with $\delta_0 = 0.125$ en $a_T = 5$, and in both cases the initial condition:

$$h(x, y, 0) + h_b(y) = H_i + \eta_{add} \tag{4.1.14}$$

with

$$\eta_{add} = \begin{cases} \frac{1}{2} h_a \left[1 + \cos(\pi\,(x - \frac{1}{2}L_e)/r)\right] e^{-((y-0.5)/y_s)^2} & \text{if } |(x - \frac{1}{2}L_e)| < r \\ 0 & \text{if } |(x - \frac{1}{2}L_e)| \geq r \end{cases} \tag{4.1.15}$$

and $H_i = 0.13$, $r = 1.0$, $h_a = 0.25$ en $y_s = 0.5$.

i. Show the water surface $h + h_b$ at the following time levels: $t = 0, 50, 100, 150, 200$ for the simulations with the bottom profiles (i) en (ii).
ii. Explain the differences between the simulations with profile (i) en (ii).
iii. Make plots of the total mass M and the potential vorticity Q.
iv. Verify your answer with simulations with at least twice as many mesh points.

References and discussion

Wright and Willmott (1992, *Journal of Physical Oceanography*, **22**, 139–159 pp) consider a similar Rossby-wave equation with an ad hoc dissipation of the form h_{xx} which ensures that sharp gradients in the waves are smoothed out.

The fluid mechanics in this assignment can also be extended to situations with a general bottom profile $h_b(x, y)$ in which the water depth h can become zero. This can result in numerical problems at the domain boundary, because the condition $h(x, y, t) = 0$ then becomes time-dependent. The Rossby-wave-equation is well suited for testing numerical schemes for moving boundaries.

vi. **Seiches and tides in channels and gulfs**

To model seiches and tides in shallow seas in an approximate manner, we can use the rotating shallow water equations

$$\partial_t u + u\,\partial_x u + v\,\partial_y u - \mathsf{f}\,v = -\,g\,\partial_x(h + h_b) \tag{4.1.16}$$

$$\partial_t v + u\,\partial_x v + v\,\partial_y v + \mathsf{f}\,u = -\,g\,\partial_y(h + h_b) \tag{4.1.17}$$

$$\partial_t h + \partial_x(h\,u) + \partial_y(h\,v) = 0 \tag{4.1.18}$$

with water depth $h(x, y, t)$, depth-averaged horizontal velocities $u(, x, y, t)$ and $v(x, y, t)$ in the horizontal directions with horizontal coordinates x and y, g the acceleration of gravity, f the constant Coriolis parameter, time t, and $h_b(x, y)$ the topography. See the sketch in Figure 4.2.

We linearize (4.1.16)–(4.1.18) around a state of rest with rest depth $h(x, y, t) = D(x, y)$. Small perturbations $\eta(x, y, t), u'(x, y, t)$ and $v'(x, y, t)$ of this state of rest are approximately governed by the linearized shallow water equations (dropping the primes on u' and v')

$$\partial_t u - \mathsf{f}\,v = -\,g\,\partial_x \eta \tag{4.1.19}$$

$$\partial_t v + \mathsf{f}\,u = -\,g\,\partial_y \eta \tag{4.1.20}$$

$$\partial_t \eta + \partial_x(D\,u) + \partial_y(D\,v) = 0 \tag{4.1.21}$$

with $h + b = H + \eta(x, y, t)$ or $h = D + \eta$.

(a) Scale both (4.1.16)–(4.1.18) and (4.1.19)–(4.1.21) using

$$\begin{aligned} u =&U\,u', \quad v = U\,v', \quad (x,y) = L_s\,(x',y') \quad t = (L_s/U)\,t' \\ h =&H\,h', \quad \eta = H\,\eta', \quad h_b = H\,h_b'. \end{aligned}$$

Show that we can thus also consider (4.1.16)–(4.1.16) and (4.1.19)–(4.1.21) as the dimensionless equations by replacing the unprimed coordinates, variables and parameters by the primed coordinates and variables and parameters f by $\mathsf{f}' = \mathsf{f}\,L_s/U$ and g by $g' = g\,H/U^2$. Take $g' = 1$ which then defines $U > 0$. Due to the scaling, the typical magnitude of the variables can be chosen to lie around unity, which is advantageous in the numerical computation.

(b) For $h_b = 0$ write down the conservation form of the linearized shallow water equations (4.1.19)–(4.1.21). (see §1.2 & 1.5.2.)

(c) Assume here that $h_b = \mathsf{f} = 0$ and that the variables u, v, η are independent of y. Identify the eigenvalues and eigenvectors of the relevant matrix for the linearized shallow water equations (4.1.19)–(4.1.21). Based on this preliminary information, make an estimate of the time step for a relevant discretization using a mesh size h_k (a length). (see §1.3.)

(d) Assume here that $h_b = \mathsf{f} = 0$ and that the variables u, v, η are independent of y. Identify the eigenvalues and eigenvectors of the relevant matrix for the nonlinear shallow water equations (4.1.16)–(4.1.18). Compare the eigenvalues and eigenvectors of the linearized and nonlinear shallow water equations. Based on this preliminary information, make an estimate of the time step for a relevant discretization with mesh size h_k. (see §1.5.2.)

(e) Write the (scaled) linearized system of shallow water equations with $h_b = 0$, $D = H$ and hereafter $(u, v, \eta) = (u(x,t), v(x,t), \eta(x,t))$ in the concise form

$$\partial_t \vec{w} + \partial_x \vec{f} = \vec{S} \tag{4.1.22}$$

with the state, flux vectors and rotation vectors

$$\vec{w} = (u, \eta, v)^T, \quad \vec{f} = (g\,\eta, H\,u, 0)^T, \quad \text{and} \quad \vec{S} = \mathsf{f}\,(v, 0, -u)^T, \tag{4.1.23}$$

in which $(\cdot,\cdot,\cdot)^T$ denotes the transpose. Consider a one-dimensional domain D of length L, so in the scaled domain $x \in [0, 1]$. (see §1.5.2.)

(f) Define the energy

$$E(t) = \frac{1}{2}\int_0^L H\,(u^2 + v^2) + g\,\eta^2 \mathrm{d}x. \tag{4.1.24}$$

Show that the simplified and linearized shallow water equations (4.1.22) and (4.1.23) conserve energy (4.1.24) for certain (say periodic) boundary conditions:

$$\mathrm{d}E/\mathrm{d}t = 0. \tag{4.1.25}$$

(Multiply the simplified and linearized shallow water equations by $(H\,u, g\,\eta, H\,v)^T$, and integrate over the domain.)

(g) Introduce N finite volume cells in the domain D. Cell k has cell edges at $x = x_{k-1/2}$ and $x = x_{k+1/2}$. Integrate (4.1.22) in space over cell k using the definition of the mean

$$\vec{U}_k(t) = \frac{1}{h_k} \int_{x_{k-1/2}}^{x_{k+1/2}} \vec{w}(x,t)\mathrm{d}x \tag{4.1.26}$$

with cell length $h_k = x_{k+1/2} - x_{k-1/2}$. Show that we obtain

$$\frac{d\vec{U}_k}{dt} + \frac{1}{h_k}\left(f(\vec{w}_{k+1/2}) - f(\vec{w}_{k-1/2})\right) = \frac{1}{h_k}\int_{x_{k-1/2}}^{x_{k+1/2}} \vec{S}(\vec{w})\mathrm{d}x \tag{4.1.27}$$

with $\vec{w}_{k+1/2} = \vec{w}(x_{k+1/2}, t)$. (Variation on §2.2.)

(h) We approximate (4.1.27) further by assuming that we only know the mean $\vec{U}_k$ in each cell. Hence, in the integral in (4.1.27) in cell k we use $\vec{S}(\vec{w}) \approx S(\vec{U}_k)$ the latter which is constant in cell k ($\mathsf{f} \neq 0$). Furthermore, we approximate the flux as follows

$$\vec{f}(\vec{w}_{k+1/2}) \approx \vec{F}(\vec{U}_k, \vec{U}_{k+1}) \tag{4.1.28}$$

with the numerical flux $\vec{F}(\cdot,\cdot)$. Consider and use the numerical fluxes

$$\vec{F}(\vec{U}_k, \vec{U}_{k+1}) = (g\,\bar{\eta}_k, H\,\bar{u}_{k+1}, 0)^T \quad \text{or} \quad \vec{F}(\vec{U}_k, \vec{U}_{k+1}) = (g\,\bar{\eta}_{k+1}, H\,\bar{u}_k, 0)^T \tag{4.1.29}$$

with the means $\vec{U}_k = (\bar{u}_k, \bar{\eta}_k, \bar{v}_k)^T$. Write down the spatial discretization. (Variation on §2.2.)

(i) Construct harmonic solutions for the the simplified and linearized shallow water equations ($\mathsf{f} \neq 0$) (4.1.22) and (4.1.23) in a periodic domain:

$$v(x,t) = -\frac{\mathsf{f}\,A}{H\,k}\cos\left(k\,x - \omega\,t\right) \tag{4.1.30}$$

$$u(x,t) = \frac{\omega\,A}{H\,k}\sin\left(k\,x - \omega\,t\right) \tag{4.1.31}$$

$$\eta(x,t) = A\,\sin\left(k\,x - \omega\,t\right) \tag{4.1.32}$$

with amplitude A, frequency $\omega = \sqrt{\mathsf{f}^2 + g\,H\,k^2}$ and wave number $k = 2\,\pi\,n/L$ for integer $n > 0$. Take $g = 9.8\,m/s^2$, $f = 10^{-4}\,s^{-1}$, $L_s = 100\,km$, $U = 5\,m/s$ and use these values to scale the equations. (Substitution.)

(j) Define the discretized energy

$$E_h = \frac{1}{2\,L}\sum_{k=1}^{N} h_k\,\left(H\,(\bar{u}_k^2 + \bar{v}_k^2) + g\,\bar{\eta}_k^2\right). \tag{4.1.33}$$

Show that it is conserved in time (for suitable boundary conditions and $\mathsf{f} \neq 0$):

$$\mathrm{d}E_h/\mathrm{d}t = 0 \tag{4.1.34}$$

by using a similar calculation as for the continuous case.

(k) As we have not yet discretized time but only space in a finite volume discretization, we need to choose a time discretization. Thus use a forward Euler discretization in time for the linearized shallow water equations. Estimate the CFL condition and hence the time step. It turns out that the forward Euler discretization of time is unstable. Why? (Use common sense and a Fourier analysis on a regular grid for $\mathsf{f} = 0$ with and without discretizing time. Take the variables $\sim exp(i\,(k\,x + \omega\,t))$ and $\lambda^n\,exp(i\,k\,x_j)$, respectively. §4.4 Morton and Mayers (2005).)

(l) Instead, we use the Störmer-Verlet time discretization (Hairer et al., 2002; see also the first exercise in chapter 4.2)

$$\begin{cases} \bar{u}_k^{n+1/2} - \mathsf{f}\,\Delta t\,\bar{v}_k^{n+1/2}/4 = \bar{u}_k^n - \frac{\Delta t}{2\,h_k}\,g\,(\bar{\eta}_k^n - \bar{\eta}_{k-1}^n) + \mathsf{f}\,\Delta t\,\bar{v}_k^n/4 \\ \bar{v}_k^{n+1/2} + \mathsf{f}\,\Delta t\,\bar{u}_k^{n+1/2}/4 = \bar{v}_k^n - \mathsf{f}\,\Delta t\,\bar{u}_k^n/4 \end{cases} \tag{4.1.35}$$

$$\bar{\eta}_k^{n+1} = \bar{\eta}_k^n - \frac{\Delta t}{h_k}\,H\,(\bar{u}_{k+1}^{n+1/2} - \bar{u}_k^{n+1/2}) \tag{4.1.36}$$

$$\begin{cases} \bar{u}_k^{n+1} - \mathsf{f}\,\Delta t\,\bar{v}_k^{n+1}/4 = \bar{u}_k^{n+1/2} - \frac{\Delta t}{2\,h_k}\,g\,(\bar{\eta}_k^{n+1} - \bar{\eta}_{k-1}^{n+1}) + \mathsf{f}\,\Delta t\,\bar{v}_k^{n+1/2}/4 \\ \bar{v}_k^{n+1} + \mathsf{f}\,\Delta t\,\bar{u}_k^{n+1}/4 = \bar{v}_k^{n+1/2} - \mathsf{f}\,\Delta t\,\bar{u}_k^{n+1/2}/4 \end{cases} \tag{4.1.37}$$

Argue that this is a time discretization for the linearized shallow waters equation. A simplified argument considering the cases $\mathsf{f} = 0$ and $g = 0$ in separation is sufficient. That is, show for the case $g = 0$ that the (kinetic) energy in each cell $\propto (\bar{u}_k^2 + \bar{v}_k^2)$ is conserved (ignore $\bar{\eta}_k$). Show how to solve for the (intermediate) velocities $u^{n+1/2}, v^{n+1/2}$ and u^{n+1}, v^{n+1} by a simple inversion in every cell k.

(m) Implement and check the numerical finite volume discretization by comparing the exact solutions with the numerical solution for different values of the resolution in several plots after two periods using $n = 1$ or $n = 2$. Use one mode as exact solution but also the sum of two or more modes each with different A, k, ω (specify). Plot the results for one resolution in one plot for each variable. Compare exact and numerical results by plotting their difference at a selected final time for several resolutions (say $20, 50$ and 100 cells). Show that the Euler forward discretization is indeed unstable even for small CFL number, while the Störmer-Verlet method is (more) stable (also try running for a longer amount of periods to check if any instabilities occur at a later time and try different CFL numbers). Take a sensible choice of (nondimensional) parameters, i.e. $A = 1, g = 1, L = 1$ and $f = 2$. Specify the values used. (Note: as an intermediate step, test your code also with $\mathsf{f} = 0$.)

(n) Investigate whether the numerical energy is conserved and provide a plot of $E_h(t)$ versus time. What is the conclusion?

(o) *Bonus question (i.e. start playing with your code):*
Perform a calculation with the right end at $x = L$ open and the left end at $x = 0$ closed with $\mathsf{f} = 0$. Hence, we model a narrow inlet with the open tidal sea conditions at $x = L$. Explain how you define the velocity needed one cell beyond the wall at $x = 0$. Clearly specify relevant incoming tidal conditions. Check an exact solution for this case, and explain your calculation, see Gill (1982).

vii. **Acoustic waves**

(a) Acoustic equations describe the propagation of sound around a state of rest. Choose a constant density as rest state. Find the acoustic equations relative to a constant rest state in the literature or on the internet. Write them down in one and two spatial dimensions. In one spatial dimension, there will be one equation for the (perturbation) density $\rho(x,t)$ and one for the velocity $u(x,t)$. The speed of sound $c_0 = \sqrt{\partial p_0/\partial \rho_0}$ evaluated at the constant rest density ρ_0 with rest pressure $p_0(\rho_0)$ will appear material property.

(b) Apply the previous exercise on shallow water seiches to these acoustic equations. Note that density ρ now plays the role of the depth h, and the acoustic wave speed c_0 plays the role of the gravity wave speed $\sqrt{g\,H}$. Hence ρ_0 plays the role of the rest depth H. That is: come up with an energy-preserving finite-volume discretization in one spatial dimension and test the scheme against exact wave solutions.

(c) How does the scheme extend to two dimensions?

(See also exercise 4.10 in Morton and Mayers 2005.)

4.2 Worked examples

i. (a) Discretize the differential equation

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} = f(u) \qquad \text{with} \qquad u(0) = u_0, \quad \frac{\mathrm{d}u}{\mathrm{d}t}(0) = v_0 \tag{4.2.1}$$

for a given function $f(u)$ and $u = u(t)$, second order in time using a central difference scheme. What is the truncation error? The resulting discretization is the symplectic, Störmer-Verlet scheme (see Hairer et al., 2002).

(b) Rewrite (4.2.1) as a system

$$\frac{\mathrm{d}v}{\mathrm{d}t} = f(u) \qquad \text{with} \qquad u(0) = u_0 \tag{4.2.2}$$

$$\frac{\mathrm{d}u}{\mathrm{d}t} = v \qquad \text{with} \qquad v(0) = v_0. \tag{4.2.3}$$

Define

$$v^n = \frac{u^{n+1} - u^{n-1}}{2\,\Delta t}, \tag{4.2.4}$$

introduce an intermediate time level $v^{n+1/2}$, and by eliminating u^{n-1} from (4.2.4) and the second order discretization of (4.2.1), obtain a three-step scheme.

Answers:

(a) The second order discretization of (4.2.1) is

$$\frac{u^{n+1} - u^n + u^{n-1}}{\Delta t^2} = f(u^n). \tag{4.2.5}$$