A mixture theory for size and density segregation in shallow granular free-surface flows

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In the last ten years a lot of work has been undertaken on developing mixture theory based continuum models to describe kinetic sieving driven size-segregation. We propose an extension to these models that allows their application to bi-disperse flows, over inclined channels, with particles varying in density and size. Our model incorporates both: a recently proposed explicit formula concerning how the total pressure is distributed among different species of particles, which is one of the key elements of mixture theory based kinetic sieving models; and a shear rate dependent drag. The resulting model is used to predict the range of sizes and densities for which the mixture segregates. This prediction is validated using discrete particles simulations and good agreement is found using a single fitting parameter which determines if the pressure is scaled with the diameter, surface area, or volume of the particle.

1. Introduction.

When free-surface granular flows with particles differing in size and/or density discharge down an inclined plane they often segregate to form complex patterns (Drahun & Bridgewater 1983; Khakhar et al. 1999). These flow-induced effects must often be avoided in production processes of the pharmaceutical, chemical, food, iron and cement industry (Duran 2000; Shinbrot et al. 1999). Therefore, a quantitative prediction of segregation is vital in improving the product quality and design of the material handling equipment. Despite its importance, the fundamentals of the phenomenon are incompletely understood.

In general, segregation or de-mixing occurs due to differences in particle properties such as size (Wiederseiner et al. 2011), density (Tripathi & Khakhar 2013), shape (Pollard & Henein 1989), inelasticity (Brito & Soto 2009), surface roughness and friction (Ulrich et al. 2007). However, differences in size and density are the primary factors for de-mixing in free-surface flows over inclined channels. Experimental studies have been considered to observe the combined effects of size and density difference (Felix & Thomas 2004; Jain et al. 2005) but few continuum models have considered these combined effects (Marks et al. 2012). Felix & Thomas (2004) experimentally analysed the size and density effects of particles for bi-dispersed mixture flows in rotating tumblers, over inclined channels and pile formations. Using a continuum approach, we present an analysis predicting the degree of segregation in a bidisperse mixture flow, over inclined channels, due to both

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size and density differences, and benchmark these against discrete particle simulations. Among several mechanisms causing segregation (Bridgewater 1976), our focus lies upon kinetic sieving which is the dominant mechanism causing segregation in gravity driven free-surface flows (Savage & Lun 1988). We use the framework of mixture theory (e.g. Morland 1992) and extend the ideas of Gray & Thornton (2005); Thornton et al. (2006) by relaxing the equal particles-species density assumption and incorporating a shear rate dependent interspecies drag with a slightly generalised pressure scaling function of that proposed by Marks et al. (2012). The resulting theory is able to predict the range of sizes and densities for which segregation will occur directly from the known particle’s size and density. Please note, previously Marks et al. (2012) stated a way to incorporate density differences, but they have not considered these effects in detail.

2. Particle segregation model.

We choose a domain consisting of a chute inclined at a constant angle $\theta$ with respect to the horizontal and a Cartesian coordinate system in which the $x$–axis points down the chute, the $y$–axis points across its width and the $z$–axis points in the upward direction normal to the chute.

2.1. Mixture framework.

Our starting point for the model is a granular mixture theory composed of two different constituents, indexed 1 and 2, whose interstitial pore space is filled with air, which has a negligible effect on these dense granular flows. Mixture theory (e.g Morland 1992) for a binary continuum postulates that all constituents of the mixture simultaneously occupy space and time. This leads to overlapping fields with partial pressures, $p^\nu$, densities, $\rho^\nu$, and velocities, $u^\nu = [u^\nu, v^\nu, w^\nu]^T$ in the three coordinate directions, corresponding to each constituent indexed $\nu = 1, 2$. Each of the constituents satisfies the following fundamental balance laws of mass and momentum for these partial fields

$$\partial_t \rho^\nu + \nabla \cdot (\rho^\nu u^\nu) = 0,$$

$$\rho^\nu (\partial_t u^\nu + u^\nu \cdot \nabla u^\nu) = -\nabla p^\nu + \rho^\nu g + \beta^\nu,$$  

(2.1)

where $g = (g_t, 0, -g_n)^T$ is the gravity vector with $g_t$ being the standard acceleration due to free fall, $g_t = g \sin \theta$ and $g_n = g \cos \theta$. The $\beta^\nu$ represent interspecies drag force resisting the motion between the constituents. As these forces are internal, from Newtons’ third law the sum of these drags must be zero, i.e., $\beta^1 + \beta^2 = 0$. Given a unit mixture volume, each of the constituents occupies a volume fraction $\phi^1$ or $\phi^2$, including the interstitial pore space. Hence, by definition, the individual volume fractions sum to unity $\phi^1 + \phi^2 = 1$.

Furthermore, the bulk density $\rho$, barycentric granular velocity or bulk velocity $\mathbf{u}$ and the bulk pressure $p$ are defined as $\rho = \rho^1 + \rho^2$, $\mathbf{u} = (\rho^1 \mathbf{u}^1 + \rho^2 \mathbf{u}^2)/\rho$, $p = p^1 + p^2$ respectively. A vital element in the mixture theory is the relation between partial and intrinsic variables. Hereby, variables such as velocity, density and pressure are related as follows

$$u^\nu = u^{\nu*}, \quad \rho^\nu = \phi^\nu \rho^{\nu*}, \quad p^\nu = \rho^{\nu*} \mu^{\nu*}$$

with $p^{\nu*} = f^\nu p$,  

(2.2)

where $^{*}$ denotes intrinsic variable. Motivated Marks et al. (2012), we assume that $f^\nu$ scales with the species size $s^\nu$ as

$$f^\nu = \frac{(s^\nu)^a}{\sum (s^\nu)^a \phi} = 1, 2 \text{ and } a > 0.$$  

(2.3)
The new definition for partial pressure (2.2) is a slight generalisation of the form used by Marks et al. (2012). When \( a = 1 \) or \( 2 \) or \( 3 \), the pressure precisely scales with the length, surface area, volume of the particle respectively.

### 2.1.1. Drag force.

As different particle species percolate past (kinetic sieving) and squeeze upwards through (squeeze expulsion) each other, the species experience interspecies friction (Savage & Lun 1988). Following Marks et al. (2012), a generalised version of Gray & Thornton (2005) interaction drag or interspecies friction is assumed to be

\[
\beta^\nu = p\nabla(\phi^\nu f^\nu) - \rho^\nu \frac{\beta}{c} f^\nu - u, \quad \nu = 1, 2, \quad (2.4)
\]

with \( c \) being a priori unknown coefficient of interspecies interaction and \( \dot{\gamma} \) being the shear rate. The momentum balance for each individual species (2.1) can be thus restated as

\[
\rho^\nu (\partial_t u^\nu + u^\nu \cdot \nabla u^\nu) = -\phi^\nu f^\nu \nabla p + \rho^\nu g - \rho^\nu \frac{\beta}{c} f^\nu - u. \quad (2.5)
\]

In shallow large scale industrial or natural granular flows, the aspect ratio, i.e., ratio of the flow quantities, like flow velocity and flow length, in the downslope and cross-slope direction to those in the normal direction, is small. Summing the momentum balance (2.5) of each species implies the flow, at leading order in aspect ratio, is in lithostatic balance, i.e., \( \partial p/\partial z = -\rho g \cos \theta \). Moreover, the down- and cross-slope velocity of the species is considered to be equal to the bulk down- and cross-slope velocity component \( u^\nu = u, \nu = v, \nu = 1, 2 \). Assuming the flow to be in viscous balance by neglecting the inertia terms, substituting the drag force and velocity definitions into the particles’ normal momentum balance equation, the species’ percolation or normal velocity is

\[
w^1 = w - \dot{\gamma} \frac{\rho}{c} \left[ \frac{(1 - \phi) (s^a - \tilde{\rho})}{\phi + (1 - \phi) s^a} \right] \quad \text{and} \quad w^2 = w + \dot{\gamma} \frac{\rho}{c} \left[ \frac{\phi (s^a - \tilde{\rho})}{\phi + (1 - \phi) s^a} \right]. \quad (2.6)
\]

Here \( \tilde{\rho} = \rho^2/\rho^1 \) and \( \tilde{s} = s^2/s^1 \) are the particle density and size ratios respectively; whereas, \( \phi \) is the volume fraction of species-1. Using the mass balance equation (2.1), and substituting the percolation velocity (2.6), we obtain the governing equation for \( \phi^1 = \phi \) as follows

\[
\frac{\partial \phi}{\partial t} + \frac{\partial (\phi u)}{\partial x} + \frac{\partial (\phi v)}{\partial y} + \frac{\partial (\phi w)}{\partial z} - \frac{\partial}{\partial z} \left( \dot{\gamma} \frac{\rho}{c} (s^a - \tilde{\rho}) \left[ \frac{\phi (1 - \phi)}{\phi + (1 - \phi) s^a} \right] \right) = 0. \quad (2.7)
\]

In general, an approximate bulk velocity \( \mathbf{u} = (u, v, w)^T \) can be computed from an existing shallow granular model, see Woodhouse et al. (2012) for a possible method of coupling.

### 2.1.2. Scaling or non-dimensionalising.

Assuming the bulk flow velocity is approximated using such models, the flow quantities are scaled by hydrostatic scalings as follows

\[
(x, y, z) = (Lx, Ly, Hz), \quad (u, v, w) = (Uu, Uv, (HU/L)\tilde{w}), \quad t = (L/U)\tilde{t}. \quad (2.8)
\]

Variables \( U, L \) and \( H \) are suitable characteristic scales for the flow velocity, length, and depth/height respectively where \((L, L, L) \gg Hz\). Substituting the above scalings and dropping the tildes, the governing equation (2.7) is restated as

\[
\frac{\partial \phi}{\partial t} + \frac{\partial (\phi u)}{\partial x} + \frac{\partial (\phi v)}{\partial y} + \frac{\partial (\phi w)}{\partial z} - \frac{\partial}{\partial z} \left[ \tilde{S} F(\phi) \right] = 0 \quad \text{with} \quad F(\phi) = \left[ \frac{\phi (1 - \phi)}{\phi + (1 - \phi) s^a} \right], \quad (2.9)
\]
where $\hat{S}_r = qL/HU$ is the non-dimensional number defined as the ratio of the mean segregation velocity, $q = q_n (\hat{s} - \hat{\rho})$, to a typical magnitude of the normal bulk velocity.

3. Solutions for limiting cases.

For the kinematic limiting cases, solutions are constructed for (2.7) for a velocity field $u = (u(z), 0, 0)$ in the domain $0 \leq z \leq 1$ and $x \geq 0$, for a mixture flow of unit height. For simplicity, initially we consider shallow flows in which the velocity profiles are almost linear (Weinhart et al. 2012); hence to a good approximation we can take the shear rate, $\dot{\gamma}$, to be constant. Therefore, with $\dot{\gamma} = 1$ Eq. 2.9 becomes

$$\frac{\partial \phi}{\partial t} + u \frac{\partial \phi}{\partial x} - \hat{S}_r \frac{\partial}{\partial z} \left( \frac{\phi (1 - \phi)}{\phi + (1 - \phi)s^a} \right) = 0,$$

(3.1)

where $\hat{S}_r$ at $\dot{\gamma} = 1$. As in Gray & Thornton (2005), we use a simplified boundary condition where there is no flux of particles at the free-surface and the base by considering $F(\phi) = 0$ at $z = 0, 1$. Solutions to (3.1) can be constructed for two types of inflow boundary conditions, prescribed at $x = 0$, termed as homogeneous mixture inflow and normally graded mixture inflow. In the former type, a homogeneous mixture of concentration $\phi_0$ enters at $x = 0$ whereas in the latter normally graded particles, i.e., a mixture in which smaller particles lie on top of the larger particles, enters at the upstream of the flow.

3.1. Analytical solutions.

Assuming the flow has reached its steady state, the differential equation (3.1) is restated as

$$u(z) \frac{\partial \phi}{\partial x} - \hat{S}_r \left[ \frac{(\phi - 1)^2 s^a - \phi^2}{(\phi + (1 - \phi)s^a)^2} \right] \frac{\partial \phi}{\partial z} = 0,$$

(3.2)

which is a quasi-linear partial differential equation. By using the method of characteristics, as applied in Gray & Thornton (2005) and Thornton et al. (2006), the above equation (3.2) is analytically solved for specific inflow boundary conditions.


An exact solution to the above equation is obtained via the method of characteristics, such that $\phi$ is constant, $\phi_\lambda$, along each characteristic curve given by

$$u(z) \frac{dz}{dx} = \hat{S}_r \left[ \frac{(\phi_\lambda - 1)^2 s^a - \phi_\lambda^2}{(\phi_\lambda + (1 - \phi_\lambda)s^a)^2} \right].$$

(3.3)

The characteristics describe the flux of information into the domain. Given a velocity field, $u(z)$, the above equation is integrable as downslope velocity $u$ is a function of $z$ alone and $\phi_\lambda$ is constant. Solutions for general velocity fields can be obtained by introducing a depth-integrated velocity coordinate $\psi$, where

$$\psi = \int_0^z u(z')dz'.$$

(3.4)

Equation (3.3) can then be integrated to give a straight line characteristics,

$$\psi = \hat{S}_r \left[ \frac{(\phi_\lambda - 1)^2 s^a - \phi_\lambda^2}{(\phi_\lambda + (1 - \phi_\lambda)s^a)^2} \right] (x - x_\lambda) + \psi_\lambda,$$

(3.5)
in terms of the mapped variables, with \((x, \psi)\) as the starting point for varying \(\phi\). We choose a scaling such that without loss of generality the mapped coordinate \(\psi = 1\) at the free surface \(z = 1\). As \(u(z) > 0\) is taken from the onset to simplify matters, the map between the physical and depth-integrated coordinates can easily be constructed for a whole class of general velocity fields.

### 3.1.2. Jump conditions.

Experimental evidence of segregation flows reveals that concentration jumps or shocks can emerge (Savage & Lun 1988). Presence of shocks implies that the segregation equation (2.9) or (3.1), is no longer valid because particle concentration \(\phi\) is then no longer continuous. Hence, a jump condition should be applied across the discontinuity (Whitham 1974). We derive the jump condition from an integral version of the conservative form of segregation equation (3.1). Integrating this from \(L_1\) to \(L_2\) with respect to \(z\) gives

\[
\frac{\partial}{\partial x} \int_{L_2}^{L_1} \phi u dx - \dot{S}_r \left[ \frac{\phi(1-\phi)}{\phi + (1-\phi)s^a} \right]_{L_1}^{L_2} = 0, \tag{3.6}
\]

where the brackets \([\ ]\)' denote the difference of the enclosed function value at upper and lower limit. Assuming a jump in \(\phi\) exists at \(z = J(x)\) and following Whitham (1974), the jump condition is

\[
\left[ \phi u J' + \dot{S}_r \left( \frac{\phi(1-\phi)}{\phi + (1-\phi)s^a} \right) \right]_{-}^{+} = 0, \tag{3.7}
\]

with \(J' = dJ/du\) and \(\cdot\)' denote the limits on either side of the discontinuity/jump \(J(x)\). The above equation can be restated as

\[
u \frac{dJ}{dx} = -\dot{S}_r \left[ \frac{(1 - (\phi^+ + \phi^-)) s^a - \phi^+ \phi^- (1 - s^a)}{\phi^+ \phi^- (s^a - 1)^2 + (\phi^+ + \phi^-)(s^a - s^{2a}) + s^{2a}} \right], \tag{3.8}
\]

when solved, gives the location of the shock. Following Gray & Thornton (2005), we state the above equation in terms of depth-integrated velocity coordinates, (3.4), i.e.,

\[
u \frac{dJ}{dx} = -\dot{S}_r \left[ \frac{(1 - (\phi^+ + \phi^-)) s^a - \phi^+ \phi^- (1 - s^a)}{\phi^+ \phi^- (s^a - 1)^2 + (\phi^+ + \phi^-)(s^a - s^{2a}) + s^{2a}} \right], \tag{3.9}
\]

independent of the assumed monotonically increasing velocity profile.

### 3.1.3. Jumps in mapped coordinates.

For a homogeneous inflow condition and purely size-based segregation, i.e., \(\rho = 1\), positions of the shocks are determined from the shock relations (3.9). By substituting \(\phi^+ = 1\) and \(\phi^- = \phi_0\) and integrating with boundary condition \(\psi = 0\) at \(x = 0\), the position of the shock is \(\psi_1 = -(g_n/c)(1 - s^a) [\phi_0/(\phi_0 + (1 - \phi_0)s^a)] x\). Similarly, by substituting \(\phi^+ = 0\) and \(\phi^- = \phi_0\) into the shock relations and integrating with boundary condition \(\psi = 1\) at \(x = 0\), the position of shock \(\psi_2 = 1 + (g_n/c)(1 - s^a) [(1 - \phi_0)(\phi_0 + (1 - \phi_0)s^a)] x\). The shock \(\psi_2\) propagates downwards to merge with shock \(\psi_1\) at the triple point, \(x_{triple} = (\phi_0 + (1 - \phi_0)s^a)((s^a - 1)(g_n/c))\) and at \(\psi = \phi_0\) in depth-integrated velocity variables, resulting in a third shock separating the 100% species-1 and species-2 regions. The shock position, \(\psi_3\), is determined by substituting \(\phi^+ = 0\) and \(\phi^- = 1\) into the shock relations which on integrating gives \(\psi_3 = \phi_0\). For \(x \geq x_{triple}\), When the shock positions \(\psi_1, \psi_2\) and \(\psi_3\) are mapped back to physical coordinates, they yield the solid lines in Fig. 1.

### 3.1.4. Physical solutions and comparison with other models.

The shock positions, in terms of the depth-integrated velocity coordinates, are valid for all velocity fields given that the fields have a well-defined map from the depth-integrated
velocity coordinate space to physical coordinate space. If we consider a linear shear profile
\( u = \alpha + 2(1 - \alpha)z \) where \( 0 \leq \alpha \leq 1 \), then it follows from (3.4) that the depth-integrated velocity coordinate \( \psi = \alpha z + (1 - \alpha)z^2 \) with \( \psi(1) = 1 \) at the free surface. The \( \psi \) coordinate can easily be mapped back to the physical space as

\[
z = \begin{cases} 
\psi, & \text{for } \alpha = 1 \\
-\alpha + \sqrt{\alpha^2 + 4(1 - \alpha)\psi} \over 2(1 - \alpha), & \text{for } \alpha \neq 1.
\end{cases}
\]

(3.10)

In case of a simple shear flow where \( \alpha = 0 \), from (3.10) we find that \( z = \sqrt{\psi} \). Thereby, the jump/shock positions are mapped back to the physical space. Fig. 1 shows the results from our model, the model of Gray & Thornton (2005) and the model of Savage & Lun (1988). To allow direct comparison of the effects of the flux functions, between our’s and Gray & Thornton (2005)’s models the segregation velocity, \( q \), is taken to be unity in both theories. Two things are clear from this comparison: firstly, the point of full segregation (location of the triple point) is further downstream for our model, by as much as a factor of two for \( \phi_0 = 0.1 \); secondly, the distance to full segregation is a function of \( \phi_0 \) in our model (in contrast to the model of Gray & Thornton (2005)). Our flux function \( F(\phi) \) is cubic and convex and it is apparent that the form of this function can have a large effect on the predicted distance to full segregation.

3.2. Numerical solutions.

Solutions can be computed to the segregation equation (2.7) for various values of shear \( \alpha \) and constant or non-constant shear rates, \( \dot{\gamma} \). Following (Marks et al. 2012), for thick enough mixture flows,

\[
\dot{\gamma} = \frac{[\tan \theta - \mu_c]}{(k\bar{s})} \sqrt{1.5g \cos \theta (1 - z)} \text{ where } \bar{s} = \sum \phi^{\nu}(s^{\nu})^{\alpha} \text{ with } \nu = 1, 2,
\]

(3.11)

leading to a Bagnold-type velocity profile. On substituting Eq. (3.11) in the segregation equation (2.7) gives

\[
\frac{\partial \phi}{\partial t} + \frac{\partial (\phi u)}{\partial x} + \frac{\partial (\phi v)}{\partial y} + \frac{\partial (\phi w)}{\partial z} - \frac{g_n c}{(\phi + (1 - \phi)\bar{s})^2} \left( \frac{\phi(1 - \phi)\sqrt{1 - z}}{(\phi + (1 - \phi)\bar{s})^2} \right) = 0,
\]

(3.12)

where \( C = (\tan \theta - \mu_c)k((s^1)^{\alpha}) \) is a material parameter. We use a high resolution shock capturing method, the space discontinuous Galerkin finite element method (space-
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Figure 2. For \( a = 1, \hat{\rho} = 0.5 \) and \( \hat{s} = 2 \), the development of volume fraction \( \phi \) is shown as a function of the downslope coordinate \( x \) and flow depth \( z \). The domain is initially filled with a mixture of \( \phi = 0.25 \) and the bulk flow is from left to right. (i) Constant shear rate \( (\gamma = 1) \) i.e. simple shear flow \( \alpha = 0 \), (a)-(b) Homogeneous mixture inflow \( (\phi = 0.6) \) and (c)-(d) normally graded mixture inflow \( (\phi = 0.6) \). (ii) Bagnold-type shear rate \( (\text{Eq. 3.11}) \): (e)-(f) Homogeneous mixture inflow \( (\phi = 0.6) \) and (g)-(h) normally graded mixture inflow \( (\phi = 0.6) \).

DGFEM) (Tassi et al. 2007), implemented in our open-source DGFEM solver hpGEM (Pesch et al. 2007). Full details of the package including the numerical codes used to generate Fig. 2 are available through the package’s website, http://einder.ewi.utwente.nl/hpGEM/. This new implementation of a DGFEM based solver is accurate, robust and has been extensively tested against both the exact solutions and the solutions presented in Gray & Thornton (2005) and Thornton et al. (2006) for flux functions of the form \( F(\phi) = \phi(1-\phi) \).

In Fig. 2, evolution to steady-state for (i) constant shear rate \( \gamma = 1 \), i.e., simple shear \( \alpha = 0 \) and (ii) Bagnold-type rate \( (\text{Eq. 3.11}) \) for both homogeneous and normally graded inflow conditions. For constant shear rate, \( \gamma = 1 \), and normally graded inflow, Fig. 2 (c)-(d), the expected expansion fan and three shock structure can be clearly seen (Thornton et al. 2006). From a superficial investigation it would appear that the convex cubic flux function does not structurally change the evolution of the solution; however, a full in-
investigation of the solution in general for this cubic convex flux is beyond the scope of this short paper. However, with Bagnold-type shear rate, for homogeneous and normally graded inflows, the upper shock seen in Fig. 2 (a)-(d) disappears and is replaced by a gradual transition zone, Fig. 2- (e)-(f) & -(g)-(h). This is similar to what is captured by the simpler cell automata model of Marks & Einav (2011).

4. No/very weak segregation.

One of the key new features of our model is the ability to predict the ratios of size and density for which no or very weak segregation should occur. Due to the non-dimensionalization used, this is simply given by the line $\hat{s}^a = \hat{\rho}$, recall $a = 1$ if we use the pressure function suggested by Marks et al. (2012).

In order to validate our no-segregation prediction, we used fully three-dimensional discrete particle simulations (DPMs), implemented in our in-house open source DPM package, MercuryDPM. This package has previously been used by Thornton et al. (2012) to investigate size-segregation in chute flows. The particle contact properties used in this work are taken to be the same as in their investigation. Full details about the MercuryDPM and the source code used for this paper can be found at the website MercuryDPM.org.

For homogeneous initial conditions (randomly mixed), particle volume fraction 50%, a series of DPM simulations for different values of $\hat{\rho}$ and $\hat{s}$ were carried out to make a phaseplot of $[\hat{\rho} \times \hat{s}]$. The simulations were carried out in a box inclined at an angle of 20° to the horizontal. The base of the box is rough, created by fixing small particles randomly, see Weinhart et al. (2012) for more details. The sensitivity to both basal
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and initial conditions on the steady-state has been thoroughly investigated and hardly any sensitivity was found. Once the flow has reached its steady state, we calculated $D_{com}(\hat{\rho}, \hat{s}) = (\text{COM}_2 - \text{COM}_B)/\text{COM}_B$, as a function of $\hat{s}$ and $\hat{\rho}$. Here COM$_2$ is the vertical centre of mass of species-2 particles and COM$_B$ is the bulks’ vertical centre of mass. For a given $\hat{s}$ and $\hat{\rho}$, the flow is steady when the function value, $D_{com}$, remains constant with time.

In Fig. 3, we plot $D_{com}$ as a function of both $\hat{\rho}$ and $\hat{s}$. When the value of $D_{com}$ is positive, particles of species-2 are near the free surface; vice-versa, when it is negative, particles of species-2 are near the base. Close inspection of the data shows very weak segregation along the solid line fitted by a value of $a = 3$, also implying that the pressure is scaled by the volume of the particle. Below the solid line the species-2 particles rise towards the free-surface and above the solid-line species-2 particles fall towards the base. The dashed-solid line corresponds to the prediction via kinetic theory for a binary mixture (Jenkins & Yoon 2002). The discrepancy between the theories could be possibly because the prediction by Jenkins & Yoon (2002) is valid only for mixtures in which large particles are dilute in a dense mixture comprising of smaller particles which is mostly not the case in our simulations. The predictions, Fig. 3, are similar to the experimental findings of Felix & Thomas (2004). Moreover, a generalised pressure scaling function could be obtained directly from the DPMs. This work is ongoing and some early results can be found in (Weinhart et al. 2013). In this paper, we have not incorporated the diffusive nature of these flows into our model (Gray & Chugunov 2006) and therefore the model predicts full segregation, eventually, on either side of the no segregation line. However, if we had included diffusion in the model then $D_{com}$ could be reinterpreted as a segregation Péclet number (ratio of segregation to diffusive strength) as in Thornton et al. (2012).

5. Summary and conclusions.

In this paper, we derived a generic continuum model to predict the extent of segregation in a bidisperse granular mixture flow due to differences in particle size and density. For a given density and size ratio, the model predicts the extent of de-mixing in gravity-driven chute flows. For purely size-based segregation, the model has been compared to two previous models. The derived model is solved analytically using the method of characteristics, and numerically using a discontinuous Galerkin finite element method. The model was used to predict the ratios of particle size and density for which no segregation would occur. This prediction is independent of the details of the drag coefficient between the particles, the bulk velocity profile of the flow and contains no fitting parameters.

To validate this prediction, we performed discrete particle simulations as an alternative to laboratory experiments of field measurements. The model performs surprisingly well, when compared with the discrete particle simulations, with the fitting parameter determined from the DPMs.

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