

Hamiltonian Wave–Vortex and Vortex Models

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We consider a generic finite-dimensional generalized Hamiltonian system with fast and slow modes, where the ratio of the fast and slow timescales defines a small parameter ε . Such a formulation includes many systems of interest in geophysical fluid dynamics, where to leading order in ε the slow motion is nonlinear vortex dynamics and the fast motion consists of linear waves.

Given such a Hamiltonian formulation, we show that it is possible to construct truncated Hamiltonian “wave–vortex” models formally correct to $\mathcal{O}(1)$. This is a non-trivial matter because of the nonlinearity of the Jacobi identity: most truncations of a generalized Hamiltonian system will destroy its Hamiltonian structure. In the first, more general, wave–vortex model the slow and fast modes are coupled only through $\mathcal{O}(\varepsilon)$ terms in the Hamiltonian with no fast–slow coupling in the Poisson bracket. This model permits the separation of the fast and slow motions in a Hamiltonian framework using techniques such as canonical averaging. In both derivations, Jacobi’s identity for the truncated model follows from that for the parent model. Furthermore, a consistent leading-order Hamiltonian slow “vortex” model emerges from both wave–vortex models at leading order (corresponding to a singular perturbation) in ε . This provides a systematic way to derive Hamiltonian models of slow, or balanced, dynamics.

As an illustration of the present approach, we derive reduced models for the rapidly rotating shallow-water equations with the Rossby number as ε .

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1. Introduction

In many applications one encounters problems with two timescales in which the dependent variables z can be divided into slow variables s and fast variables f after suitable scaling. The ratio of fast to slow timescales then defines a small parameter ε . Following Warn *et al.* (1995), we thus consider the generic system (written in the slow timescale)

$$\begin{aligned} \frac{ds}{dt} &= \mathcal{S}(s, f; \varepsilon) \\ \frac{df}{dt} + \frac{1}{\varepsilon} \Gamma f &= \mathcal{F}(s, f; \varepsilon). \end{aligned} \tag{1.1}$$

Here \mathcal{S} and \mathcal{F} can be considered as functions or operators, and Γ is a constant invertible skew-hermitian matrix (or a linear operator with purely imaginary eigen-

values). The last property implies that f undergoes rapid energy-conserving oscillations in the limit $\varepsilon \rightarrow 0$. Hence the fast motions are waves, not damped motion; this property distinguishes the present approach from centre manifold theory (Carr, 1981).

In many physical systems, particularly in inviscid models of geophysical fluid dynamics, there is an important additional property: The system (1.1) often possesses a (generalized) Hamiltonian structure, meaning that it can be written as

$$\frac{dz}{dt} = \{z, H\} \quad (1.2)$$

for a *Hamiltonian* $H(z)$ and a *Poisson bracket* $\{\cdot, \cdot\}$ that will both be defined below.

Examples of systems of the form (1.1) and (1.2) can be found, e.g., in Olver (1986) and Shepherd (1990); Bokhove (2002a) shows examples in which the prototypical singular form emerges from existing Hamiltonian formulations. In particular, the isopycnic or isentropic, hydrostatic rapidly-rotating multi-layer equations of motion describing large-scale atmosphere or ocean dynamics can be brought into the desired archetypal Hamiltonian formulation (using, e.g., Bokhove, 2002b), as can the compressible, isentropic equations of motion. In the first case, the Rossby number appears as small parameter ε as a result of rapid rotation, and in the second case the Mach number appears as ε .

Approximating generalized Hamiltonian systems while preserving their Hamiltonian structure is a non-trivial matter because of the nonlinearity of the Jacobi identity (Olver, 1984, 1996). Our first aim is to derive Hamiltonian approximate models containing both vortical and wave motions (henceforth, a wave–vortex model); in this paper we derived two such models. Unlike the weak-wave model of Nore and Shepherd (1997), however, here we do not assume that the waves are weak. Next, we derive a model containing only vortical motion (henceforth, a vortex model), which corresponds to the classical “balance” limit, by taking the wave motion to be zero (to within the order of the approximation). For the two physical systems described above, the respective vortex models are the quasi-geostrophic and the two- or three-dimensional incompressible Euler equations in Hamiltonian form (see Bokhove, 2002a). Wave–vortex models for the one-layer isopycnic or shallow-water equations are presented in section 4.

For their more heuristically derived weak-wave model, Nore and Shepherd (1997) proved nonlinear stability theorems and derived saturation bounds for unstable basic states. While the cosymplectic operator in our first wave–vortex model is essentially the same as the one in Nore and Shepherd’s, although the meaning of the variables is different at $\mathcal{O}(\varepsilon)$, the Hamiltonian is much more complicated (and the model is formally one order higher in accuracy). Hence, it remains open whether one can derive nonlinear stability theorems for the wave–vortex models derived here. Finally, the decoupling of slow and fast variables in the bracket may be useful in designing Hamiltonian discretizations and in canonical averaging techniques (e.g., Wirosoetisno, 1999).

2. Archetypal Generalized Hamiltonian Formulation

In this section and the following we work with finite-dimensional systems for simplicity, although the example in section 4 suggests that in many important cases the

results can be carried over to continuous systems. Let $s \in \mathbf{R}^p$ be the slow variables and $f \in \mathbf{R}^q$ be the fast variables, and let $z = (s, f)$. We denote $\partial_s := \partial/\partial s$ and $\partial_f := \partial/\partial f$. Individual components will be denoted by superscripts: $z^i, s^i, f^i, \partial_s^i$ and ∂_f^i . Moreover, $(\partial_f g)^{ij} := \partial g^j/\partial f^i$, etc. Repeated indices are understood to be summed over the relevant ranges.

For the asymptotics we adopt the following notation: Any function written in the form $F(z; \varepsilon)$ is understood to be of $\mathcal{O}(1)$, meaning that $\lim_{\varepsilon \rightarrow 0} F(z; \varepsilon)$ is finite. By $\mathcal{O}(\varepsilon^n)$ we mean $\varepsilon^n F(z; \varepsilon)$ for some $F(z; \varepsilon)$.

The generalized (this qualifier is understood implicitly henceforth) Poisson bracket $\{F, G\}$ of two functions $F(z)$ and $G(z)$ is a derivation, $\{F, GK\} = \{F, G\}K + G\{F, K\}$, which is antisymmetric, $\{F, G\} = -\{G, F\}$, and which obeys Jacobi's identity, $\{F, \{G, K\}\} + \{G, \{K, F\}\} + \{K, \{F, G\}\} = 0$ (e.g., Olver, 1986). More specifically, we write

$$\{F, G\} = (\partial F/\partial z^i)\{z^i, z^j\}(\partial G/\partial z^j). \quad (2.1)$$

In our problem, $\{z^i, z^j\}$ is by hypothesis given by

$$\begin{aligned} \{s^i, s^j\} &= J^{ij} &= J_0^{ij}(s) + \varepsilon J_1^{ij}(s, f; \varepsilon) \\ \{f^i, s^j\} &= L^{ij} &= L_0^{ij}(s, f) + \varepsilon L_1^{ij}(s, f; \varepsilon) \\ \{f^i, f^j\} &= -\frac{1}{\varepsilon}T^{ij} + Y^{ij} &= -\frac{1}{\varepsilon}T^{ij} + Y_0^{ij}(s) + \varepsilon Y_1^{ij}(s, f; \varepsilon). \end{aligned} \quad (2.2)$$

We note that T is a constant invertible skew-symmetric matrix, and that antisymmetry of the Poisson bracket dictates that $\{s, f\} = -L^T$, where the superscript T denotes matrix transpose. Here and in the rest of this paper, J_0, T, Y_0 , and also A, R_0, g and h introduced below are understood to denote fixed *functions* of their arguments.

For conciseness and readability, whenever possible we shall use a vector-matrix notation in which (2.2a) reads $\{s, s\} = J$ and so on. That J_0 cannot depend on f can be seen by considering Jacobi's identity $\{s, \{s, f\}\} + \dots = 0$ at $\mathcal{O}(1/\varepsilon)$. That Y_0 depends only on s is a condition that we impose, because many Poisson brackets of hydrodynamic type obey this condition.

We consider an archetypal Hamiltonian formulation of (1.1), viz.,

$$\frac{dz^i}{dt} = \{z^i, H\} = \{z^i, z^j\}(\partial H/\partial z^j). \quad (2.3)$$

Taking $H(z; \varepsilon)$ to be of the form

$$H(s, f; \varepsilon) = \frac{1}{2}f^T A f + R_0(s) + \varepsilon R_1(s, f; \varepsilon), \quad (2.4)$$

where A is a constant symmetric matrix, (2.3) and the Poisson bracket (2.2) then imply the equations of motion

$$\begin{aligned} \frac{ds}{dt} &= J \partial_s H - L^T \partial_f H \\ \frac{df}{dt} &= L \partial_s H - \frac{1}{\varepsilon}T \partial_f H + Y \partial_f H. \end{aligned} \quad (2.5)$$

For (2.5) to take the form (1.1) we must have $T = \Gamma A^{-1}$.

3. $\mathcal{O}(1)$ Hamiltonian Perturbation

In perturbation theory one constructs a system

$$\begin{aligned} \frac{ds}{dt} &= \mathcal{S}'(s, f; \varepsilon) \\ \frac{df}{dt} + \frac{1}{\varepsilon} \Gamma f &= \mathcal{F}'(s, f; \varepsilon), \end{aligned} \quad (3.1)$$

which seeks to approximate the original system (1.1) to a given order in ε . We say that the system (3.1) is an $\mathcal{O}(\varepsilon^n)$ *regular perturbation* of the original system (1.1) if $\mathcal{S}'(s, f; \varepsilon) - \mathcal{S}(s, f; \varepsilon) = \mathcal{O}(\varepsilon^{n+1})$ and $\mathcal{F}'(s, f; \varepsilon) - \mathcal{F}(s, f; \varepsilon) = \mathcal{O}(\varepsilon^{n+1})$ for all values of (s, f) . By an $\mathcal{O}(1)$ *singular perturbation* of (1.1) we mean a system of the form (3.1) such that $\mathcal{S}'(s, f; \varepsilon) - \mathcal{S}(s, f; \varepsilon) = \mathcal{O}(\varepsilon)$ and $\mathcal{F}'(s, f; \varepsilon) - \mathcal{F}(s, f; \varepsilon) = \mathcal{O}(\varepsilon)$ on the ‘‘slow manifold’’ $(s, f) = 0$. We refer the reader to Warn *et al.* (1995) and Wirosoetisno *et al.* (2002) for more background.

The basis of our Hamiltonian perturbation theory is a change of variables $z = (s, f) \mapsto \tilde{z} = (\tilde{s}, \tilde{f})$. In terms of the new variable \tilde{z} , the generalized Poisson bracket of two functions $F(\tilde{z})$ and $G(\tilde{z})$ is given by

$$\{F, G\} = (\partial F / \partial \tilde{z}^i) \{z^i, z^j\} (\partial G / \partial \tilde{z}^j). \quad (3.2)$$

The following lemma is of central importance:

Lemma 3.1. *Given the Poisson bracket (2.2), one can find functions g and h (locally) such that with*

$$\tilde{s} := s + \varepsilon g(s, f) \quad \text{and} \quad \tilde{f} := f + \varepsilon h(s, f), \quad (3.3)$$

one has

$$\{\tilde{s}, \tilde{s}\} = J_0(\tilde{s}) + \mathcal{O}(\varepsilon), \quad \{\tilde{s}, \tilde{f}\} = \mathcal{O}(\varepsilon) \quad \text{and} \quad \{\tilde{f}, \tilde{f}\} = -\frac{1}{\varepsilon} T + \mathcal{O}(\varepsilon). \quad (3.4)$$

We note that g and h , which we choose to satisfy [cf. (3.14) and (3.19)]

$$\partial_f g = T^{-1} L_0 \quad \text{and} \quad \partial_f h = \frac{1}{2} T^{-1} Y_0, \quad (3.5)$$

are not unique. The proof of this lemma will appear at the end of this section; we first discuss some of its consequences. We note that the inverse of (3.3) is

$$s = \tilde{s} - \varepsilon g(\tilde{s}, \tilde{f}) + \mathcal{O}(\varepsilon^2) \quad \text{and} \quad f = \tilde{f} - \varepsilon h(\tilde{s}, \tilde{f}) + \mathcal{O}(\varepsilon^2). \quad (3.6)$$

Using this, we compute the Hamiltonian in the (\tilde{s}, \tilde{f}) variables,

$$\begin{aligned} \tilde{H}(\tilde{s}, \tilde{f}; \varepsilon) &= H(\tilde{s} - \varepsilon g, \tilde{f} - \varepsilon h) + \mathcal{O}(\varepsilon^2) \\ &= \frac{1}{2} \tilde{f}^T A \tilde{f} - \varepsilon \tilde{f}^T A h(\tilde{s}, \tilde{f}) \\ &\quad + R_0(\tilde{s}) - \varepsilon \tilde{\partial}_s R_0(\tilde{s}) g(\tilde{s}, \tilde{f}) + \varepsilon R_1(\tilde{s}, \tilde{f}; 0) + \mathcal{O}(\varepsilon^2), \end{aligned} \quad (3.7)$$

where $\tilde{\partial}_s = \partial / \partial \tilde{s}$.

Let us now introduce the $\mathcal{O}(1)$ Poisson bracket: For any $F(\tilde{z})$ and $G(\tilde{z})$, let

$$\{F, G\}_1 := (\tilde{\partial}_s F)^T J_0(\tilde{s}) \tilde{\partial}_s G - \frac{1}{\varepsilon} (\tilde{\partial}_f F)^T T \tilde{\partial}_f G \quad (3.8)$$

where J_0 and T are defined in (2.2), and $\tilde{\partial}_f = \partial/\partial\tilde{f}$. From this definition it is clear that $\{\cdot, \cdot\}_1$ is an antisymmetric derivation; the fact that it also satisfies Jacobi's identity follows from considering Jacobi's identity for the original bracket $\{\cdot, \cdot\}$ at leading order in ε .

Hence, we arrive at the following result.

Theorem 3.2. *Let the Poisson bracket $\{\cdot, \cdot\}_1$ be that defined in (3.8), let \tilde{s} and \tilde{f} be those in lemma 3.1, with g and h satisfying (3.5), and let*

$$\tilde{H}_1(\tilde{s}, \tilde{f}) := \frac{1}{2}\tilde{f}^T A \tilde{f} + R_0(\tilde{s}) - \varepsilon \tilde{\partial}_s R_0(\tilde{s})g(\tilde{s}, \tilde{f}) - \varepsilon \tilde{f}^T A h(\tilde{s}, \tilde{f}) + \varepsilon R_1(\tilde{s}, \tilde{f}; 0). \quad (3.9)$$

Then the Hamiltonian system

$$\frac{d\tilde{z}}{dt} = \{\tilde{z}, \tilde{H}_1\}_1 \quad (3.10)$$

is an $\mathcal{O}(1)$ regular perturbation of the original system (2.2), (2.3) and (2.4).

That the system (3.10) is Hamiltonian follows from its construction; that it is indeed an $\mathcal{O}(1)$ perturbation of the original system can be checked by direct computation of the equations of motion. We note that this Hamiltonian system is more ‘‘complicated’’ than a non-Hamiltonian $\mathcal{O}(1)$ perturbation. This seems to be the price that one has to pay to retain the Hamiltonian structure (which does impose strong constraints on the system). One important advantage of the system (3.10) is that its Poisson bracket is essentially canonical in the fast variable \tilde{f} .

If we now perform a singular perturbation to the system (3.10), we find that, to leading order, the slow manifold is $(\tilde{s}, \tilde{f} = 0)$ and the dynamics is governed by

$$\frac{d\tilde{s}}{dt} = J_0(\tilde{s})\tilde{\partial}_s R_0(\tilde{s}). \quad (3.11)$$

This is the analogue of theorem 3.2, but the fact that $\tilde{f} = 0$ allows us to go one step further: Transforming back to the original (s, f) variables, we find that the Hamiltonian form (3.11) is invariant to $\mathcal{O}(1)$, giving us the following

Theorem 3.3. *For any $F(s)$, the Hamiltonian system*

$$\frac{ds}{dt} = \{s, H_0(s)\}_0 = J_0(s)\partial_s R_0 \quad (3.12)$$

with $\{F, G\}_0 := (\partial_s F)^T J_0(s)\partial_s G$ and $H_0(s) := R_0(s)$ is an $\mathcal{O}(1)$ singular perturbation of the original system (2.2), (2.3) and (2.4).

Proof of lemma 3.1. (3.4a) follows directly from (2.2) and (3.3). We next compute

$$\begin{aligned} \{\tilde{f}, \tilde{s}\} &= \{f + \varepsilon h, s + \varepsilon g\} \\ &= \{f, s\} + \varepsilon\{f, g\} + \varepsilon\{h, s\} + \varepsilon^2\{h, g\} \\ &= L_0(s, f) - T \partial_f g + \mathcal{O}(\varepsilon). \end{aligned} \quad (3.13)$$

Thus, (3.4b) will hold if we can find a $g(s, f)$ satisfying

$$\partial_f g = T^{-1}L_0. \quad (3.14)$$

By lemma 4.1 (see Appendix A), and turning to index notation, a necessary and sufficient condition for this is that

$$\partial_f^k (T^{-1} L_0)^{ij} = \partial_f^i (T^{-1} L_0)^{kj}. \quad (3.15)$$

To show that this holds, we consider the Jacobi identity

$$\begin{aligned} 0 &= \{s^i, \{f^j, f^k\}\} + \{f^j, \{f^k, s^i\}\} + \{f^k, \{s^i, f^j\}\} \\ &= \{s^i, Y^{jk}\} + \{f^j, L^{ki}\} - \{f^k, L^{ji}\} \\ &= -\frac{1}{\varepsilon} T^{jl} \partial_f^l L_0^{ki} + \frac{1}{\varepsilon} T^{kl} \partial_f^l L_0^{ji} + \mathcal{O}(1). \end{aligned} \quad (3.16)$$

Since Jacobi's identity holds for any ε , the $\mathcal{O}(1/\varepsilon)$ terms above must vanish independently of the $\mathcal{O}(1)$ terms,

$$T^{kl} \partial_f^l L_0^{ji} - T^{jl} \partial_f^l L_0^{ki} = 0. \quad (3.17)$$

Multiplying this equation by $(T^{-1})^{mj} (T^{-1})^{nk}$ followed by some manipulations gives us (3.15) and establishes the existence of g .

We now turn to

$$\begin{aligned} \{\tilde{f}, \tilde{f}\} &= \{f + \varepsilon h, f + \varepsilon h\} \\ &= -\frac{1}{\varepsilon} T + Y - (\partial_f h)^T T - T \partial_f h + \mathcal{O}(\varepsilon). \end{aligned} \quad (3.18)$$

Therefore, the condition that needs to be satisfied for $\{\tilde{f}, \tilde{f}\} = -(1/\varepsilon)T + \mathcal{O}(\varepsilon)$ is, with a slight rewriting,

$$T \partial_f h - (T \partial_f h)^T = Y_0(s). \quad (3.19)$$

Note that the l.h.s. of (3.19) is antisymmetric, as is $Y_0(s)$ directly from (2.2). Now any h that solves

$$\partial_f h = \frac{1}{2} T^{-1} Y_0 \quad (3.20)$$

will also solve (3.19), since T is antisymmetric and invertible. The last equation is solvable if its r.h.s. satisfies the condition of lemma 4.1, i.e., if

$$\partial_f^k (T^{-1} Y_0)^{ij} = \partial_f^i (T^{-1} Y_0)^{kj}. \quad (3.21)$$

But since $\partial_f Y_0(s) \equiv 0$, this is trivial, thus proving the existence of h . \square

Note that we have been unable to prove consistency and sufficiency conditions on h in the general case where Y_0 depends on f as well as on s . However, if we can find an h (twice differentiable) satisfying $T \partial_f h - (T \partial_f h)^T = Y_0(s, f)$, Jacobi's identity $0 = \{\tilde{f}^i, \{\tilde{f}^j, \tilde{f}^k\}\} + \{\tilde{f}^j, \{\tilde{f}^k, \tilde{f}^i\}\} + \{\tilde{f}^k, \{\tilde{f}^i, \tilde{f}^j\}\}$ follows.

In certain cases, it is possible to "truncate" the Poisson bracket *without* a change of variables. This is summarized in the following result.

Theorem 3.4. *Consider the original Hamiltonian system (2.2), (2.3) and (2.4) with $L_0 = L_0(s)$ in (2.2b). Then the Hamiltonian system*

$$\frac{dz}{dt} = \{z, H\}_{1'} \quad (3.22)$$

with the Poisson bracket

$$\begin{aligned} \{F, G\}_{1'} &:= (\partial_s F)^\top J_0(s) \partial_s G + (\partial_f F)^\top L_0(s) \partial_s G \\ &\quad - (\partial_s F)^\top [L_0(s)]^\top \partial_f G + (\partial_f F)^\top \left[-\frac{1}{\varepsilon} T + Y_0(s)\right] \partial_f G \end{aligned} \quad (3.23)$$

and the Hamiltonian (2.4) is an $\mathcal{O}(1)$ regular perturbation of the original system.

Proof of theorem 3.4. That the dynamical system (3.22)–(3.23) is an $\mathcal{O}(1)$ perturbation of the original system can be verified directly by writing down the equations of motion. That the bracket (2.2) satisfies Jacobi’s identity is less obvious: It can be verified by observing that in Jacobi’s identity for (2.2) [with $L_0 = L_0(s)$] at leading order, that is, at $\mathcal{O}(1)$, only functional derivatives of the bracket with respect to s appear and no $\mathcal{O}(1/\varepsilon)$ terms (beside the usual second-order derivatives of F, G, K with respect to s, f which cancel due to the antisymmetric nature of the bracket). Hence, it coincides with Jacobi’s identity for the bracket (3.23) truncated to $\mathcal{O}(1)$. \square

Theorem 3.4 is actually slightly more general: It holds for Hamiltonian systems of the form (2.2), (2.3) and (2.4) with f everywhere replaced by a “mixed” slow–fast variable $u = f + Ms$ for a matrix (or linear operator) M . This is the form relevant to subsection 4a(ii) below.

4. Example: Shallow-Water Equations

As an example of the results in the previous section, here we derive Hamiltonian vortex and wave–vortex models for the rapidly rotating one-layer isopycnic or shallow-water equations in a periodic domain. We note that, since our system is infinite-dimensional, lemma 3.1 as proved above does not apply, but the example here suggests that, possibly with additional hypotheses, it can be extended to continuous systems (where g and h are nonlinear operators acting on f and s).

The shallow-water equations can be written in the following Hamiltonian form (cf. Shepherd, 1990),

$$\frac{d\mathcal{F}}{dt} = \{\mathcal{F}, \mathcal{H}\} \quad (4.1)$$

with Poisson bracket

$$\{\mathcal{F}, \mathcal{G}\} = \int \left\{ q \hat{\mathbf{z}} \cdot \frac{\delta \mathcal{F}}{\delta \mathbf{v}} \times \frac{\delta \mathcal{G}}{\delta \mathbf{v}} + \frac{1}{\varepsilon \text{Fr}} \frac{\delta \mathcal{G}}{\delta \mathbf{v}} \cdot \nabla \frac{\delta \mathcal{F}}{\delta \eta} - \frac{1}{\varepsilon \text{Fr}} \frac{\delta \mathcal{F}}{\delta \mathbf{v}} \cdot \nabla \frac{\delta \mathcal{G}}{\delta \eta} \right\} dx dy \quad (4.2)$$

and Hamiltonian

$$\mathcal{H} = \frac{1}{2} \int \left\{ (1 + \varepsilon \text{Fr} \eta) |\mathbf{v}|^2 + \text{Fr} \eta^2 \right\} dx dy. \quad (4.3)$$

Here \mathbf{v} is the horizontal velocity, non-dimensionalized using a velocity scale U ; η is the deviation of the free surface from rest, where the total depth $1 + \varepsilon \text{Fr} \eta$ has been non-dimensionalized by a mean water depth d_w ; the spatial and temporal derivatives have been non-dimensionalized by a horizontal length scale ℓ and by U . Also, $q = (1/\varepsilon + \hat{\mathbf{z}} \cdot \nabla \times \mathbf{v})/(1 + \varepsilon \text{Fr} \eta)$, $\hat{\mathbf{z}}$ is the vertical unit vector, and $\delta \mathcal{F}/\delta \eta$ denotes the functional derivative of \mathcal{F} with respect to η , etc. The non-dimensional

parameters are the Rossby number $\varepsilon = U/(f\ell)$ and the rotational Froude number $\text{Fr} = f^2 \ell^2 / (g d_w)$, where here f is the Coriolis parameter and g the gravitational acceleration.

(a) *Wave-Vortex Models*

(i) *Slow and Fast Variables*

The prototypical singular form (1.1) of the shallow-water equations can be obtained by transforming the dependent variables (\mathbf{v}, η) to (Q, D, Υ) , where

$$\begin{aligned} Q &= \nabla^2 \psi - \text{Fr} \eta \\ D &= \nabla \cdot \mathbf{v} =: \nabla^2 \chi \\ \Upsilon &= \nabla^2 \psi - \nabla^2 \eta \end{aligned} \quad (4.4)$$

are the linearized potential vorticity, divergence, and geostrophic imbalance, respectively (Warn *et al.*, 1995). Here the streamfunction ψ and velocity potential χ are related to the horizontal velocity \mathbf{v} by

$$\mathbf{v} = \hat{\mathbf{z}} \times \nabla \psi + \nabla \chi. \quad (4.5)$$

In the variables (4.4) the Poisson bracket (4.2) takes the archetypal form (2.2), i.e.,

$$\begin{aligned} \{\mathcal{F}, \mathcal{G}\} &= \int \left\{ \frac{Q}{1 + \varepsilon \text{Fr} \eta} \left[J \left(\frac{\delta \mathcal{F}}{\delta D}, \frac{\delta \mathcal{G}}{\delta D} \right) + J \left(\frac{\delta \mathcal{F}}{\delta \Upsilon} + \frac{\delta \mathcal{F}}{\delta Q}, \frac{\delta \mathcal{G}}{\delta \Upsilon} + \frac{\delta \mathcal{G}}{\delta Q} \right) \right. \right. \\ &\quad \left. \left. + \left(\nabla \frac{\delta \mathcal{F}}{\delta D} \right) \cdot \nabla \left(\frac{\delta \mathcal{G}}{\delta \Upsilon} + \frac{\delta \mathcal{G}}{\delta Q} \right) - \left(\nabla \frac{\delta \mathcal{G}}{\delta D} \right) \cdot \nabla \left(\frac{\delta \mathcal{F}}{\delta \Upsilon} + \frac{\delta \mathcal{F}}{\delta Q} \right) \right] \right. \\ &\quad \left. + \frac{1}{\varepsilon} \left[\left(\nabla^2 \frac{\delta \mathcal{F}}{\delta D} \right) \mathcal{P} \frac{\delta \mathcal{G}}{\delta \Upsilon} - \left(\nabla^2 \frac{\delta \mathcal{G}}{\delta D} \right) \mathcal{P} \frac{\delta \mathcal{F}}{\delta \Upsilon} \right] \right\} dx dy, \end{aligned} \quad (4.6)$$

where $\mathcal{P} := (\text{Fr})^{-1} \nabla^2 - 1$, and $J(a, b) := (\partial_x a)(\partial_y b) - (\partial_x b)(\partial_y a)$ is the Jacobian.

The change of variables that brings the Poisson bracket (4.6) into the desired form [cf. (3.4)] can be computed in the following manner. The continuous analogue of (3.14), written as $T\partial_f g = L_0$, is

$$\begin{aligned} &\int \left[\left(\mathcal{P} \frac{\delta \mathcal{F}}{\delta \Upsilon} \right) \nabla^2 \left(\frac{\widehat{dg}}{dD} \frac{\delta \mathcal{G}}{\delta Q} \right) - \left(\nabla^2 \frac{\delta \mathcal{F}}{\delta D} \right) \mathcal{P} \left(\frac{\widehat{dg}}{d\Upsilon} \frac{\delta \mathcal{G}}{\delta Q} \right) \right] dx dy \\ &= \int Q \left[J \left(\frac{\delta \mathcal{F}}{\delta \Upsilon}, \frac{\delta \mathcal{G}}{\delta Q} \right) + \nabla \frac{\delta \mathcal{F}}{\delta D} \cdot \nabla \frac{\delta \mathcal{G}}{\delta Q} \right] dx dy. \end{aligned} \quad (4.7)$$

We denote the Fréchet derivative of g with respect to D by dg/dD , and the adjoint by an overhat; similarly for $dg/d\Upsilon$. Several integrations by parts lead us to

$$\begin{aligned} \frac{dg}{d\Upsilon} u &= \nabla \cdot (Q \nabla \nabla^{-2} \mathcal{P}^{-1} u) \\ \frac{dg}{dD} u &= J(Q, \mathcal{P}^{-1} \nabla^{-2} u) \end{aligned} \quad (4.8)$$

for any function u , giving (we can obviously add any function of Q alone to g)

$$g(Q, D, \Upsilon) = \nabla \cdot (Q \nabla \nabla^{-2} \mathcal{P}^{-1} \Upsilon) + J(Q, \mathcal{P}^{-1} \nabla^{-2} D). \quad (4.9)$$

Upon computing h in a similar fashion, we arrive at

$$\begin{aligned}\tilde{Q} &= Q + \varepsilon [J(Q, \mathcal{P}^{-1}\nabla^{-2}D) + \nabla \cdot (Q\nabla\nabla^{-2}\mathcal{P}^{-1}\Upsilon)] \\ \tilde{D} &= D - \frac{\varepsilon}{2} [J(Q, \nabla^{-2}\mathcal{P}^{-1}\Upsilon) - \nabla \cdot (Q\nabla\mathcal{P}^{-1}\nabla^{-2}D)] \\ \tilde{\Upsilon} &= \Upsilon + \frac{\varepsilon}{2} [J(Q, \mathcal{P}^{-1}\nabla^{-2}D) + \nabla \cdot (Q\nabla\nabla^{-2}\mathcal{P}^{-1}\Upsilon)].\end{aligned}\quad (4.10)$$

As noted above, to $\mathcal{O}(\varepsilon)$ one can compute (Q, D, Υ) from $(\tilde{Q}, \tilde{D}, \tilde{\Upsilon})$ by replacing ε with $-\varepsilon$ and switching the roles of the tilde and non-tilde variables. As in theorem 3.2, it follows that our $\mathcal{O}(1)$ approximate wave–vortex model is given by the Poisson bracket

$$\{\mathcal{F}, \mathcal{G}\}_1 = \int \left\{ \tilde{Q} J\left(\frac{\delta\mathcal{F}}{\delta\tilde{Q}}, \frac{\delta\mathcal{G}}{\delta\tilde{Q}}\right) + \frac{1}{\varepsilon} \left[(\nabla^2 \frac{\delta\mathcal{F}}{\delta\tilde{D}}) \mathcal{P} \frac{\delta\mathcal{G}}{\delta\tilde{\Upsilon}} - (\nabla^2 \frac{\delta\mathcal{G}}{\delta\tilde{D}}) \mathcal{P} \frac{\delta\mathcal{F}}{\delta\tilde{\Upsilon}} \right] \right\} dx dy, \quad (4.11)$$

and the Hamiltonian

$$\tilde{\mathcal{H}}_1(\tilde{Q}, \tilde{D}, \tilde{\Upsilon}) = \frac{1}{2} \int \{ |\mathbf{v}|^2 + \text{Fr} \eta^2 + \varepsilon \text{Fr} \eta |\mathbf{v}|^2 \} dx dy. \quad (4.12)$$

Here \mathbf{v} and η are to be regarded as functions of $(\tilde{Q}, \tilde{D}, \tilde{\Upsilon})$ using $\mathbf{v} = \hat{\mathbf{z}} \times \nabla\psi + \nabla\chi$, (4.4), and the inverse of (4.10); to our $\mathcal{O}(1)$ approximation, one may replace η and \mathbf{v} in the last term by $\tilde{\eta}$ and $\tilde{\mathbf{v}}$ in order to simplify the Hamiltonian.

The equations of motion are given by

$$\begin{aligned}\frac{\partial\tilde{Q}}{\partial t} + J\left(\frac{\partial\tilde{\mathcal{H}}_1}{\partial\tilde{Q}}, \tilde{Q}\right) &= 0 \\ \frac{\partial\tilde{D}}{\partial t} - \frac{1}{\varepsilon} \nabla^2 \mathcal{P} \frac{\delta\tilde{\mathcal{H}}_1}{\delta\tilde{\Upsilon}} &= 0 \\ \frac{\partial\tilde{\Upsilon}}{\partial t} + \frac{1}{\varepsilon} \mathcal{P} \nabla^2 \frac{\delta\tilde{\mathcal{H}}_1}{\delta\tilde{D}} &= 0.\end{aligned}\quad (4.13)$$

When one computes explicitly the functional derivatives $\delta\tilde{\mathcal{H}}_1/\delta\tilde{Q}$, etc., a large number of terms appear. As mentioned above, this appears to be the necessary price for keeping the Hamiltonian structure of the original system.

(ii) Linearized Potential Vorticity and Velocity Variables

In analogy with theorem 3.4, we now present a continuous example where the Poisson bracket can be “truncated” directly without a change of variable. At $\mathcal{O}(1)$, we find a model close to the shallow-water equations, but with a simpler Poisson bracket. By further truncating the Hamiltonian, we find a “weak-wave–vortex” model consisting of linear wave dynamics coupled to nonlinear vortex dynamics.

We start with the Poisson bracket (4.2) in terms of dependent variables (Q, \mathbf{v}) , where $Q = \nabla^2\psi - \text{Fr} \eta$ is the linearized potential vorticity [cf. (4.4a)] and \mathbf{v} is the velocity,

$$\begin{aligned}\{\mathcal{F}, \mathcal{G}\} &= \int \left\{ \frac{Q}{1 + \varepsilon \text{Fr} \eta} J\left(\frac{\delta\mathcal{F}}{\delta Q}, \frac{\delta\mathcal{G}}{\delta Q}\right) + \left(\frac{1}{\varepsilon} + \frac{Q}{1 + \varepsilon \text{Fr} \eta}\right) \hat{\mathbf{z}} \cdot \frac{\delta\mathcal{F}}{\delta\mathbf{v}} \times \frac{\delta\mathcal{G}}{\delta\mathbf{v}} \right. \\ &\quad \left. + \frac{Q}{1 + \varepsilon \text{Fr} \eta} \left[\frac{\delta\mathcal{G}}{\delta\mathbf{v}} \cdot \nabla \frac{\delta\mathcal{F}}{\delta Q} - \frac{\delta\mathcal{F}}{\delta\mathbf{v}} \cdot \nabla \frac{\delta\mathcal{G}}{\delta Q} \right] \right\} dx dy.\end{aligned}\quad (4.14)$$

If we expand the r.h.s. in powers of ε and drop terms of $\mathcal{O}(\varepsilon)$, we find

$$\begin{aligned} \{\mathcal{F}, \mathcal{G}\}_{1'} = & \int \left\{ Q J\left(\frac{\delta\mathcal{F}}{\delta Q}, \frac{\delta\mathcal{G}}{\delta Q}\right) + \left(\frac{1}{\varepsilon} + Q\right) \hat{\mathbf{z}} \cdot \frac{\delta\mathcal{F}}{\delta\mathbf{v}} \times \frac{\delta\mathcal{G}}{\delta\mathbf{v}} \right. \\ & \left. + Q \left[\frac{\delta\mathcal{G}}{\delta\mathbf{v}} \cdot \nabla \frac{\delta\mathcal{F}}{\delta Q} - \frac{\delta\mathcal{F}}{\delta\mathbf{v}} \cdot \nabla \frac{\delta\mathcal{G}}{\delta Q} \right] \right\} dx dy. \end{aligned} \quad (4.15)$$

This turns out to be a good Poisson bracket, as can be directly verified, cf. theorem 3.4.

Let us now consider a model defined by the full Hamiltonian (4.3) and the Poisson bracket (4.15). The equations of motion are, after some calculation,

$$\begin{aligned} \frac{\partial Q}{\partial t} + \nabla \cdot (Q \mathbf{v}) &= \varepsilon \text{Fr} \nabla \cdot (\eta Q \mathbf{v}) \\ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} + \frac{1}{\varepsilon} (\hat{\mathbf{z}} \times \mathbf{v} + \nabla \eta) &= -\varepsilon \text{Fr} \eta Q (\hat{\mathbf{z}} \times \mathbf{v}). \end{aligned} \quad (4.16)$$

This model differs from the shallow-water equations only in that the $\mathcal{O}(\varepsilon)$ terms on the r.h.s. are zero in the latter model.

While the Poisson bracket is simplified, the equations of motion are largely similar to the ones for the original shallow-water equations. However, by truncating the Hamiltonian to

$$\mathcal{H}_{1'} = \frac{1}{2} \int \{ |\mathbf{v}|^2 + \text{Fr} \eta^2 \} dx dy \quad (4.17)$$

a simpler coupled Hamiltonian weak-wave–vortex model emerges,

$$\begin{aligned} \frac{\partial Q}{\partial t} + \nabla \cdot (Q \mathbf{v}) &= 0 \\ \frac{\partial \mathbf{v}}{\partial t} + \frac{1}{\varepsilon} (\hat{\mathbf{z}} \times \mathbf{v} + \nabla \eta) &= 0. \end{aligned} \quad (4.18)$$

In this model, the Q equation has the same form as that in the shallow-water equations, but the velocity dynamics is completely linear. Thus this model describes sufficiently weak waves, coupled to vortex dynamics. Note that the model is accurate to $\mathcal{O}(1/\varepsilon)$ when $\text{Fr} = \mathcal{O}(1)$, but it is accurate to $\mathcal{O}(1/\sqrt{\varepsilon})$ when $\text{Fr} = \mathcal{O}(\sqrt{\varepsilon})$. (This is because the potential energy density $\text{Fr} \eta^2/2 = \mathcal{O}(1/\text{Fr})$ when written in terms of Q and \mathbf{v} .)

In contrast to the weak-wave model of Nore and Shepherd (1997), where the slow equation reads $\partial Q/\partial t + J(\psi, Q) = 0$ with $Q = \nabla^2 \psi - \text{Fr} \eta$, the present weak-wave–vortex model retains all terms in the unapproximated equation for Q , including advection by the full velocity and the term $Q \nabla \cdot \mathbf{v}$. The present formulation in terms of the velocity \mathbf{v} is also more convenient when one studies the effects of solid boundaries on the dynamics, such as Kelvin waves.

(b) Vortex Model

A singular perturbation of the shallow-water equations (cf. theorem 3.3) yields the well-known Hamiltonian formulation of the quasi-geostrophic equations:

$$\frac{\partial Q}{\partial t} = \{Q, \mathcal{H}_0(Q)\}_0, \quad (4.19)$$

where
$$\{\mathcal{F}, \mathcal{G}\}_0 = \int Q J \left(\frac{\delta \mathcal{F}}{\delta Q}, \frac{\delta \mathcal{G}}{\delta Q} \right) dx dy \quad (4.20)$$

and
$$\mathcal{H}_0 = \frac{1}{2} \int (|\nabla \psi|^2 + \text{Fr} \psi^2) dx dy. \quad (4.21)$$

It can also be obtained from the wave–vortex model (4.13) by setting $\tilde{D} = \tilde{\Upsilon} = 0$ and $\tilde{Q} = Q$, or from the weak-wave–vortex model (4.18) at leading order in $\mathcal{O}(\varepsilon)$. In both cases we have $Q = (\nabla^2 - \text{Fr}) \psi$, which is the quasi-geostrophic potential vorticity.

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Appendix A.

In this appendix we derive a result concerning the local solvability of certain systems of first-order partial differential equations with constant coefficients. As in the main text, here the summation convention is in effect.

Lemma 4.1. *Let $x \in \mathbf{R}^d$ and let $K : \mathbf{R}^d \rightarrow \mathbf{R}^d$ be a given smooth function (of x). Then the system*

$$\frac{\partial g}{\partial x^i} = K^i \quad (\text{A } 1)$$

is locally solvable for $g : \mathbf{R}^d \rightarrow \mathbf{R}$ if and only if $\partial K^i / \partial x^j = \partial K^j / \partial x^i$.

The “only if” part is obtained immediately from the equality of mixed partial derivatives by taking $\partial / \partial x^j$ of both sides. To prove the “if” part we use the following

Theorem 4.2. [cf. theorem 1.40 in Olver (1986)] *Let $\{v_1, \dots, v_r\}$ be a system of smooth vector fields in \mathbf{R}^d , with members $v_1(x) = (v_1^1(x), \dots, v_1^d(x))$ where $x \in \mathbf{R}^d$. The system $\{v_1, \dots, v_r\}$ is integrable (that is, the vectors are tangent to an integral submanifold) if and only if it is in involution, namely, there exist smooth functions $c_{ij}^k(x)$ such that*

$$\begin{aligned} [v_i, v_j] &:= (v_i^n \partial_n v_j^m - v_j^n \partial_n v_i^m) \partial_m \\ &= c_{ij}^k(x) v_k. \end{aligned} \quad (\text{A } 2)$$

Here we have used the notation $\partial_i = \partial / \partial x^i$ and we have adopted $(\partial_1, \dots, \partial_d)$ as a basis for the tangent space $T\mathbf{R}^d$.

Proof of lemma 4.1. We construct the following d vector fields in \mathbf{R}^{d+1} : Let

$$v_i^m = \begin{cases} K^i & m = d + 1 \\ 1 & m = i \\ 0 & \text{otherwise.} \end{cases} \quad (\text{A } 3)$$

Since the v_i are linearly independent, the integral submanifold of the system $\{v_1, \dots, v_d\}$, when it exists, is a hypersurface in \mathbf{R}^{d+1} . Alternatively, this hypersurface can be regarded as the graph of a function $g : \mathbf{R}^d \rightarrow \mathbf{R}$ which satisfies (A 1).

Computing the Lie bracket appearing in (A 2), the only non-zero terms are

$$\begin{aligned} [v_i, v_j] &= (v_i^n \partial_n v_j^{d+1} - v_j^n \partial_n v_i^{d+1}) \partial_{d+1} \\ &= (v_i^i \partial_i K^j - v_j^j \partial_j K^i) \partial_{d+1} \\ &= (\partial_i K^j - \partial_j K^i) \partial_{d+1}. \end{aligned} \quad (\text{A } 4)$$

From the last line it is clear that if $\partial_i K^j - \partial_j K^i = 0$, in other words, if ∇K is symmetric, we can simply take $c_{ij}^k(x) = 0$ in (A 2), proving the existence of the integral submanifold and of the function g . \square

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