of the finite-gap theory in many dimensions, what we shall explain below should convince the reader that there is a nice class of multidimensional Schrödinger operators exhibiting many if not all of the characteristic properties of the one-dimensional algebro-geometric operators.

The approach we will follow has been developed starting from the work of Veselov and the author [1]. It was motivated by the results of Dubrovin–Krichever–Novikov [7] and Veselov–Novikov [8, 9], who suggested a nice theory of the operators in two variables which are algebro-geometric on *one energy level*. That made natural the following question posed by Novikov: find those of these operators which are algebro-geometric on *every* energy level. The theory [7]–[9] predicted that such operators must be rare and exceptional, which made a task of finding them rather tricky. A starting point for us was the idea that, perhaps, such examples might be related to other exceptional mathematical objects like simple Lie algebras or root systems. And while this has been partly confirmed in the follow-up papers [2, 10] (see also a recent paper [11]), the whole story turned out to be more complex than we initially thought. Before proceeding, let us also mention the papers [12]–[19] devoted to other approaches to generalizing the finite-gap theory to higher dimensions.

We start by recalling that one of the characteristic features of the algebro-geometric operators $L = -\frac{d^2}{dx^2} + u(x)$ is the existence of nontrivial operators commuting with L. Namely, such L are completely characterized by the property that they admit a commuting operator A of odd order:

$$[L, A] = 0, \qquad A = \frac{d^{2n+1}}{dx^{2n+1}} + \dots . \tag{1.2}$$

In other words, L is a member of a nontrivial commutative ring of ordinary differential operators. An elementary lemma due to Burchnall and Chaundy says that L and A are related by an algebraic relation $P(L, A) = 0$, so the commutative ring generated by L, A is isomorphic to $\mathbb{C}[\lambda, \mu]/\{P(\lambda, \mu) = 0\}$, the coordinate ring of an affine algebraic curve. This is a part of the famous Burchnall–Chaundy–Krichever correspondence between the commutative rings of ordinary differential operators and algebraic curves, see [34]. This theory is the most complete in the case of rings of *rank one*, when the common eigenspace of the operators from the ring is one-dimensional. This suggests that similarly one should be interested in commutative rings R of partial differential operators in n variables such that $\mathrm{Spec} R$ is a n-dimensional affine variety and that the common eigenspace of the operators from R is one-dimensional (see Section 3.2 below for a precise definition). A Schrödinger operator $L = -\Delta + u$ is called *algebraically integrable* if it is a member of such a ring.

One can construct trivial examples of algebraically integrable operators by taking direct sums $L = L_1(x, \partial_x) + L_2(y, \partial_y)$ of two one-dimensional algebro-geometric L_1, L_2. The first nontrivial examples were found in [1, 10]. They are related to the quantum Calogero–Moser system and have the form

$$L = -\Delta + u(x), \qquad u = \sum_{i<j} 2m(m+1)(x_i - x_j)^{-2}, \tag{1.3}$$

where m is integer. One can associate a version of the Calogero–Moser system to any root system or Coxeter group [20], and this gives more examples of algebraically integrable Schrödinger operators [2].

Note that the Calogero–Moser operator (1.3) can be viewed as a multivariable version of the operator

$$L = -\frac{d^2}{dx^2} + \frac{m(m+1)}{x^2} .$$

The latter is a member of a nice family, related to the rational solutions of KdV. They all have the form

$$u(x) = \sum_{i=1}^{N} m_i(m_i + 1)(x - x_i)^{-2}, \quad m_i \in \mathbb{Z}_+ , \tag{1.4}$$

and can be characterized by the property [21] that the solutions of the equation

$$-f'' + uf = \lambda f$$

are single-valued in the complex domain for any λ. The latter property is equivalent to the following overdetermined system of algebraic relations on the position of the poles x_i and their 'multiplicities' m_i:

$$\sum_{i:\,i\neq j} \frac{m_i(m_i + 1)}{(x_i - x_j)^{2s+1}} = 0 \quad \text{for every} \quad j = 1, \dots, N \quad \text{and} \quad s = 1, \dots, m_j . \tag{1.5}$$

A surprising fact, first observed in [3] and further explored in [4], was that very similar conditions are responsible for the algebraic integrability in higher dimension. To illustrate this parallel, let us formulate one result. We will assume that the potential $u(x)$ has the following form, generalizing (1.4) in a natural way:

$$u(x) = \sum_{\alpha \in \mathcal{A}} \frac{m_\alpha (m_\alpha + 1)(\alpha, \alpha)}{(\alpha, x)^2} \, . \tag{1.6}$$

Here $\mathcal{A}$ is a finite collection of non-parallel, non-isotropic vectors $\alpha \in \mathbb{C}^n$ with prescribed multiplicities $m_\alpha \in \mathbb{Z}_+$, and $(\cdot, \cdot)$ stands for the standard Euclidean product in $\mathbb{C}^n$.

Impose next the following conditions ('locus relations') onto the set $\mathcal{A}$ and the multiplicities m_α: for each $\alpha \in \mathcal{A}$ and every $s = 1, \ldots, m_\alpha$

$$\sum_{\beta: \, \beta \neq \alpha} \frac{m_\beta (m_\beta + 1)(\beta, \beta)(\alpha, \beta)^{2s-1}}{(\beta, x)^{2s+1}} \equiv 0 \qquad \text{along the hyperplane} \ \ (\alpha, x) = 0 \, . \tag{1.7}$$

The parallel between these conditions and (1.5) is transparent. Such configurations $\{\alpha, m_\alpha\}$ were called 'locus configurations' in [4].

Before formulating the theorem, let us introduce the ring of *quasi-invariants* $Q^{\mathcal{A}}$, as the ring of polynomials $f(k)$, $k \in \mathbb{C}^n$, satisfying the following conditions: for every $\alpha \in \mathcal{A}$

$$f(s_\alpha k) - f(k) = o((\alpha, k)^{2m_\alpha}) \quad \text{near the hyperplane } (\alpha, k) = 0 \, .$$

Here s_α is the orthogonal reflection with respect to the hyperplane $(\alpha, k) = 0$.

The following theorem follows from the results of [4], see Theorems 3.11, 3.15 below.

Theorem 1.1. *For any locus configuration the corresponding operator $L = -\Delta + u(x)$ is algebraically integrable. Moreover, L is a member of a commutative ring of partial differential operators, which is isomorphic to the ring $Q^{\mathcal{A}}$ of quasi-invariants.*

As we will explain below (see Theorem 3.15), this ring is a *maximal* commutative ring. Furthermore, the commuting operators L_f and their common eigenfunction can be given by rather explicit formulas. Notice that the ring $Q^{\mathcal{A}}$ replaces the coordinate ring of the *spectral curve* $P(\lambda, \mu) = 0$ which we have in dimension one. Thus, one should think of $X = \mathrm{Spec} Q^{\mathcal{A}}$ as the 'spectral variety' for L. Informally speaking, X looks like $\mathbb{C}^n$ with cusps along hyperplanes $(\alpha, x) = 0$. These varieties have been studied in many important cases, see [26]–[29].

Theorem 1.1 and its generalizations are the main subject of the present survey which is organized as follows. In Section 2 we review the one-dimensional algebro-geometric operators (1.1) for the cases when u is rational, trigonometric and elliptic, highlighting those of their features which extend to higher dimension. In Section 3 we consider the case of $L = -\Delta + u$ with rational potential. In the process, we define algebraic integrability and prove Theorem 1.1. Trigonometric and elliptic cases are treated in Sections 4 and 5, respectively. Note that as a by-product, we obtain a proof of a conjecture due to Gezstesy–Weikard [30], see Theorem 5.9 below. In Section 6 we list known irreducible examples of the operators studied in Sections 3-5. The last section contains some concluding remarks.

2. One-dimensional case

In this section we recall known facts about the algebro-geometric operators in dimension one. Let $L = -\frac{d^2}{dx^2} + u(x)$ be a one-dimensional Schrödinger operator with (locally) analytic potential u. It is called **algebro-geometric**, or 'finite-gap', if there is an ordinary differential operator A of odd order, commuting with L. This terminology is motivated by a remarkable fact, observed by Novikov [22], that in the case when u is real-valued smooth and periodic, the existence of A implies that the spectrum of L in $L^2(\mathbb{R})$ has a finite number of instability intervals ('gaps'). Further, according to the celebrated Its–Matveev formula [23], u can be expressed via multivariable theta-function on the Jacobian of an algebraic curve, thus 'algebro-geometric'. The following three families of potentials have a rich intrinsic algebraic structure associated to them and are of particular interest, namely, rational, trigonometric and elliptic u. These and their higher dimensional analogues will be the main subject of this survey.

Let us begin with the class of rational $u(x)$. In that case one can give the following characterization of the algebro-geometric operators.

Theorem 2.1. *Assume that $L = -\frac{d^2}{dx^2} + u(x)$ has a rational potential with $u(\infty) = 0$. Then the following statements about L are equivalent:*

(i) L is algebro-geometric;

(ii) L can be obtained from $L_0 = -\frac{d^2}{dx^2}$ by applying classical Darboux transformations at the zero level;

(iii) There exists a differential operator D with rational coefficients such that $LD = DL_0$ where $L_0 = -\frac{d^2}{dx^2}$;

(iv) (exact solvability) L has an eigenfunction $\psi(k,x)$ of the form $\psi = P(k,x)e^{kx}$ such that $L\psi = -k^2\psi$ where $P(k,x)$ is polynomial in k and rational in x;

(v) (trivial local monodromy) For any $\lambda \in \mathbb{C}$ all solutions $f(x)$ of the equation $Lf = \lambda f$ are meromorphic in the complex domain.

Equivalence of $(i) - (ii)$ is well known from the theory of the KdV equation, see [24, 25]; (iii) follows from (ii) since D factors as a product of successive first-order Darboux transformations; equivalence of (iii) and (iv) is expressed by the formula $\psi = D(e^{kx})$; an implication from (iv) to (v) is almost obvious; finally, implication from (v) to (ii) is the result of Duistermaat and Grünbaum [21].

Following [21], one can give an explicit formulation of the property (v) in the theorem. First, (v) implies that L has regular singularities. This, together with the decay at infinity, leads to the formula (1.4). After that the local analysis at each pole leads to the conditions (1.5). In generic situation all $m_i = 1$ and the number of the poles is $N = m(m+1)/2$ for some $m \in \mathbb{Z}$. In that case $u = \sum_{i=1}^{N} 2(x-x_i)^{-2}$ and the system of equations (1.5) reduces to the following:

$$\sum_{i:\, i \neq j} (x_i - x_j)^{-3} = 0 \quad \text{for every} \quad j = 1, \ldots, N. \tag{2.1}$$

This is the so-called *rational locus* introduced in [24]. It describes the equilibrium points of the classical rational Calogero–Moser system of N particles. The set of

solutions to (2.1) is empty unless $N = m(m+1)/2$, while for each $N = m(m+1)/2$ the solutions depend on m complex parameters and can be given explicitly, see e.g. [25].

Remark 2.2. Theorem 2.1 has a nice generalization [31] to the case when $u - x^2 \to 0$ as $x \to \infty$, see also [32] for the multidimensional version.

Let us turn now to the trigonometric case, i.e. when u is a rational function of $z = e^{2ix}$. We can formulate the following analog of the previous result.

Theorem 2.3. *Assume that $L = -\frac{d^2}{dx^2} + u(x)$ has a trigonometric potential $u(x)$, such that $u(-i\infty) = u(+i\infty) = 0$. Then the following statements about L are equivalent:*

(i) L is algebro-geometric;

(ii) L can be obtained from $L_0 = -\frac{d^2}{dx^2}$ by applying classical Darboux transformations at the levels of the form $\lambda = -n^2$ with $n \in \mathbb{Z}$;

(iii) There exists a differential operator D with trigonometric coefficients such that $LD = DL_0$ where $L_0 = -\frac{d^2}{dx^2}$;

(iv) (exact solvability) L has an eigenfunction $\psi(k, x)$ of the form $\psi = P(k, x)e^{kx}$ such that $L\psi = -k^2\psi$ where $P(k, x)$ is polynomial in k and trigonometric in x;

(v) (trivial local monodromy) For any $\lambda \in \mathbb{C}$ all solutions $f(x)$ of the equation $Lf = \lambda f$ are meromorphic in the complex domain.

Essentially, this is a result of [33]. An implication from (v) to (iv) is less straightforward (it does not follow from the standard facts of the KdV theory). An alternative proof of this follows from Theorem 4.2 below.

Again, one can reformulate (v), using the local analysis of the equation $Lf = \lambda f$. We will make use of the following definition.

Definition 2.4. Given a point $x_0 \in \mathbb{C}$ and $m \in \mathbb{Z}_+$, let us say that a function $f(x)$ is m-quasi-invariant, or simply **quasi-invariant** at x_0 if $f(sx) - f(x) = o((x - x_0)^{2m})$ near $x = x_0$, where s denotes the symmetry across x_0, i.e. $sx = 2x_0 - x$.

This property means the absence of odd terms, up to $(x - x_0)^{2m+1}$, in the Laurent expansion of f at x_0. Under such terminology, the conditions (1.5) express the quasi-invariance of u at each of the poles.

The assumptions of the theorem together with (v) imply that u has the form

$$u(x) = \sum_{i=1}^{N} m_i(m_i + 1)\sin^{-2}(x - x_i), \quad m_i \in \mathbb{Z}_+, \qquad (2.2)$$

with x_i distinct modulo π, and similar to (1.5), we conclude that u must be m_i-quasi-invariant at each of the poles x_i.

Since the algebro-geometric operators of the form (2.2) can be obtained from $L_0 = -\frac{d^2}{dx^2}$ by a sequence of Darboux transformations, one gets an explicit formula for the potential $u(x)$. As a result, the set of such operators breaks into separate families, each labeled by a finite sequence of integers $0 < k_1 < \cdots < k_n$. Such a family depends on n extra nonzero complex parameters ('phases'), and the generic $u(x)$ in this family has $N = k_1 + \cdots + k_n$ poles x_i with all $m_i = 1$ (in general, $k_1 + \cdots + k_n = \sum m_i(m_i + 1)/2$). For these poles we have the following system of

equations ('trigonometric locus' from [24]):

$$\sum_{i:\,i\neq j} \cos(x_i - x_j)\sin^{-3}(x_i - x_j) = 0 \quad \text{for every } j = 1, \ldots, N. \qquad (2.3)$$

Thus, in contrast with the rational case, for a fixed N the variety (2.3) has several components, corresponding to different partitions of N (each being $(\mathbb{C}^\times)^n$ topologically), with the KdV flows acting transitively on every component.

Finally, let us consider the elliptic case, i.e. when u is an elliptic function. The following elegant characterization of the algebro-geometric operators with elliptic potential is the result of Gesztesy and Weikard [30].

Theorem 2.5. *An operator $L = -\frac{d^2}{dx^2} + u(x)$ with an elliptic potential is algebro-geometric if and only if it has trivial local monodromy, i.e. for any $\lambda \in \mathbb{C}$ all solutions $f(x)$ of the equation $Lf = \lambda f$ are meromorphic in the complex domain.*

Again, using the local analysis, one shows that such u must be of the form

$$u(x) = \sum_{i=1}^{N} m_i(m_i + 1)\wp(x - x_i), \quad m_i \in \mathbb{Z}_+, \qquad (2.4)$$

with x_i distinct modulo periods, and similar to (1.5), u must be m_i-quasi-invariant at each of its poles x_i.

Generically, all $m_i = 1$ and the quasi-invariance of u reduces to the following system ('elliptic locus' from [24]):

$$\sum_{i:\,i\neq j} \wp'(x_i - x_j) = 0 \quad \text{for every } j = 1, \ldots, N. \qquad (2.5)$$

The structure of the components of the elliptic locus is much more complicated, see [24] and [35]–[37].

Summing up, we see that in all three cases (rational, trigonometric, elliptic) triviality of the local monodromy singles out the algebro-geometric operators. Note that this happens only for specific analytic classes of potentials, otherwise this would not be very restrictive, for instance, any operator with holomorphic u has trivial local monodromy (it has no singular points at all).

In the next sections we will generalize these results to higher dimension, following [3]–[5]. In particular, this will provide alternative proofs in dimension one.

3. RATIONAL CASE

3.1. **Exact solvability.** Let us consider a Schrödinger operator in $\mathbb{C}^n$ $L = -\Delta + u(x)$ with rational potential **decaying at infinity**, i.e. $u = p(x)/q(x)$ where $\deg q > \deg p$. For convenience and in analogy with the one-dimensional case, we will denote this symbolically as $u(\infty) = 0$.

Definition 3.1. Let us call $L = -\Delta + u$ with meromorphic potential u **exactly solvable** if it admits an eigenfunction $\psi(k, x) = P(k, x)e^{(k, x)}$, $k, x \in \mathbb{C}^n$, with $L\psi = -k^2\psi$, where P is polynomial in k and meromorphic in x. Equivalently, there exists a PDO $D(x, \partial/\partial x)$ such that $L \circ D = D \circ (-\Delta)$.

We want to give a characterization of all exactly solvable operators L, similar to Theorem 2.1. The first interesting fact is that the highest symbol $-\Delta$ of L puts serious restrictions onto geometry of singularities of u. The following result is due to Berest and Veselov [38, 39].

Theorem 3.2. *If $L = -\Delta + u(x)$ with meromorphic u is exactly solvable then the singularities of u form a union of hyperplanes in $\mathbb{C}^n$.*

Next, we need the following analytic result, see [4], Theorem 2.2.

Lemma 3.3. *Let $L = -\Delta + u$ be exactly solvable. Let Π be one of the hyperplanes where u has poles. Then*

(1) Π is non-isotropic, thus in suitable Cartesian coordinates $x_1, \ldots, x_n$ it is given by the equation $x_1 = 0$;

(2) u has the form $u = m(m+1)x_1^{-2} + u_0(x)$ where m is an integer and u_0 is regular along Π;

(3) the first m odd derivatives of u_0 with respect to x_1, i.e. $\partial_1 u_0, \ldots, \partial_1^{2m-1} u_0$, vanish along Π.

Properties (2) and (3) mean that the potential u is **m-quasi-invariant** with respect to the hyperplane $\Pi : x_1 = 0$, in the sense of the following definition.

Definition 3.4. Given a non-isotropic hyperplane $\Pi \subset \mathbb{C}^n$ and an integer $m \in \mathbb{Z}_+$, we say that a function f of $x \in \mathbb{C}^n$ is m-quasi-invariant with respect to Π, if $f(x) - f(sx)$ has zero of order $\geq 2m+1$ along Π. Here s is the orthogonal reflection with respect to Π.

In dimension one, the quasi-invariance of u guarantees the triviality of the local monodromy. A similar interpretation exists in higher dimension [4].

Suppose now that $u(x)$ is rational with $u(\infty) = 0$ and such that $L = -\Delta + u$ is exactly solvable. Then, according to the previous two results, its only singularities are the second-order poles along a finite collection of hyperplanes. Each of these hyperplanes in standard coordinates can be given by its equation $\alpha(x) := a_0 + a_1 x_1 + \cdots + a_n x_n = 0$, so α is, effectively, a pair (α_0, a_0) where $a_0 = \alpha(0) \in \mathbb{C}$ and $\alpha_0 = \mathrm{grad}\,\alpha = (a_1, \ldots, a_n)$. Thus, we arrive at a finite set $\mathcal{A} = \{\alpha\}$ of affine-linear functions, encoding the poles of the potential. Further, due to the lemma, the singular part of u along $\Pi_\alpha : \alpha(x) = 0$ must be of the form $m_\alpha(m_\alpha + 1)(\alpha_0, \alpha_0)(\alpha(x))^{-2}$, for some integer m_α. Together with the decay at infinity, this allows us to conclude that the potential $u(x)$ of any exactly solvable operator must be of the form

$$u(x) = \sum_{\alpha \in \mathcal{A}} \frac{m_\alpha(m_\alpha + 1)(\alpha_0, \alpha_0)}{((\alpha(x))^2}, \quad m_\alpha \in \mathbb{Z}_+. \tag{3.1}$$

Additionally, part (3) of the lemma says that u must be m_α-quasi-invariant with respect to $\Pi_\alpha : \alpha(x) = 0$.

These are necessary conditions for exact integrability of L. Remarkably, they turn out to be sufficient, as was shown in [3].

Theorem 3.5. *If the potential u of a Schrödinger operator L has the form (3.1) and is m_α-quasi-invariant with respect to each Π_α, then L is exactly integrable.*

For the proof see [3]. The proof is effective and gives an explicit formula for the eigenfunction $\psi(k, x)$ such that $L\psi = -k^2 \psi$. Namely, one has:

$$\psi = (2^M M!)^{-1}(L + k^2)^M \left[\prod_{\alpha \in \mathcal{A}} (\alpha(x))^{m_\alpha} e^{(k,x)} \right], \quad M = \sum_{\alpha \in \mathcal{A}} m_\alpha. \tag{3.2}$$

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In this formula the differential operator $(L+k^2)^M$ is applied to the quasipolynomial in the brackets. Such formula was first discovered for the Calogero–Moser systems by Berest [40], and it has inspired the results of [3].

Remark 3.6. From (3.2) it is easy to deduce that ψ has a form $\psi = P(k,x)e^{(k,x)}$ where P is rational in x and polynomial in k with the highest term $P_0(k) = \prod_{\alpha \in \mathcal{A}} (\alpha_0, k)^{m_\alpha}$. Equivalently, one can say that ψ has the form $\psi = De^{(k,x)}$ where D is differential operator in x with the highest symbol $P_0(\partial) = \prod_{\alpha \in \mathcal{A}} (\alpha_0, \partial)^{m_\alpha}$.

If all the hyperplanes pass through the origin, they are determined by the set $\mathcal{A}$ of their normals α, i.e. each of the hyperplanes is of the form $\Pi_\alpha : (\alpha, x) = 0$. In this case $u(x)$ is given by the formula (1.6) and the quasi-invariance reduces to (1.7).

Combining all these results together we get the following multidimensional generalization of Theorem 2.1.

Theorem 3.7. *Let $L = -\Delta + u$ be a Schrödinger operator with rational potential $u(x)$ such that $u(\infty) = 0$. Then L is exactly solvable if and only if u has a form (3.1) and is m_α-quasi-invariant with respect to each $\Pi_\alpha : \alpha(x) = 0$.*

3.2. **Algebraic integrability.** In this section we discuss the notion of algebraic integrability for the Schrödinger operators following [1, 2, 10, 41]. In particular, we demonstrate that all operators from Theorem 3.7 are algebraically integrable, which strengthens the analogy between the results above and the one-dimensional case.

We start by recalling that a **quantum completely integrable** (or Liouville integrable) **system** (QCIS) with n degrees of freedom is given by a set of n algebraically independent, pairwise commuting partial differential operators in n variables $L_1, \ldots, L_n$: $[L_i, L_j] = 0$. Many examples of such systems are obtained as a quantization of classical completely integrable Hamiltonian systems. In many interesting cases the quantized Hamiltonian takes the form of a Schrödinger operator $L_1 = -\Delta + u(x)$, and $L_2, \ldots, L_n$ should be thought of as *quantum integrals* of the corresponding quantum system. The main problem usually is to compute the spectrum and the eigenstates of the system, that is the eigenfunctions ψ in an appropriate Hilbert space such that $L_i \psi = \lambda_i \psi$.

From the algebraic point of view, however, we can think of L_i as operators with analytic coefficients in some open chart of $\mathbb{C}^n$ or, more generally, some smooth algebraic variety X. From that point of view, a QCIS is an embedding

$$\mathbb{C}[\lambda_1, \ldots, \lambda_n] \to D(X), \quad \lambda_i \mapsto L_i,$$

of the ring of polynomials into the ring of differential operators on X. More generally, one can consider an embedding

$$\theta : \mathcal{O}(\Lambda) \to D(X),$$

where Λ is an affine variety of $\dim(\Lambda) = n$ and $\mathcal{O}(\Lambda)$ denotes the coordinate ring of Λ. (The discussion above corresponds to the case of Λ being an affine space $\mathbb{C}^n$.)

Given a quantum system like this, one considers for any point $\lambda \in \Lambda$ the corresponding 'eigenvalue problem' $\theta(g)\psi = g(\lambda)\psi, \forall g \in \mathcal{O}(\Lambda)$. The dimension of the local solution space of this system at generic point of X for generic λ is called the **rank** of a QCIS. Recall further, that a QCIS $S = (\Lambda, \theta)$ is **algebraically integrable** if it is dominated by another QCIS $S' = (\Lambda', \theta')$ of **rank one** (S is

dominated by S' if there is an algebra homomorphism $h : \mathcal{O}(\Lambda) \to \mathcal{O}(\Lambda')$ such that $\theta = \theta' \circ h$). This is the definition from [41], the original definition in [1, 2] was slightly different, though essentially equivalent to it for the case when the operators have constant highest symbols. If the rank of the original system is greater than one (as it usually is), then its algebraic integrability implies that, apart from the operators $L_1, \ldots, L_n$, we have additional commuting operators which are not algebraic combinations of L_i (though they are algebraically dependent with L_i). In dimension $n = 1$ this is equivalent to saying that L is a member of a rank one commutative subring of $D(X)$; this is known to coincide with the class of algebro-geometric operators.

Let us turn to the exactly integrable operators L from Theorem 3.7 in order to prove their algebraic integrability.

First, using the idea of [42], we will construct sufficiently many operators commuting with L. Recall that L has an eigenfunction $\psi(k, x)$ of the form $\psi = De^{(k,x)}$ where D is a differential operator in x with the highest symbol $P_0(\partial) = \prod_{\alpha \in \mathcal{A}}(\alpha_0, \partial)^{m_\alpha}$. In the proposition below D^* will denote the *formal adjoint* of D, i.e. the result of applying to D an anti-involution $x \to x$, $\partial \to -\partial$. For example, $L^* = L$.

Theorem 3.8 ([42]). *(1) For any constant coefficient operator $p(\partial)$, the operator $A_p = Dp(\partial)D^*$ commutes with L. Moreover, all these operators for various polynomials p pairwise commute and $A_p\psi = \lambda_p\psi$ where $\lambda_p(k) = P_0(k)p(k)P_0(-k)$.*
(2) D^*D is a constant coefficient operator: $D^*D = P_0(\partial)P_0(-\partial)$.

In [42] the proof is given for the Calogero–Moser system, but it applies to our situation as well. Notice also that any algebraic relation between the eigenvalues $\lambda_p(k)$ of the operators forces the same relation between A_p's. This follows from a simple fact that any partial differential operator $A(x, \partial)$ annihilating the quasi-polynomial family $\psi(k, x)$ must be zero.

Now take $n + 1$ operators $A_0, A_1, \ldots, A_n$ corresponding to $p_0 = 1$ and $p_i = \partial_i$. Let us denote the polynomial $P_0(k)P_0(-k)$ which appears in the theorem above as $R(k)$:

$$R(k) = \prod_{\alpha \in \mathcal{A}}(\alpha_0, k)^{m_\alpha}(\alpha_0, -k)^{m_\alpha}, \quad k \in \mathbb{C}^n. \tag{3.3}$$

It is homogeneous of degree $2M = 2\sum_{\alpha \in \mathcal{A}} m_\alpha$. According to the theorem, we have that $A_i\psi = \lambda_i\psi$ where

$$\lambda_0 = R(k) \quad \text{and} \quad \lambda_i = R(k)k_i \quad \text{for } i = 1, \ldots, n. \tag{3.4}$$

The eigenvalues $\lambda_0, \ldots, \lambda_n$ are subject to the relation

$$\lambda_0^{2M+1} = R(\lambda_1, \ldots, \lambda_n) \tag{3.5}$$

and we have the same relation between A_0 and $A_1, \ldots, A_n$. It is clear that $\lambda_1, \ldots, \lambda_n$ and the operators $A_1, \ldots, A_n$ are algebraically independent, so they define a QCIS. Notice also that the map

$$\pi : \ k = (k_1, \ldots, k_n) \mapsto \lambda = (\lambda_0, \ldots, \lambda_n) \tag{3.6}$$

from $\mathbb{C}^n$ onto the hypersurface (3.5) in $\mathbb{C}^{n+1}$ is bijective outside the locus $R(k) = 0$.

We will prove now that the holonomic system of equations $A_i f = \lambda_i f$ ($i = 0, \ldots, n$) for generic point λ of the hypersurface (3.5) has a unique (up to a factor) locally analytic solution, which is $f = \psi(x, k)$ with $k = \pi^{-1}(\lambda)$. Clearly, this

would imply the algebraic integrability of the QCIS defined by $A_1, \ldots, A_n$, and the algebraic integrability of L would follow from that.

So, let us consider, for a fixed $k \in \mathbb{C}^n$, the system

$$A_i f = \lambda_i(k) f, \quad i = 0, \ldots, n, \tag{3.7}$$

and an associated system with *constant coefficients*:

$$\lambda_i(\partial) g = \lambda_i(k) g, \quad i = 0, \ldots, n. \tag{3.8}$$

Proposition 3.9. *The partial differential operator D^* takes the (locally analytic) solutions $f(x)$ of (3.7) to the solutions $g = D^* f$ of the system (3.8). If the spectral parameter k is such that $\lambda_0 = R(k) \neq 0$, then the inverse map is given by $f = (R(k))^{-1} D g$.*

To prove the proposition, we use the fact that the constant coefficient operators in the left-hand side of (3.8) have the form $\lambda_0(\partial) = D^* D$ and $\lambda_i(\partial) = D^* D \partial_i$ for $i > 0$. This is due to the second part of Theorem 3.8. It follows that $\lambda_0(\partial) g = D^* D D^* f = D^* \lambda_0(k) f = \lambda_0(k) g$. Similarly, for $i > 0$ we get: $\lambda_i(\partial) g = D^* D \partial_i D^* f = D^* A_i f = D^* \lambda_i(k) f = \lambda_i(k) g$. This proves the first claim. Now since $D g = D D^* f = A_0 f$, which is $\lambda_0(k) f$ by the first equation in (3.7), the second claim follows.

Corollary 3.10. *Any (locally analytic) solution $f(x)$ to (3.7) with $R(k) \neq 0$ is a constant multiple of $\psi = D e^{(k,x)}$.*

To prove that, take any solution $f(x)$ analytic in some open chart of $\mathbb{C}^n$, and apply D^* to get a solution $g(x) = D^* f$ to (3.8). But this constant coefficient system has the only solution $e^{(k,x)}$ in case $R(k) \neq 0$. Indeed, by the Nullstellensatz applied to the commutative ring $\mathbb{C}[\partial]$, the system of equations $(\lambda_i(\partial) - c_i) g = 0$ implies that $(\partial_i - k_i) g = 0$ for all $i = 1, \ldots, n$, as soon as the system of equations $\lambda_i(k) = c_i$ has $k = (k_1, \ldots, k_n)$ as its only solution. And this is the case if $R(k) \neq 0$. Thus, we get that $\partial_i g = k_i g$ for all $i = 1, \ldots, n$. And this implies that g is a constant multiple of $\exp(k_1 x_1 + \cdots + k_n x_n)$.

Summing up, we get the following theorem.

Theorem 3.11. *Any exactly solvable Schrödinger operator $L = -\Delta + u$ from Theorem 3.7 is algebraically integrable.*

3.3. **Baker–Akhiezer function and quasi-invariant polynomials.** The commutative ring constructed in Theorem 3.8 is not a maximal one, and can be extended in a way we are going to describe now. An important tool is a multidimensional Baker–Akhiezer function, which plays a central role in the approach of [1, 2, 4].

For simplicity, let us restrict to the case when the configuration of the poles of u is central, i.e. when $u(x)$ has the form (1.6):

$$u(x) = \sum_{\alpha \in \mathcal{A}} \frac{m_\alpha (m_\alpha + 1)(\alpha, \alpha)}{(\alpha, x)^2}. \tag{3.9}$$

Here $\mathcal{A}$ is a finite collection of non-parallel vectors $\alpha \in \mathbb{C}^n$ with prescribed multiplicities $m_\alpha \in \mathbb{Z}_+$ which satisfies the locus conditions (1.7).

According to Theorem 3.5, the Schrödinger operator $L = -\Delta + u$ has an eigenfunction $\psi(k, x)$, quasipolynomial in the spectral parameter k. This function is a common eigenfunction of the commutative ring constructed in Theorem 3.8 and, as such, it is unique up to a factor depending on k. A nontrivial fact [4] is that

there is a distinguished way to normalize ψ to achieve a symmetry between x and k. The resulting function $\psi(k, x)$ has the following form:

$$\psi = \delta^{-1}(x)\delta^{-1}(k)P(k, x)e^{(k, x)}, \qquad (3.10)$$

where $\delta(x) = \prod_{\alpha \in \mathcal{A}}(\alpha, x)^{m_\alpha}$ and $P(k, x)$ is polynomial in k, x with the highest term $\delta(x)\delta(k)$.

Theorem 3.12 ([4]). *The (properly normalized) common eigenfunction ψ of the commutative ring constructed in Theorem 3.8 has the form* (3.10) *and is symmetric in k, x in the sense that $\psi(k, x) = \psi(x, k)$. As a result, ψ satisfies the same equations in the spectral parameter.*

The last property is called *bispectrality* following the fundamental paper by Duistermaat–Grünbaum [21]. In [4] it was also shown that, in the spirit of the finite-gap theory, ψ can be uniquely characterized by (3.10) together with certain analytic properties. Namely, for each of the hyperplanes $(\alpha, k) = 0$ the function $(\alpha, k)^{m_\alpha}\psi(k, x)$ must be m_α-quasi-invariant with respect to this hyperplane, in the sense of Definition 3.4. This will allow us to extend the commutative ring from Theorem 3.8. Before formulating the result, let us make one remark which we will use below.

Remark 3.13. Note that $L\psi = -k^2\psi$. From this equation and from the type of the singularity that u has along a hyperplane $(\alpha, x) = 0$, one easily derives that ψ should either have a pole of order m_α or a zero of order $m_\alpha + 1$ along this hyperplane. But the latter clearly contradicts (3.10). Thus, we conclude that ψ must have a pole of order m_α. By the symmetry between k and x we get the same properties in k. In particular, this implies that the polynomial $P(k, x)$ in (3.10) is irreducible as polynomial in k.

Definition 3.14. Given a configuration $\mathcal{A}$, define the ring of quasi-invariants $Q^{\mathcal{A}}$ as the ring of polynomials $f(k)$ which are m_α-quasi-invariant with respect to the hyperplane $(\alpha, k) = 0$ for all $\alpha \in \mathcal{A}$.

Theorem 3.15. *(1) For each homogeneous quasi-invariant polynomial $f(k) \in Q^{\mathcal{A}}$ there exists a partial differential operator L_f with the highest symbol $f(\partial)$ such that $L_f\psi = f(k)\psi$. All these operators pairwise commute.*

(2) This commutative ring is a maximal commutative subring of the ring of analytic differential operators on any open chart $U \subset \mathbb{C}^n$ outside the singular locus $\delta(x) = 0$.

Part (1) is the result of [4]. Note that there is the following explicit formula for L_f, due to Berest: $L_f = C\mathrm{ad}_L^N[f]$ where $N = \deg f$ and C is an appropriate constant, see [40, 4].

To prove the second claim, let A be a differential operator on U that commutes with all of A_i from Theorem 3.8. Then $A\psi$ must be proportional to ψ since it is the common eigenfunction of $A_0, \ldots, A_n$, which is unique according to Corollary 3.10. Thus, $A\psi = f(k)\psi$ for some polynomial in k. We want to prove that f is quasi-invariant polynomial, thus $A = L_f$. To this end, observe that since $A(x, \partial_x)$ does not involve k, the function $f(k)\psi = A\psi$ has the same analytic properties in k as ψ. Namely, $f(k)(\alpha, k)^{m_\alpha}\psi$ must be m_α-quasi-invariant as a function of k with respect to the hyperplane $(\alpha, k) = 0$. From the remark above, we conclude that the function $(\alpha, k)^{m_\alpha}\psi$ is not zero along this hyperplane. And we know that multiplying it by

$f(k)$ gives another m_α-quasi-invariant function. This immediately implies that f itself must be m_α-quasi-invariant with respect to the hyperplane $(\alpha, k) = 0$. Since this must be true for any hyperplane, we conclude that $f \in Q^{\mathcal{A}}$ and we are done.

Remark 3.16. In papers [26, 29] a result close to part (2) of Theorem 3.15 is proved in some important special cases.

Remark 3.17. The question whether or not the ring $Q^{\mathcal{A}}$ is finitely generated is not quite trivial: it seems to be finitely generated in most cases, except some 'degenerate' ones. (See [43] for an example of an infinitely generated maximal commutative ring of partial differential operators.) Now, assuming that $Q^{\mathcal{A}}$ is finitely generated, we may think of it as the coordinate ring of an affine algebraic variety; this is analogous to the spectral curve in dimension one. In many important cases these varieties were shown to have a number of very interesting properties [26]–[29].

4. Trigonometric case

Let $L = -\Delta + u(x)$ be an n-dimensional Schrödinger operator with trigonometric potential. By this we mean that u is a rational function of $e^{2i(\nu_1, x)}, \ldots, e^{2i(\nu_n, x)}$, where $\{\nu_s\}$ generate a rank n lattice $\mathcal{L}^*$ in $\mathbb{C}^n$. Since we will not be using the real structure, there will be no difference between the trigonometric and hyperbolic settings. Clearly, the periods of $u(x)$ form the reciprocal lattice $\mathcal{L}$ generated by $\pi\omega_1, \ldots, \pi\omega_n$ with $(\nu_i, \omega_j) = \delta_{ij}$. We will impose the following assumption on u which is analogous to the assumption $u(+i\infty) = u(-i\infty) = 0$ in Theorem 2.3. Namely, we will assume that u decays at infinity along any ray $x(t) = x + iyt$, $t \in \mathbb{R}$, with generic $y \in \mathcal{L} \otimes \mathbb{R}$. For example, $u = \sin^{-2}(\nu_s, x)$ satisfies this assumption, as does $u = \sin^{-2}(\alpha, x)$ for any $\alpha \in \mathcal{L}^*$. In analogy with $n = 1$ case, let us denote this property symbolically as $u(\pm i\infty) = 0$.

We want to characterize those L that are exactly solvable. Similar to the rational case, one uses Theorem 3.2 and Lemma 3.3 to conclude that the potential u of such an operator must have second order poles along the collection of (pairwise distinct) hyperplanes $\Pi_{(\alpha,l)} : \alpha(x) = \pi l$ with $l \in \mathbb{Z}$, for a finite set $\mathcal{A} = \{\alpha\}$ of affine-linear functions of the form $\alpha(x) = a_0 + (\alpha_0, x)$, with non-isotropic α_0. Since the configuration of the poles must be periodic, we assume that all the vectors α_0 belong to the lattice $\mathcal{L}^*$. As a result, we get that the singularities of u are encoded in a finite set $\mathcal{A}$ with the assumptions as above, with prescribed multiplicities $m_\alpha \in \mathbb{Z}_+$. Due to the condition $u(\pm i\infty) = 0$, the potential u is completely determined by its singularities and is described by the formula

$$u(x) = \sum_{\alpha \in \mathcal{A}} \frac{m_\alpha(m_\alpha + 1)(\alpha_0, \alpha_0)}{\sin^2 \alpha(x)}, \quad m_\alpha \in \mathbb{Z}_+. \qquad (4.1)$$

Remark 4.1. As an example, note that a one-dimensional $u = k(k+1)\sin^{-2} x + 4l(l+1)\sin^{-2} 2x$ with $k, l \in \mathbb{Z}$ does not meet the requirements above, since $\sin x$ and $\sin 2x$ have common zeros. Instead, one should consider $u = k(k+1)\sin^{-2} x + l(l+1)\sin^{-2}(x + \frac{\pi}{2}) = (k-l)(k+l+1)\sin^{-2} x + 4l(l+1)\sin^{-2} 2x$.

Now, Lemma 3.3 tells us that the potential (4.1) of an exactly solvable operator $L = -\Delta + u$ must be m_α-quasi-invariant with respect to each of the hyperplanes

$\Pi_{(\alpha,l)}$: $\alpha(x) = \pi l$, $l \in \mathbb{Z}$. These are necessary conditions, which turn out to be sufficient according to the next theorem.

Theorem 4.2 ([3]). *If the potential $u(x)$ of a Schrödinger operator $L = -\Delta + u$ has the form (4.1) with the assumptions as above, and u is quasi-invariant with respect to each of its pole hyperplanes, then L is exactly solvable.*

Proof. We outline the proof since in [3] it was omitted. Suppose $L = -\Delta + u$ has a quasi-invariant potential of the form (4.1). We will effectively construct an eigenfunction of L of the form $\psi(x,k) = P(k,x)e^{(k,x)}$. Let V denote a subspace of meromorphic functions $f(x)$ with the following properties:

(1) The poles of f must be along some of the hyperplanes $\Pi_{(\alpha,l)}$: $\alpha(x) = \pi l$, $l \in \mathbb{Z}$, with the order of the pole $\leq m_\alpha$;

(2) For each $\alpha \in \mathcal{A}$ the function $f(x) \sin^{m_\alpha} \alpha(x)$ must be m_α-quasi-invariant with respect to $\Pi_{(\alpha,l)}$.

The central observation [3] is that the operator L preserves the space V: $L(V) \subseteq V$. This follows directly from the quasi-invariance of u. As a result, L will preserve the subspace V_0 of V consisting of those f having the form $f = \delta^{-1}Q(k,x)e^{(k,x)}$ where $Q \in \mathbb{C}[e^{\pm i(\nu_1,x)}, \ldots, e^{\pm i(\nu_n,x)}]$ is a trigonometric polynomial in x and polynomial in k, and $\delta(x)$ is as follows:

$$\delta = \prod_{\alpha \in \mathcal{A}} \sin^{m_\alpha} \alpha(x). \tag{4.2}$$

Now, for a trigonometric polynomial $Q = \sum_{\nu \in \mathcal{L}} a_\nu e^{i(\nu,x)}$, by its **support** we will mean the convex hull of those ν with $a_\nu \neq 0$. Then one can show that an application of L to $f \in V_0$ does not increase the support of Q, i.e. if $f = \delta^{-1}Q(k,x)e^{(k,x)}$ then $\widehat{f} = Lf$ has the form $\widehat{f} = \delta^{-1}\widehat{Q}(k,x)e^{(k,x)}$ where $\text{supp}(\widehat{Q}) \subseteq \text{supp}(Q)$. Moreover, one can compute how an application of L affects the coefficients a_ν at the vertices of $\text{supp}(Q)$. More precisely, let $f = \delta^{-1}Q(k,x)e^{(k,x)}$ be from V_0 and ν be a vertex of the support of the trigonometric polynomial Q, with a_ν being the corresponding coefficient. Then the coefficient $\widetilde{a}_\nu$ in $\widetilde{f} = Lf$ is given by the formula $\widetilde{a}_\nu = -(k - \nu - \rho)^2 a_\nu$. Here λ^2 stands for (λ, λ) and ρ is one of the vertices of the support of $\delta(x)$. (One can say exactly which vertex should be taken, but we will not need this.)

Now we can construct $\psi(k,x)$ as follows. Take $f_0 = \delta(x)e^{(k,x)}$, so $f_0 = \delta^{-1}Q_0 e^{(k,x)}$ with $Q_0 = \delta^2$. Clearly, f_0 belongs to V_0. Now let us start applying to f_0 operators of the form $L + c$ successively to decrease the support of Q at each step. To achieve this we just choose any vertex ν of the support of Q and take $c = (k - \nu - \rho)^2$, in accordance with the paragraph above. Since the support is finite, repeating this sufficiently many times will produce zero. The last nonzero function f in this process will be a quasipolynomial in k, such that $(L + (k - \nu)^2)f = 0$ for some $\nu \in \mathcal{L}^*$. Shifting k by ν will give us $\psi = P(k,x)e^{(k,x)}$, quasipolynomial in k and such that $L\psi = -k^2\psi$. This finishes the proof. $\square$

Remark 4.3. One can refine the arguments used in the proof above, and get an explicit formula for the eigenfunction $\psi(k,x)$, generalizing the Berest's formula (3.2). See [44], section 5.6 for the Calogero–Moser case.

Combining this theorem with the previous discussion we obtain the following result.

Theorem 4.4. *Let $L = -\Delta + u$ be a Schrödinger operator with trigonometric potential such that $u(\pm i\infty) = 0$. Then L is exactly solvable if and only if the potential u has the form (4.1) and is m_α-quasi-invariant with respect to each of the hyperplanes $\Pi_{(\alpha,l)}: \; \alpha(x) = \pi l$.*

Finally, by the arguments similar to those used in section 3.2 we obtain the algebraic integrability of all these operators.

Theorem 4.5. *All exactly solvable operators $L = -\Delta + u(x)$ from the previous theorem are algebraically integrable.*

The structure of the corresponding commutative ring of partial differential operators is described below. For simplicity, we will restrict to the case when u has the form

$$u(x) = \sum_{\alpha \in \mathcal{A}} \frac{m_\alpha(m_\alpha + 1)(\alpha, \alpha)}{\sin^2(\alpha, x)} \,. \tag{4.3}$$

Here $\mathcal{A}$ is a finite collection of non-parallel vectors $\alpha \in \mathbb{C}^n$ with prescribed multiplicities $m_\alpha \in \mathbb{Z}_+$. We keep all the assumptions of Theorem 4.2, namely, the vectors α are assumed to belong to a lattice $\mathcal{L}^*$ of rank n, and u must be m_α-quasi-invariant with respect to each of the hyperplanes $\Pi_{(\alpha,l)}$.

Definition 4.6. For $\omega \neq 0$ define the ring $Q_\omega^{\mathcal{A}}$ as the ring of polynomials $f(k)$ such that for all $\alpha \in \mathcal{A}$ and any $j = 1, \ldots, m_\alpha$

$$f(k - \omega j\alpha) \equiv f(k + \omega j\alpha) \quad \text{for } (\alpha, k) = 0 \,.$$

Remark 4.7. For a given $\mathcal{A}$ all rings $Q_\omega^{\mathcal{A}}$ are isomorphic, say, to $Q_1^{\mathcal{A}}$. The reason we introduce ω is that then this definition can be viewed as a deformation of the definition 3.14, with the latter corresponding to the limit $\omega \to 0$. In that sense, we may think of the ring $Q^{\mathcal{A}}$ in Theorem 3.15 as $Q_0^{\mathcal{A}}$. This suggests that $Q_0^{\mathcal{A}}$ coincides with the associated graded ring for $Q_\omega^{\mathcal{A}}$. It is easy to prove the inclusion $\mathrm{gr}(Q_1^{\mathcal{A}}) \subseteq Q_0^{\mathcal{A}}$, but we do not know a general proof of the fact that they coincide.

Now, according to Theorem 4.2, $L = -\Delta + u$ has an eigenfunction $\psi(k, x)$ which is quasipolynomial in the spectral parameter k. More precisely, $\psi(k, x) = \delta^{-1}(x)P(k, x)e^{(k,x)}$, where δ is given by (4.2) and P is polynomial in k and trigonometric polynomial in x. The theorem below is a trigonometric version of Theorem 3.15.

Theorem 4.8. *(1) For any polynomial $f(k) \in Q_1^{\mathcal{A}}$ there exists a partial differential operator L_f such that $L_f\psi = f(k)\psi$. All these operators commute, and the operator corresponding to $f = -k^2$ coincides with $L = -\Delta + u(x)$.*

(2) The commutative ring consisting of all L_f with $f \in Q_1^{\mathcal{A}}$ is a maximal commutative subring (of rank one) inside the ring of analytic operators on any open chart $U \subset \mathbb{C}^n$ outside the singular locus $\delta(x) = 0$. (Here δ is the function (4.2).)

The proof of the theorem is similar to the proof of Theorem 3.15. The main idea is to characterize $\psi(k, x)$ in a unique way by its analytic properties in x and k. This can be done by exploiting the technique developed in [45, 44]. A similar result is true for any $L = -\Delta + u$ from Theorem 4.2, not necessarily for those of the form (4.3). Details will appear in [46].

Remark 4.9. As a by-product, one gets [46] that ψ satisfies certain difference equations in the spectral parameter k; this is a deformation of the symmetry between

k, x that we had in the rational case. As an example, for the Sutherland–Calogero–Moser operator L these dual difference operators are nothing but (a rational version of) the Macdonald–Riujsenaars operators [47, 48]. For other root systems one gets similarly the Macdonald difference operators, see [45]. For the deformed root systems this bispectral duality gives a natural way to deform the Macdonald–Ruijsenaars operators. For $A_{n,1}(m)$ this was done in [45], for the other cases see [49].

5. Elliptic case

In this section we consider the operators $L = -\Delta + u$ with elliptic potential, following [5]. Our results are parallel to those for the rational and trigonometric cases, though they are less complete.

Let us first describe the class of potentials $u(x)$ we will be considering. They are the following elliptic analogues of the trigonometric potentials (4.1):

$$u(x) = \sum_{\alpha \in \mathcal{A}} c_\alpha \wp(\alpha(x)|\tau), \quad c_\alpha = m_\alpha(m_\alpha + 1)(\alpha_0, \alpha_0). \tag{5.1}$$

Here $\wp(z|\tau)$ is the Weierstrass $\wp$-function with the periods $1, \tau$, $\text{Im}(\tau) > 0$. As in the trigonometric case, $\mathcal{A}$ stands for a finite set of affine-linear functions on $\mathbb{C}^n$, each given by its equation $\alpha(x) = a_0 + (\alpha_0, x)$, and the multiplicities m_α are non-negative integers. We assume that the vectors α_0 are non-isotropic and belong to a rank n lattice $\mathcal{L}^* \subset \mathbb{C}^n$. In that case the period lattice for u will be of rank $2n$ and can be described as $\mathcal{L} + \tau \mathcal{L}$ where $\mathcal{L}$ is the reciprocal of $\mathcal{L}^*$, i.e. $\mathcal{L} := \{v \in \mathbb{C}^n : (v, w) \in \mathbb{Z} \text{ for all } w \in \mathcal{L}^*\}$. Thus, u may be viewed as a meromorphic function on a (compact) torus $T = V/\mathcal{L} + \tau \mathcal{L}$, which is isomorphic to the product of n copies of the elliptic curve $\mathcal{E} = \mathbb{C}/\mathbb{Z} + \tau \mathbb{Z}$. Singularities of $u(x)$ are the second order poles along the following set of the hyperplanes (or hypertori):

$$\text{Sing}(u) = \bigcup_{\alpha \in \mathcal{A}} \bigcup_{m,n \in \mathbb{Z}} \Pi_\alpha^{m,n}, \qquad \Pi_\alpha^{m,n} := \{x : \alpha(x) = m + n\tau\}.$$

Similarly to the trigonometric case, we have to assume that all the hyperplanes $\Pi_\alpha^{m,n}$ are *pairwise distinct*. Next, motivated by the results above, let us require the quasi-invariance of u.

Definition 5.1. Let us call a Schrödinger operator $L = -\Delta + u$ a **generalized Lamé operator**, if u has the form (5.1) and is m_α-quasi-invariant with respect to each of the hyperplanes $\Pi_\alpha^{m,n} \in \text{Sing}(u)$.

As we already know, in dimension one this definition singles out all elliptic algebro-geometric operators $L = -\frac{d^2}{dz^2} + u(z)$, with the Lamé operator $L = -\frac{d^2}{dz^2} + m(m+1)\wp(z)$ being one of them. This, together with the previous results for the rational and trigonometric cases, makes the following conjecture very natural.

Conjecture 5.2 ([5]). *Any generalized Lamé operator is algebraically integrable.*

While this conjecture remains open in general, in [5] it was proved under additional assumption of the complete integrability of L (in a slightly stronger sense, see below). This is an important result since the complete integrability in many cases is easier to establish. In particular, as we will explain, this allows one to

prove the conjecture of [1] about the algebraic integrability of the quantum elliptic Calogero–Moser systems for all root systems.

Let us formulate the result of [5] precisely. Let L be a generalized Lamé operator. Recall that L is **completely integrable** if it is a member of a commutative family of differential operators $L_1 = L, L_2, \ldots, L_n$ which are algebraically independent. We assume that the L_i's have meromorphic coefficients and are periodic with respect to the same lattice, which makes them (singular) differential operators on the torus $T = \mathbb{C}^n/\mathcal{L} + \tau\mathcal{L}$.

Definition 5.3. Let us say that a completely integrable Schrödinger operator $L = -\Delta + u$ in $V = \mathbb{C}^n$ is **strongly integrable** if the commuting operators $L_1 = L, \ldots, L_n$ have algebraically independent homogeneous constant highest symbols $s_1, \ldots, s_n$ such that $\mathbb{C}[V]$ is finitely generated as a module over the ring generated by $s_1, \ldots, s_n$ or, equivalently, that the system $s_1(\xi) = 0, \ldots, s_n(\xi) = 0$ has a unique solution $\xi = 0$.

Remark 5.4. For example, in dimension $n = 1$ any $L = -\frac{d^2}{dx^2} + u(x)$ is strongly integrable. On the other hand, consider the QCIS defined by $L_1, \ldots, L_n$ constructed in Section 3.2. Then this system is not strongly integrable since the highest symbols are divisible by $R(k)$.

Theorem 5.5 ([5]). *Any generalized Lamé operator L which is strongly integrable is algebraically integrable.*

Let us explain the main idea behind the proof of this theorem. It uses the following important result of Braverman, Etingof and Gaitsgory [41], which links the algebraic integrability to the differential Galois theory. Namely, for a given QCIS system defined by $L_1, \ldots, L_n$ consider the eigenvalue problem

$$L_i\psi = \lambda_i\psi, \quad i = 1, \ldots, n, \tag{5.2}$$

with $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$. Then for generic λ this system is holonomic [41]. Next, for a fixed λ one defines the differential Galois group of (5.2). The main result of [41] is the following characterization of algebraically integrable systems.

Theorem 5.6 ([41]). *A quantum completely integrable system is algebraically integrable if and only if the differential Galois group of (5.2) is commutative for generic λ.*

Now let L be a strongly integrable generalized Lamé operator, so we have the commuting operators $L_1 = L, \ldots, L_n$ with meromorphic coefficients on the torus $T = \mathbb{C}^n/\mathcal{L}+\tau\mathcal{L}$. Then one can show that our assumptions on the highest symbols of L_i imply that the holonomic system (5.2) has regular singularities, see [5]. Next, fix a point $x_0 \in T\backslash\mathtt{Sing}(u)$ and consider the finite-dimensional space of locally analytic solutions. Then the monodromy group of this system gives us a representation of the fundamental group of $T \setminus \mathtt{Sing}(u)$ in the local solution space. Now, from the quasi-invariance of the potential $u(x)$ one derives that the solutions to (5.2) have no branching along the hypertori from $\mathtt{Sing}(u) \subset T$, i. e. they are *locally meromorphic* functions on T. As a result, the monodromy representation factors through the fundamental group of T, which is abelian. Thus, the monodromy group of the system (5.2) is abelian. However, a general result due to Deligne [50] says that for a regular holonomic system on a smooth projective variety the differential Galois group is the Zariski closure of the monodromy group. (See [5] for a simple proof

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of this fact in our situation.) As a result, we conclude that the differential Galois group of (5.2) is abelian for any λ, and Theorem 5.5 follows.

Moreover, according to [41, 5], in that situation the solution space of (5.2) for generic λ will be generated by the *quasiperiodic* solutions:

Corollary 5.7. *There exist meromorphic 1-forms ω_i on the torus T, with first order poles and depending analytically on λ, such that the functions $\psi_i = e^{\int \omega_i}$ give a basis of the solution space of* (5.2) *for generic λ.*

Each of these functions will be a *double-Bloch solution*, in the following sense (cf.[52]):
$$\psi_i(x + \omega_j) = a_{ij}\psi_i(x), \qquad \psi_i(x + \tau\omega_j) = b_{ij}\psi_i(x), \tag{5.3}$$
for appropriate $a_{ij}, b_{ij} \in \mathbb{C}^*$, where $\omega_1, \ldots, \omega_n$ is the basis of the lattice $\mathcal{L}$. See [5], section 5, for a general procedure of computing these double-Bloch solutions.

Remark 5.8. Our proof applies to a more general situation when one has commuting operators $L_1, \ldots, L_n$ on an abelian variety, such that the system $L_i\psi = \lambda_i\psi$, $i = 1, \ldots, n$, is regular holonomic. Then the triviality of its local monodromy around singularities implies that this system is algebraically integrable.

Let us finish this section by considering some examples when Theorem 5.5 applies.

5.1. **One-dimensional case.** We consider $L = -\frac{d^2}{dx^2} + u(x)$ with elliptic potential u. In this case u must have a form (2.4):
$$u = \sum_{i=1}^{N} m_i(m_i + 1)\wp(x - x_i)$$
with distinct x_i (modulo the periods) and with $m_i \in \mathbb{Z}_{>0}$. The quasi-invariance of u boils down to the following $M = \sum m_i$ conditions:
$$\sum_{i:\, i\neq j} m_i(m_i + 1)\wp^{(2s-1)}(x_j - x_i) = 0 \quad \text{for every } j \text{ and } s = 1, \ldots, m_j. \tag{5.4}$$

By doing the local analysis (cf. [21]), one can see that in the one-dimensional case our definition of a generalized Lamé operator is equivalent to the property that all solutions of the equation $Lf = \lambda f$ are meromorphic in the complex domain.

Thus, we get an alternative proof of Theorem 2.5. Moreover, it generalizes easily to the higher order case. Indeed, let
$$L = \frac{d^n}{dx^n} + a_{n-2}(x)\frac{d^{n-2}}{dx^{n-2}} + \cdots + a_0(x) \tag{5.5}$$
be an ordinary differential operator of order n. It is called *agebro-geometric* if it is a member of a rank one maximal commutative ring of differential operators. The rank is defined as the dimension of the common eigenspace or, equivalently, as the greatest common divisor of the orders of the operators from the ring. In other words, L is algebro-geometric if there exists another operator A of order prime to n such that $[L, A] = 0$, see [34]. The above results imply immediately the following theorem, conjectured by Gesztesy and Weikard in [30].

Theorem 5.9 ([5]). *Suppose that L given by* (5.5) *has elliptic coefficients and all solutions of the equation $Lf = \lambda f$ are meromorphic in the complex domain for any $\lambda \in \mathbb{C}$. Then L is algebro-geometric.*

Converse is also true, due to the general formula expressing the Baker function via τ-function for the KP hierarchy, see [30, 53].

5.2. **Calogero–Moser problem.** A natural example of a generalized Lamé operator in dimension > 1 is given by the quantum Calogero–Moser problem with integer coupling constants. Recall that the quantum elliptic Calogero–Moser problem is described by the following n-dimensional Schrödinger operator:

$$L = -\Delta + \sum_{i<j} 2m(m+1)u(x_i - x_j), \quad u(z) = \wp(z|\tau). \tag{5.6}$$

Here m is the coupling constant, which is an arbitrary complex number. In physical literature m is usually assumed to be real; also, to ensure that u is real-valued, the period τ of $\wp$ is often taken to be purely imaginary. The singularities of L are the second order poles along the hyperplanes $x_i - x_j = m + n\tau$ and clearly L is symmetric with respect to any of these hyperplanes. As a result, for any **integer** m, the Calogero–Moser operator (5.6) gives a particular example of a generalized Lamé operator. In the same fashion we may consider a generalization of the Calogero–Moser problem for any root system R, due to Olshanetsky–Perelomov [20]:

$$L = -\Delta + \sum_{\alpha \in R_+} m_\alpha(m_\alpha + 1)(\alpha, \alpha)u((\alpha, x)), \quad u(z) = \wp(z|\tau). \tag{5.7}$$

Here R_+ is a positive half of a root system $R = \{\alpha\}$, and the coupling constants m_α are chosen in a W-invariant way, where W is the Weyl group of R. Specializing the parameters m_α to integers, we get more examples of the generalized Lamé operators. The standard Calogero–Moser problem corresponds to $R = A_{n-1}$.

In [1] Veselov and the author conjectured that the quantum Calogero–Moser problems related to the root systems are algebraically integrable if the coupling constants are integers. This conjecture, clearly, is a special case of a more general conjecture 5.2. For the A_n-case it was proved by Braverman, Etingof and Gaitsgory in [41]. To prove it in full generality, one uses the fact, due to Cherednik [54] that the operator (5.7) is completely integrable. See also [55] where the set of commuting operators $L_1, \ldots, L_n$ was given explicitly for all classical root systems. One easily checks then that the highest symbols of $L_1, \ldots, L_n$ satisfy our condition of strong integrability, thus the following result:

Theorem 5.10 ([5], conjectured in [1]). *For any root system R and any integer W-invariant coupling constants m_α the quantum elliptic Calogero–Moser operator (5.7) is algebraically integrable.*

A similar result is true for the Inozemtsev operator, which is a BC_n version of (5.7), see in the next section. Also, the algebraic integrability has been checked in [5] for all known generalized Lamé operators in dimension two. This provides an extra evidence in support of Conjecture 5.2.

6. EXAMPLES

In the previous sections we considered algebraically integrable Schrödinger operators $L = -\Delta + u$ with three classes of potential: rational, trigonometric and elliptic. Known (irreducible) examples of these operators in dimension > 2 fall into three categories: (1) Calogero–Moser systems related to Coxeter groups and root systems [20]; (2) Berest-Loutsenko family [56], this is an interesting series of

two-dimensional examples with rational u; (3) generalized Calogero–Moser systems related to the deformed root systems, the first examples of which appeared in [57, 4].

6.1. **Calogero–Moser systems.** These examples are related to the Calogero–Moser systems with integer coupling parameters m_α. The corresponding operator L is given by the formula (5.7) with R_+ being a positive half of a root system $R = A_n, \ldots, G_2$, and u being $z^{-2}, \sin^{-2} z$ or the Weierstrass' $\wp(z)$, respectively.

In $R = BC_n$ the corresponding operator in the trigonometric case involves 3 parameters $k, l, m \in \mathbb{Z}_+$ and looks as follows [20]:

$$L = -\Delta + \sum_{i<j} \frac{2m(m+1)}{\sin^2(x_i - x_j)} + \frac{2m(m+1)}{\sin^2(x_i + x_j)} + \sum_{i=1}^n \frac{k(k+1)}{\sin^2 x_i} + \frac{l(l+1)}{\cos^2 x_i}. \quad (6.1)$$

Its elliptic version due to Inozemtsev involves 5 parameters m, g_0, g_1, g_2, g_3 and looks as follows:

$$L = -\Delta + 2m(m+1) \sum_{i<j} (\wp(x_i - x_j) + \wp(x_i + x_j)) + \sum_{i=1}^n \sum_{s=0}^3 g_s(g_s+1)\wp(x_i + \omega_s), \quad (6.2)$$

with $\omega_s (s = 0 \ldots 3)$ denoting the half-periods $0, 1/2, \tau/2, (1+\tau)/2$ of $\wp$. According to [5], this operator is algebraically integrable for any integer m, g_0, g_1, g_2, g_3.

In the rational case we may consider any finite reflection group W in $\mathbb{R}^n$, and take R to be the set of the unit normals to its reflection hyperplanes, choosing $m_\alpha \in \mathbb{Z}_+$ in a W-invariant way. The corresponding Schrödinger operator is given by the same formula (5.7) with $u = z^{-2}$. Thus, in the rational case apart from $A_n, \ldots, G_2$ cases we have also I_n (dihedral groups), H_3 and H_4.

6.2. **Berest–Loutsenko family.** This family was discovered in [56] in relation to the Huygens' principle and Hadamard problem. It is in dimension two, so L has the form $L = -\partial_x^2 - \partial_y^2 + U(x, y)$ where the potential U is rational homogeneous of degree -2. The Berest–Loutsenko operators L are in one-to-one correspondence with π-*periodic soliton potentials* $u(\varphi)$ (we discussed those in Section 2). Namely, for any such $u(\varphi)$ we define U in polar coordinates (r, φ) as $U = r^{-2}u(\varphi)$. The singularities of $U(x, y)$ form a collection of lines passing through the origin, and the positions of these lines correspond to the poles of $u(\varphi)$. For instance, taking $u(\varphi) = m(m+1)N^2 \sin^{-2} N\varphi + l(l+1)N^2 \cos^{-2} N\varphi$ we get the dihedral case of the Calogero–Moser problem. An interesting feature of this family is that it has continuous parameters, all other known irreducible examples in higher dimension appear in discrete families. Note that the notion of irreducibility we refer to is not quite straightforward and involves *isotropic reduction*. See [56, 4] for more details and examples.

6.3. **Deformed root systems.** Other known irreducible examples of the generalized Lamé operators in dimension > 1 are related to the deformed root systems. Below we describe the set of linear functions $\mathcal{A} = \{\alpha\}$ and the corresponding multiplicities m_α. For convenience, we will allow m_α to be negative, with L given by the same formula as above:

$$L = -\Delta + \sum_{\alpha \in \mathcal{A}} m_\alpha(m_\alpha + 1)(\alpha_0, \alpha_0)u(\alpha(x)) \quad (6.3)$$

where $u(z) = z^{-2}, \sin^{-2} z$ or $\wp(z|\tau)$, respectively.

(1) $A_{n,1}(m)$ system [4].
It consists of the following linear functions in $\mathbb{C}^{n+1}$:

$$\begin{cases} x_i - x_j, & 1 \le i < j \le n, & \text{with } m_\alpha = m, \\ x_i - \sqrt{m}x_{n+1}, & i = 1, \ldots, n, & \text{with } m_\alpha = 1. \end{cases}$$

Here m is an integer parameter.

(2) $C_{n,1}(m,l)$ system [4].
It consists of the following linear functions in $\mathbb{C}^{n+1}$:

$$\begin{cases} x_i \pm x_j, & 1 \le i < j \le n, & \text{with } m_\alpha = k, \\ 2x_i, & i = 1, \ldots, n, & \text{with } m_\alpha = m, \\ x_i \pm \sqrt{k}x_{n+1}, & i = 1, \ldots, n, & \text{with } m_\alpha = 1, \\ 2\sqrt{k}x_{n+1} & & \text{with } m_\alpha = l. \end{cases}$$

Here k, l, m are integer parameters related by $k = \frac{2m+1}{2l+1}$. In two-dimensional case $n = 1$ the first group of roots is vacuous and there is no restriction for k to be integer.

(3) Here is a BC_n-type generalization of the previous example for the trigonometric and elliptic cases [5]. For the elliptic case we denote by $\omega_s(s = 0 \ldots 3)$ the half-periods, as in (6.2). The set of linear functions $\alpha \in \mathcal{A}$ and the corresponding multiplicities look as follows:

$$\begin{cases} x_i \pm x_j, & 1 \le i < j \le n, & \text{with } m_\alpha = k, \\ x_i + \omega_s, & i = 1, \ldots, n, & \text{with } m_\alpha = m_s \ (s = 0 \ldots 3), \\ x_i \pm \sqrt{k}x_{n+1}, & i = 1, \ldots, n, & \text{with } m_\alpha = 1, \\ \sqrt{k}x_{n+1} + \omega_s & & \text{with } m_\alpha = l_s \ (s = 0 \ldots 3). \end{cases}$$

Here k, l_s, m_s are nine integer parameters related through $k = \frac{2m_s+1}{2l_s+1}$ for all $s = 0 \ldots 3$. The previous case corresponds to $m_s \equiv m$ and $l_s \equiv l$. Again, in the case $n = 1$ the first group of roots is not present and k does not have to be integer.

In the trigonometric case we have the same formulas, but with just two half-periods $\omega_0 = 0$ and $\omega_1 = \frac{\pi}{2}$. Respectively, we will have 5 integer parameters k, l_0, l_1, m_0, m_1 with $k = \frac{2m_s+1}{2l_s+1}$ for $s = 0, 1$.

(4) Hietarinta operator [58].
In this case we have three covectors in $\mathbb{C}^3$,

$$\alpha = a_1 x_1 - a_2 x_2, \ \beta = a_2 x_2 - a_3 x_3, \ \gamma = a_1 x_1 - a_3 x_3,$$

with $m_\alpha = m_\beta = m_\gamma = 1$. Here a_i are arbitrary complex parameters such that $a_1^2 + a_2^2 + a_3^2 = 0$. Notice that the system is essentially two-dimensional since $\alpha + \beta + \gamma = 0$.

(5) $A_{n-1,2}(m)$ system [59].
It consists of the following linear functions in $\mathbb{C}^{n+2}$:

$$\begin{cases} x_i - x_j, & 1 \le i < j \le n, & \text{with } m_\alpha = m, \\ x_i - \sqrt{m}x_{n+1}, & i = 1, \ldots, n, & \text{with } m_\alpha = 1, \\ x_i - \sqrt{-1-m}x_{n+2}, & i = 1, \ldots, n, & \text{with } m_\alpha = 1, \\ \sqrt{m}x_{n+1} - \sqrt{-1-m}x_{n+2} & & \text{with } m_\alpha = 1. \end{cases}$$

Here m is integer. Notice that for $m = 1$ this system coincides with the system $A_{n,1}(-2)$ above.

7. Conclusion

We described above three classes of multidimensional Schrödinger operators which generalize the algebro-geometric operators $L = -\frac{d^2}{dx^2} + u$ with rational, trigonometric and elliptic potential $u(x)$. Here we would like to mention some open questions related to that approach.

One of the main questions is to describe completely all operators in those classes, for instance, by proving that the list of examples we gave exhausts all irreducible cases. This would be very important for solving the Hadamard problem, see [4].

Secondly, it would be very desirable to find more explicit description of the spectral variety in the elliptic case. For the A_n-case, Felder and Varchenko construct in [60] the Hermite–Bloch variety parametrizing the double-Bloch eigenfunctions, but its relation to the spectral variety has yet to be clarified. In [5] the Hermite–Bloch variety was also (implicitly) constructed for the generalized Lamé operators, and we have the same question in that case. Also, for the A_n-case, the double-Bloch solutions were constructed in [60] in a rather effective way, via a version of Bethe ansatz. It would be very interesting to generalize the Bethe ansatz of [60] to other root systems and their deformations.

Finally, it is conceptually clear that there is a q-analogue of this story, related to the commutative rings of partial difference operators, but it is yet to be worked out in detail. For the root systems this is related to the operators of Macdonald–Ruijsennars type, and their algebraic integrability (in trigonometric case) was established in [61] for A_n-case and in [44] in general. This has nice applications to the Macdonald theory, see [61, 44]. The deformed $A_{n,1}(m)$ case was treated in [44], and the methods used there are applicable to all deformed root systems. The elliptic case, however, is more difficult and very little is proven in dimension > 1. Let us mention here the paper [62] where the Bethe ansatz of Felder–Varchenko was extended to the elliptic Ruijsenaars system $(R = A_n)$, also see some very explicit formulas in [5] for a particular case related to $R = BC_2$. Perhaps, a first natural step in that direction would be to try to generalize the results of [41] to the difference setting.

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School of Mathematics, University of Leeds, Leeds, LS2 9JT, UK

E-mail address: oleg@maths.leeds.ac.uk