Nonlinear Wave Problems that I can’t Control

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1 Introduction

2 Three Control Challenges

3 Variational Space-Plus-Time Water Waves

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1. Introduction

Some hydrodynamics ... rogue waves are anomalously high waves defined relative to a significant wave height $H_s$.

- Index (Khariff et al. ’09, Dysthe et al. ’08):
  \[ AI = \frac{H_{rw}}{H_s} > 2 \quad \text{or} \quad AI = \frac{\eta_{rw}}{H_s} > 1.25 \]

- Relevance in maritime & coastal engineering — ship design & safety offshore structures

- Pyramidal rogue wave (Faulkner 2001):

- spatial wave focussing due to refraction or wave caustics, or wave focussing in coastal convergences
- crossing seas, nearly standing waves with pyramidal waves.
Here, emphasis will lie on rogue waves & wave focussing:

- in crossing seas
- due to coastal or submarine convergences.

Moreover, (rogue) wave energy devices based on wave focussing will be introduced or discussed:

- the TapChan or tapered channel device
- the Oscillating Water Column (OWC) device with wind turbine
- a new device with a more direct energy conversion?
2. Three Control Challenges

Employ port-Hamiltonian methods to control:

- a wave maker to create the highest rogue wave?
- geometry and dynamo in a new rogue wave energy device?
- maximum berm formation by breaking waves on beaches?
Bore Soliton Splash

- **Water channel**: $L = 43.5 \pm 0.25\,\text{m}$, $D = 1.20\,\text{m}$, $w = 2 \pm 0.05\,\text{m}$

- **Sluice** compartment & gate, lifted with $2.5\,\text{m/s}$

- **V-shaped or linear convergence** at other end with $c = 2.7\,\text{m}$

- **Start at rest** with water levels $h_0 \in [0.32, 0.47]\,\text{m}$ and $h_1 \in [0.67, 1.02]\,\text{m}$. 
Bore Soliton Splash

- 7+2 Cases of which 3 repeats (reproducibility):

<table>
<thead>
<tr>
<th>#</th>
<th>( h_0 ) (m) ±0.01m</th>
<th>( h_1 ) (m) ±0.01m</th>
<th>( H_s ) ±0.05m</th>
<th>( H_{rw} ) ±0.5m</th>
<th>Peak #</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.32</td>
<td>0.67</td>
<td>-</td>
<td>0.6</td>
<td>-</td>
<td>bore</td>
</tr>
<tr>
<td>2</td>
<td>0.38</td>
<td>0.74</td>
<td>-</td>
<td>2.5</td>
<td>-</td>
<td>good splash</td>
</tr>
<tr>
<td>3</td>
<td>0.41</td>
<td>0.9</td>
<td>0.35</td>
<td>3.25</td>
<td>2nd</td>
<td>thin jet</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>cf. 6 &amp; 8</td>
</tr>
<tr>
<td>4</td>
<td>0.47</td>
<td>1.0</td>
<td>0.35</td>
<td>1</td>
<td>2nd</td>
<td>bore &amp; low</td>
</tr>
<tr>
<td>5</td>
<td>0.41</td>
<td>1.02</td>
<td>0.40</td>
<td>1.5</td>
<td>1st</td>
<td>bore &amp; low</td>
</tr>
<tr>
<td>6</td>
<td>0.41</td>
<td>0.9</td>
<td>0.35</td>
<td>3.5</td>
<td>2nd</td>
<td>BSS cf. 3 &amp; 8</td>
</tr>
<tr>
<td>7</td>
<td>0.45</td>
<td>0.8</td>
<td>0.35</td>
<td>2.5</td>
<td>2nd</td>
<td>good splash</td>
</tr>
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<tr>
<td>8</td>
<td>0.41</td>
<td>0.9</td>
<td>0.35</td>
<td>3.5</td>
<td>2nd</td>
<td>BSS &amp; ++</td>
</tr>
<tr>
<td>9</td>
<td>0.43</td>
<td>0.9</td>
<td>0.45</td>
<td>1.8</td>
<td>1st</td>
<td>bubble</td>
</tr>
</tbody>
</table>
Estimated wave amplitudes $H_s \in [0.35, 0.45]$m

After opening sluice gates three solitary waves emerge: large, medium and small

First highest soliton often breaks into spilling breaker

**Variety of outcomes**: minor/major reflections, resonances between waves, smooth waves, sheets, pyramidal waves

$H_{rw} \in [0.6, 3.5]$m.
Highest Cases 3, 6 & 8:

- $H_s = 0.35\text{m} \quad & \quad H_{rw} = [3.25, 3.5, 3.5]\text{m}$
- *bore soliton splash*
  
  Wout Zweers youtube channel:
  [https://www.youtube.com/watch?v=YSXsXNX4zW0](https://www.youtube.com/watch?v=YSXsXNX4zW0)

- Truly rogish:

$$AI = \frac{H_{rw}}{H_s} = \frac{3.5}{0.35} \approx 10.$$
Bore-Soliton-Splash

Three Control Challenges

Wave Control

Introduction

Variational Space-Plus-Time Water Waves

Conclusion

Appendix
Control the Bore-Soliton-Splash?

- Control the wave maker, including the resulting:
  - nonlinearity
  - geometry
  - dispersion
  - solitons
  - spilling breaker or bore
- to obtain the highest bore soliton splash?
Control of Soliton-Splash with KP?

- For 3 KdV solitons with amplitude $A$, there is “V-shaped channel geometry” with angles $\theta_{max}$ for which $A_{max} = 9$
- Exact KP solution by Kodama (idea, Dresden ’11):

$$\partial_x(\partial_t \eta + \frac{3}{2} \eta \partial_x \eta + \frac{1}{6} \partial_x^3 \eta) + \frac{1}{2} \partial_y^2 \eta = 0$$

- Fine in open sea, but inconsistent in water wave channel!
- Solution: bidirectional Boussinesq model.
Rogue Wave Energy Device

Combine geometric wave focussing with linear dynamo:

- Dynamo above water on a mast attached to floater.
- More direct energy conversion!
- Mechanically robust?
- Working prototype (using flash lights charged by shaking).
- Link: http://www1.maths.leeds.ac.uk/~obokhove/WaveEnergy.m4v
3. Space-plus-Time Finite Element Water Waves

Key geometric issues:

- in the **spatial discretization** of Miles’ variational principle for nonlinear water waves, and
- in the **temporal discretization** of the non-autonomous spatially discrete variational principle.

Verification and validation against measurements of **nonlinear waves** driven by a piston wave maker.
Consider the (2D) potential flow water wave problem:

- gravity as restoring force with acceleration $g$
- velocity $u = (u, w)^T = \nabla \phi$ with velocity potential $\phi = \phi(x, z, t)$
- domain $\Omega(t)$ with bottom $S_B$, wall $S_L : x = L$ and piston wave maker at $x = R(t)$:
  \[ \Omega(t) : R(t) < x < L \quad \text{and} \quad -H(x) < z < \eta(x, t) \quad (3) \]
- free surface variables $\eta(x, t)$ & $\phi_s = \phi(x, z = \eta(x, t), t)$
- kinematic **single-valued** free surface $z = \eta(x, t)$:
  \[ \partial_t \eta + \partial_x \phi \partial_x \eta = \partial_z \phi \quad \text{at} \quad z = \eta(x, t) \quad (4) \]
- dynamic free surface condition: continuity of pressure.
Miles’ Variational Principle

- Miles 1977 (see also Cotter & B. 2010):

\[
0 = \delta \int_0^T \mathcal{L}[\phi, \phi_s, \eta, t] dt \\
= \delta \int_0^T \phi_s \partial_t \eta - \mathcal{H}[\phi, \phi_s, \eta, t] dt \\
= \delta \int_0^T \int_{\mathcal{L}} \int_{\mathcal{R}(t)} \phi_s \partial_t \eta - \frac{1}{2} g \eta^2 \\
- \int_{\Gamma} \eta(x, t) \frac{1}{2} |\nabla \phi|^2 dz dx - \int_{-H(x)}^{\eta_w} \frac{dR}{dt} \phi_w dz dt
\]

- **piston wave-maker** \( R(t) \) with \( \phi_w \equiv \phi(R(t), z, t) \), \( H_w = H(R(t)) \), \( \eta_w = \eta(R(t), t) \)
- **non-autonomous** due to piston wave-maker
- canonical Hamiltonian system.
Miles’ Variational Principle

Variations yield, using

\[ \delta \eta(x, 0) = \delta \eta(x, T) = 0 \]
\[ \delta \phi_s = (\delta \phi)_s + (\partial_z \phi)_s \delta \eta \]
\[ n_s = (-\eta_x, 1)^T / \sqrt{1 + (\partial_x \eta)^2}, \]

**dynamic/kinematic conditions** and **Laplace’s equation**, BC’s:

\[ \delta \eta(x, t) : \partial_t \phi_s + \frac{1}{2}(\partial_x \phi)^2 + g \eta - \frac{1}{2}(\partial_z \phi)_s^2(1 + (\partial_x \eta)^2) = 0 \]
\[ \delta \phi_s(x, t) : \partial_t \eta + \partial_x \phi_s \partial_x \eta - (\partial_z \phi)_s(1 + (\partial_x \eta)^2) = 0 \]
\[ \delta \phi(x, z, t) : \nabla^2 \phi = 0 \]
\[ \delta \phi_w : dR/dt - (\partial_x \phi)_w = 0 \]
\[ \delta \phi_{B,L} : n \cdot \nabla \phi = 0 \]
Geometric Wave Modeling: Space FEM

FEM formulation of Miles’ variational principle.

- FEM test/basis functions $\tilde{\phi}_j(x, z, t), \hat{\phi}_k(x, t)$ with $i, j$ in $\Omega$
- Indices $k, l, r$ at free surface, $i', j'$ in interior & $m, \tilde{m}$ at wave maker.
- Substitute finite element expansions directly into VP:

$$
\phi_h(x, z, t) = \phi_j(t)\tilde{\phi}_j(x, z, t) \\
\phi_{sh}(x, t) = \phi_l(t)\varphi_l(x, t) \\
\eta_h(x, t) = \eta_k(t)\varphi_k(x, t)
$$
Moving mesh:

- **Issue 1**: realize that variations of interior nodes of mesh are (dynamically) coupled to variations of free surface nodes.

- **Issue 2**: find robust (variational) time integrators for (non-)autonomous system.

Otherwise: nonlinearly unstable discretization (cf. DGFEM in 2007).
The resulting **spatially discrete** non-autonomous VP:

\[
0 = \delta \int_0^T L[\phi_j, \eta_k, t] dt \\
0 = \delta \int_0^T M_{kl} \phi_k \frac{d\eta_l}{dt} - H[\phi_j, \eta_k, t] dt \\
= \delta \int_0^T M_{kl} \phi_k \frac{d\eta_l}{dt} - D_{kl} \phi_k \eta_l - \\
- \frac{1}{2} g M_{kl} \eta_k \eta_l - \frac{1}{2} A_{ij} \phi_i \phi_j - W_m \phi_m dt.
\]

- \(M_{kl}, D_{kl}, A_{ij}, W_m\) depend on \(\{\eta_k(t), t\}\): mesh and wave maker movements.
Geometric Wave Modeling: Space FEM

- The matrices and vectors depend on \( \{ \eta_k(t), t \} \) as follows:

\[
M_{kl}(t) = \int_{R(t)}^{L} \varphi_k \varphi_l dx
\]

\[
D_{kl}(t) = -\int_{R(t)}^{L} \frac{\partial \varphi_k}{\partial t} \varphi_l dx,
\]

\[
A_{ij}(\eta_r, t) = \int_{\Omega_h} \nabla \tilde{\varphi}_i \cdot \nabla \tilde{\varphi}_j dxdz
\]

\[
W_m(\eta_1, t) = \int_{-H_w}^{\eta_1} \frac{dR}{dt} \tilde{\varphi}_m \bigg|_{x=R(t)} dz,
\]

- due to wave maker via \( R(t) \) & mesh movement via \( \eta(x, t) \).
Geometric Wave Modeling: Space FEM

**Issue 1:** variation yields

\[ \delta \phi_k : M_{kl} \frac{d\eta_l}{dt} - D_{kl} \eta_l - A_{ik} \phi_i - W_1 \delta_{1k} = 0 \]

\[ \delta \eta_l : \frac{d(M_{kl} \phi_k)}{dt} + D_{kl} \phi_k + M_{kl} \eta_l + \frac{1}{2} \frac{\partial A_{ij}}{\partial \eta_l} \phi_i \phi_j + \frac{\partial W_m}{\partial \eta_l} \delta_{1l} \phi_m = 0 \]

\[ \delta \phi_i : A_{ij} \phi_i + W_{m'} \delta_{m'j'} = 0 \]

**Hence,** we can eliminate interior degree of freedom:

\[ A_{i'j'} \phi_{i'} = -A_{lj'} \phi_l + W_{m'} \delta_{m'j'} \implies \]

\[ \phi_{i'} = -A_{lj'} A_{j'i'}^{-1} \phi_l - W_{m'} A_{m'i'}^{-1}. \]
Eliminate $\phi_{j'}$: **spatially discrete** VP at free surface:

\[
0 = \delta \int_0^T L[\phi_k, \eta_k, t] dt
\]

\[
0 = \delta \int_0^T M_{kl} \phi_k \frac{d\eta_l}{dt} - H[\phi_k, \eta_k, t] dt
\]

\[
= \delta \int_0^T M_{kl} \phi_k \frac{d\eta_l}{dt} - D_{kl} \phi_k \eta_l - \frac{1}{2} g \eta_k \eta_l - \frac{1}{2} B_{kl} \phi_k \phi_l - C_l \phi_l - F_w dt.
\]

Cf. discrete **Boundary Element Method**

Function $F_w(t) = -\frac{1}{2} W_{\tilde{m}'m'} A_{\tilde{m}'m'}^{-1} W_{m'}$ & Schur complement

\[
B_{kl}(\eta_r, t) = (A_{kl} - A_{ki'} A_{i'j'}^{-1} A_{j'1})
\]

\[
C_l(\eta_r, t) = W_1 \delta_{l1} - A_{lj'} A_{j'm'}^{-1} W_{m'}.
\]
Geometric Wave Modeling: Time FEM

- Substitute a discontinuous Galerkin finite element expansion into the VP in each time slab $[t_n, t_{n+1}]:$

  $$\eta_l = \eta_l^{n,+} \left( \frac{t_{n+1} - t}{\Delta t_n} \right) + \eta_l^{n+1,-} \left( \frac{t - t_n}{\Delta t_n} \right)$$

  $$\phi_k = \phi_k^{n+1/2} 2 \left( \frac{t - t_n}{\Delta t_n} \right) + \phi_k^{n,+} \left( \frac{t_n + t_{n+1} - 2t}{\Delta t_n} \right).$$

- Define the delta function due to $M_{k\ell} \phi_k d\eta_l/dt$–term by splitting node $n$ in a narrow element with $C^0$-connection, and calculating its contribution in the limit of zero width:

  $$p = M_{k\ell} \phi_k = p_L + (p_R - p_L)(4\tau - 3\tau^2)$$

  $$q = \eta_l = q_L + (q_R - q_L)\tau$$

  $$\int_0^1 p \frac{dq}{dt} d\tau = p_R(q_R - q_L),$$
Issue 2: algebraic VP with “extended Hamiltonian” $H$:

$$
0 = \delta \sum_{n=0}^{N} L[\phi_k^{n,+}, \phi_k^{n+1/2}, \eta_k^{n,+}, \eta_k^{n+1,-}] dt \\
= \delta \sum_{n=0}^{N} M_{kl}^{n+1/2} \phi_k^{n+1/2} (\eta_l^{n+1,-} - \eta_l^{n,+}) \\
- \frac{\Delta t}{2} \left( H^{n+1/2}[\phi_k^{n+1/2}, \eta_k^{n,+}] + H^{n+1/2}[\phi_k^{n+1/2}, \eta_k^{n+1,-}] \right) \\
+ M_{kl}^{n+1,+} \phi_k^{n+1,+} (\eta_l^{n+1,+} - \eta_l^{n+1,-})
$$
Variations (evaluation $\uparrow = n + 1/2$):

- $\delta \phi_k^{n+1,+}$: $\eta_l^{n+1,+} = \eta_l^{n+1,-}$ continuous!

- $\delta \eta_l^{n,+} : M_{kl}^{n+1/2} \phi_k^{n+1/2} = M_{kl}^{n} \phi_k^{n,+}$

  $$- \frac{1}{2} \Delta t \frac{\partial H^{n+1/2}[\phi_k^{n+1/2}, \eta_k^{n,+}]}{\partial \eta_l^{n,+}}$$

- $\delta \phi_k^{n+1/2}$: $M_{kl}^{n+1/2} \eta_l^{n+1,-} = M_{kl}^{n+1/2} \eta_l^{n,+}$

  $$+ \frac{1}{2} \Delta t \left( \frac{\partial H[\phi_k^{n+1/2}, \eta_k^{n,+}]}{\partial \phi_k^{n+1/2}} + \frac{\partial H[\phi_k^{n+1/2}, \eta_k^{n+1,-}]}{\partial \phi_k^{n+1/2}} \right)$$

- $\delta \eta_l^{n+1,-} : M_{kl}^{n+1/2} \phi_k^{n+1,+} = M_{kl}^{n} \phi_k^{n+1/2}$

  $$- \frac{1}{2} \Delta t \frac{\partial H^{n+1/2}[\phi_k^{n+1/2}, \eta_k^{n+1,-}]}{\partial \eta_l^{n+1,-}}$$
Geometric Wave Modeling: Unfolding

- Reduces to 2nd-order symplectic Störmer-Verlet time integrator in the autonomous case
- “Unfold” the kinetic energy term again to avoid direct inversion of the matrix $A_{i'j'}$
- This defines the nature of the time discretization of $\phi_{i'}$ in the interior
- Use Newton iteration and Petc linear algebra routines.
Geometric Wave Modeling: Validation

- **Driven wave focusing** in MARIN’s wave tank
  [http://www1.maths.leeds.ac.uk/~obokhove/202002_zoom_splash_2.avi](http://www1.maths.leeds.ac.uk/~obokhove/202002_zoom_splash_2.avi)

- Entire *wave tank* MARIN with false vertical wall instead of *beach*.

- Comparison with measured data MARIN is good:

![Fourier Transform](http://www1.maths.leeds.ac.uk/~obokhove/fft_202002.png)
4. Conclusion

- **Posed control challenges**: bore soliton splash, rogue wave energy device.
- **Key issues** in variational nonlinear water wave modelling.
- **Discontinuous Galerkin variational time approach generalizes** to new pseudo-symplectic integrators.
- All extends to 3D & more general mesh movement.
- **UK project in preparation**: flood control coupled to ensemble predictions of local precipitation and water flow in uplands & river systems.
References

- http://www1.maths.leeds.ac.uk/~obokhove/presentation, movies & eprints?
- Gagarina, Ambati, Nurijanyan & Bokhove 2014: time (dis)continuous Galerkin FEM. In preparation
Ad. 2a. Maximize Shingle Beach Formation

Hele-Shaw beach dynamics

- Sketch set-up Hele-Shaw cell:

![Diagram of Hele-Shaw cell with labels: Z, L, B sub 0, H sub 0, particles, free surface, wedge, 1 sub p, theta w, x w, 0, x, 1 sub w, g.]

- All breaking wave types observed.
- *Beaches, berms & sand bars* emerge in min/hr by waves with wave forcing $T \approx 1s$. 
Ad. 2b. Smooth Case 9: No Rogue Wave

Take \( h_0 = 0.43 \text{m} \) instead of \( h_0 = 0.41 \text{m} \), same \( h_1 = 0.9 \text{m} \):
Smooth Case 9: No Rogue Wave
Smooth Case 9: No Rogue Wave
Smooth Case 9: No Rogue Wave
Smooth Case 9: No Rogue Wave
Smooth Case 9: No Rogue Wave
Smooth Case 9: No Rogue Wave
Smooth Case 9: No Rogue Wave
Ad. 2c. Bore-Soliton-Splash with KP?

- **Kadomtsev-Petviashvili equation** is unidirectional dispersive wave equation with weak lateral or $y$–dependence (Kodama 2010):

  \[
  \partial_x \left( \partial_t \eta + \frac{3}{2} \eta \partial_x \eta + \frac{1}{6} \partial_x^3 \eta \right) + \frac{1}{2} \partial_y^2 \eta = 0 \quad (5)
  \]

- Free surface deviation $\eta(x, t)$ (scaled).
When 2 \textit{KdV} sech\(^2\)-solitons with \( A = 2c \) & phase speed \( c \): 

\[
\eta(x, t) = 2c \text{ sech}^2\left(\sqrt{3c/2}(x - x_0 - ct)\right)
\]  

(6)

approach each other under an angle \( 2\theta \) (or one soliton grazes a wall with slant \( \theta \)): then there is an angle \( \theta_{max} \) for which \( A_{max} = 4 \) (Yeh & Kodama JFM 2011):