Granular flows in inclined channels with a linear contraction

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We consider a monodisperse dry granular mixture flowing down an inclined channel, with a localised contraction: theoretically and numerically. Using the depth-averaged shallow granular theory with an empirically determined constitutive friction law, validated by discrete particle simulations, we present a novel extended one-dimensional (depth- and width-averaged) granular hydraulic theory. For steady flows, the model predicts multiple flow regimes, like smooth flows without jumps or steady jumps/shocks in the contraction, which for supercritical and subcritical flows is verified by solving the two-dimensional (only depth-averaged) shallow granular equations using the discontinuous Galerkin finite element method. Despite the strong flow inhomogeneities in the contraction region, the one- and two-dimensional solutions (averaged across the channel width) surprisingly match for, both, supercritical and subcritical flows.

Key words: Granular media, shallow granular flows, inclined channel flows, depth-averaged, depth- and width-averaged

1. Introduction

A considerable number of industrial processes involve raw materials in a granular form, where grains of dissimilar properties are often mixed, fed, or separated by a variety of devices. Often, partially filled rotating drums and blenders are used in pharmaceutical and food production industries (Shinbrot et al. 1999), whereas rotary kilns and inclined cylinders (Davidson et al. 2000) are associated with chemical processes involving sinter, cement and iron production due to their ability for continuous material feed. Amongst several particle transport mechanisms associated with industrial processes, this work concerns the efficient modelling of dense rapid free surface granular flows in inclined channels.

In reality, the majority of granular flows in nature (avalanches, landslides, etc.) and industries dealing with inclined channel flows are shallow, i.e. the ratio of the characteristic length scales in the normal (H) to the streamwise direction (L) is small, H/L << 1. Although qualitative understanding of monodisperse mixture flows over inclined channels has existed for some time: several avalanche models, by exploiting the shallowness aspect, proved to be successful in quantitatively analysing these granular flows. In essence, an avalanche model utilises the already existing shallow water theory from the fluids

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community and extends it to model shallow granular free-surface flows. However, one needs to know the corresponding granular constitutive relations (friction law) to relate the normal and the tangential stresses. To our knowledge, the earliest known extension of the shallow water theory was implemented by Grigorian et al. (1967) to predict the snow avalanche paths in the Ural mountains. The formal existence of Shallow Granular (SG) theory was established by Savage & Hutter (1989), who averaged the mass and momentum balance equations in the depth-direction (depth-averaging) and assumed a Mohr-Coulomb rheology with a constant Coulomb basal friction law. In depth-averaging, one averages out the depth-dependency from the flow quantities, such as the flow height and velocity, for details see (Bokhove & Thornton 2012). As years progressed, studies extended and generalised SG theory to consider flows for several applications. For a detailed review of the extensions and applications, see (Vreman et al. 2007). Here, we focus on utilising the shallow granular theory for inclined channel flows through channels with downslope deflecting walls (contraction) (Hákonardóttir & Hogg 2005; Vreman et al. 2007; Cui et al. 2007; Rhebergen et al. 2009), see Fig. 1.

In Vreman et al. (2007), the motion of granular matter flowing over a smooth inclined channel, constrained by contracting sidewalls, was investigated by means of theoretical, numerical and experimental analysis. Results revealed upstream-moving bores or shocks, a stable reservoir state and weak oblique shocks. Flow states and flow regimes were explained via a unique one-dimensional granular “hydraulic” theory described by a set of equations obtained by extending the asymptotic analysis of Gray et al. (2003) from two to one-dimension. Instead of granular material, Akers & Bokhove (2008) theoretically and experimentally analysed water flow on a horizontal plane constrained downstream by contracting sidewalls. In this manuscript, the same one-dimensional hydraulic theory is extended to the granular case, including frictional effects. These one-dimensional shallow water equations combined with a theoretical and experimentally determined constitutive friction law lead to a one-dimensional shallow granular continuum model. Thereby, for closure, we use a friction law (constitutive law/closure relation) determined by Pouliquen & Forterre (2002), which was further investigated and verified through discrete particle simulations (Weinhart et al. 2012a; Thornton et al. 2012). The discrete particle simulations enable one to construct a map between micro-scale and macro-scale variables and functions, thereby determining the closure relations needed in the continuum model. As a result, Weinhart et al. (2012a) utilised an accurate micro-macro mapping technique called coarse-graining (Weinhart et al. 2012b; Tunuguntla et al. 2015) to verify the friction law of Pouliquen & Forterre (2002). Additionally, the same closure/friction law has also been extended to characterise self-channelising
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unconfined flows (Deboeuf et al. 2006) and bidisperse mixtures (Voortwis 2013). This enables one to apply these closure relations to more realistic flow scenarios. For example, when considering a flowing bi- or polydisperse mixture flowing down an inclined channel, an efficient continuum model would be when a closed SG model is combined with a particle segregation model (e.g. Tunuguntla et al. 2016, 2014; Marks et al. 2012; Gray & Ancey 2011) to predict the corresponding flow dynamics (Woodhouse et al. 2012; Edwards & Friendl 2016; Baker et al. 2016; Denissen et al. 2017). However, note that the closure relation must also be adapted to account for the varying mixture (e.g. Voortwis 2013) or basal (e.g. Thornton et al. 2012) properties. Moreover, given its importance, an extensive amount of research is undertaken to understand rapid dense mixture flows, using experiments (e.g. Wiederseiner et al. 2011; Johnson et al. 2012; Kokelaar et al. 2014; van der Vaart et al. 2015; Edwards & Friendl 2016) and simulations (e.g. Jing et al. 2016; Zhou et al. 2016; Tunuguntla et al. 2016; Guillard et al. 2016). However, summarising all of this is beyond the scope of our current focus.

Nevertheless, in this manuscript, through our one-dimensional asymptotic model, the flow regimes observed for granular flow in an inclined channel – with a linear contraction – are illustrated in an $F_0 - B_c$ plane, where $F_0$ is the channel upstream Froude number and $B_c$ is the channel-opening width ratio. The latter is defined as the ratio of the channel-opening width $W_c$ and the upstream channel width $W_0$. Although an approximate description of the flow characteristics, the one-dimensional asymptotic theory gives us a thorough description of the possible flow regimes. Moreover, the results obtained via the one-dimensional asymptotic theory are later verified by solving the two-dimensional shallow granular equations using a discontinuous Galerkin finite element method (DGFEM). Solving the two-dimensional shallow granular equations through DGFEM not only helps in verification of the asymptotic theory, but more importantly it scrutinises the validity of the constitutive friction law in two-dimensional space.

2. Asymptotic theory

In rapid free surface shallow granular flows, the ratio of the characteristic length- and velocity-scales in the normal to streamwise direction is small ($\ll 1$). By utilising the asymptotic analysis from Gray et al. (2003), along with a series of approximations listed by Bokhove & Thornton (2012), we obtain the depth-averaged shallow granular equations. The dimensional two-dimensional shallow granular model is stated below

\begin{align}
( hu )_t + ( hu^2 + K \frac{h^2}{2} g_n )_x + ( hvu )_y &= g_n h \left( \tan \theta - \mu(h, \vec{u}) \frac{|u|}{|\vec{u}|} \right) - g_n h \frac{db}{dx}, \\
( hv )_t + ( huv )_x + ( hv^2 + K \frac{h^2}{2} g_n )_y &= -g_n h \mu(h, \vec{u}) \frac{|v|}{|\vec{u}|} - g_n h \frac{db}{dy}.
\end{align}

The above system of equations (2.1) represent the conservation of mass and momentum in terms of the flow depth $h = h(x, y, t)$ and depth-averaged velocity $\vec{u} := (u(x, y, t), v(x, y, t))$ in a channel of width $W = W(x)$ with basal topography $b(x, y)$, where $x$ and $y$ is down- and cross-slope direction, respectively (see Fig. 1). Moreover, the variable $t$ denotes time, $\mu(h, \vec{u})$ is the basal friction coefficient and $K$ a material constant denoting stress anisotropy. The subscripts $t$, $x$, and $y$ denote the respective partial derivatives and $g_n = g \cos \theta$ is the normal acceleration due to gravity. The variable $\theta$ denotes the chute angle, which is chosen such that the average inter-particle and particle-wall forces are in balance with the downstream force of gravity...
acting on the granular particles, leading to a uniform flow in the absence of a contraction. Using the same aspect ratio argument as used for depth-averaging, the flow quantities are also averaged across the channel, width-averaged, see Appendix A.1 in Tunuguntla (2015).

After averaging (2.1) across the inclined channel, i.e. averaging out the y-dependence, the two-dimensional system of equations is reduced to a one-dimensional depth- and width-averaged shallow granular model, which, for a constant basal profile is

\[ (hW)_t + (huW)_x = 0, \]

\[ (hW)_t + (huW^2)_x + \frac{1}{2} g_n K W (h^2)_x = g_n h W \left[ \tan \theta - \mu(h, u) \right]. \]  

(2.2)

The flow quantities \( h \) and \( u \) are independent (averaged out) of \( y \), i.e. \( h = h(x, t), u = u(x, t) \). Besides width-averaging, we consider the contracting channel as illustrated in Fig. 1. The inclined channel has a constant channel-width \( W(x) = W_0 \) for \( x_0 < x < x_m \), where \( x_0 \) and \( x_m \) denote the \( x \)-coordinate of the channel’s and the contraction’s entrance, respectively. From \( x \geq x_m \), the channel-width linearly decreases from \( W(x_m) = W_0 \) to a minimum channel-width at the channel’s exit, i.e. \( W(x_c) = W_c \) with \( x_c \) the \( x \)-coordinate of the channel exit. Given the above, before we introduce the dimensionless variables denoted by primes,

\[ t = \frac{W_0}{u_l} t', \quad x = W_0 x', \quad u = u_l u', \quad h = h_l h', \quad W = W_0 W', \]

\[ F_l = \frac{u_l}{\sqrt{g_n h_l}}, \quad (\tan \theta - \mu(h, u)) = \frac{h_l}{W_0} (\tan \theta - \mu(h, u))'. \]  

(2.3)

The variables \( W_0, u_l \) and \( h_l \) are suitable characteristic length-scales for our flow domain (\( u_l \) only a reference value for the channel width), flow velocity and flow depth, respectively. In our model, \( W_0, u_l \) and \( h_l \) are the values defined upstream of the contraction at any chosen point \( x = x_l < x_m \). We also define an upstream Froude number as \( F_l = u_l/\sqrt{g_n h_l} \). The average slope of the contraction is given \( \alpha = (W_0 - W_c) / (x_c - x_m) \).

After substituting the scaled variables and dropping the primes, see Appendix A.1.2 in Tunuguntla (2015), we obtain a non-dimensional depth- and width-averaged shallow-layer model for isotropic, \( K = 1 \), granular flow,

\[ (hW)_t + (huW)_x = 0, \]

\[ u_t + uu_x + \frac{1}{F_l^2} h_x = F_l \left[ \tan \theta - \mu(h, u) \right]. \]  

(2.4)

The one-dimensional shallow-layer model (2.4) consists of the continuity equation and the downslope momentum equation. One could also arrive at these equations via a standard control volume analysis of a column of granular material viewed as a continuum from the base to the free surface, using Reynolds-stress averaging and a leading order closure. Moreover, in order to have a closed system of shallow-layer equations we need a constitutive friction law, which we shall briefly describe in the following section.

2.1. Constitutive law/Closure relation

The basic difference between the shallow-layer fluid model and a granular one, i.e. (2.1), is the presence of the basal friction coefficient \( \mu \), where \( \mu \) is the ratio of the shear to normal traction at the base. Some of the previously developed dry granular models incorporated a dry Coulomb-like friction law (Savage & Hutter 1989). However, the Coulomb-like friction law holds only in two cases (as listed in Pouliquen (1999)):
(i) When the inclined channel is smooth, fully developed uniform flows are found to exist at one critical inclination angle (Hunger & Morgenstern 1984; Patton et al. 1987; Ahn et al. 1991). Above this angle the material accelerates and below this angle the flowing material eventually stops. The rheological properties of flows over smooth channels are well described by a constant friction constant, which equals the tangent of the angle of friction between the material and the base \( \delta \), i.e. \( \mu = \tan \delta \).

(ii) Similarly, experimental studies also show that the constant friction coefficient holds for accelerating flows over rough channels at higher inclinations (Augenstein & Hogg 1974; Hunger & Morgenstern 1984). Experimental measurements of the shear forces at the bed show the friction coefficient to be independent of the velocity.

For an intermediate range of angles where steady uniform flows reside (Pouliquen 1999; Vallance 1994; Azanza et al. 1999), the simple Coulomb friction, however, fails to describe the flow rheology on channels with rough beds. Using accurate experimental measurement techniques Pouliquen (1999); Forterre & Pouliquen (2003) empirically determined a scaling which allows one to predict the variation in the mean (depth-averaged) velocity as a function of the channel inclination, flow depth and channel roughness,

\[
F = \frac{u}{\sqrt{gh}} = \beta \frac{h}{h_{\text{stop}}(\theta)} - \gamma, \quad (2.5)
\]

where \( \beta \) and \( \gamma \) are constants and \( F \) is the Froude number. The effects of changing the channel roughness, channel inclination and other features like mixture particle size is captured in \( h_{\text{stop}}(\theta) \) without any experimental velocity measurements. The variable \( h_{\text{stop}}(\theta) \) denotes the critical thickness where the flow arrests or comes to a halt. Each channel inclination has a unique critical thickness, which depends on the channel roughness and particle size, see (Pouliquen 1999) for more details concerning the measurement of \( h_{\text{stop}}(\theta) \). With this scaling law at hand, Pouliquen & Forterre (2002) further expressed the stoppage height, as a function of the angle of inclination,

\[
\frac{h_{\text{stop}}(\theta)}{Ad} = \frac{\tan(\delta_2) - \tan(\theta)}{\tan(\theta) - \tan(\delta_1)}; \quad \delta_1 < \theta < \delta_2, \quad (2.6)
\]

where \( d \) is the grain diameter and \( A \) is a characteristic dimensionless length scale over which the friction varies. In addition, the above empirical friction law (2.6) is characterised by two angles: the angle at which the material comes to rest \( \delta_1 \), below which friction dominates over gravity and the angle \( \delta_2 \) above which the material accelerates as gravity dominates friction. It is between these two angles that steady flow is possible. One can obtain the constitutive friction law on combining (2.5) and (2.6). On assuming the steady state flow assumption \( \mu = \tan \theta \) to hold (approximately) in the dynamic case as well, one obtains an improved friction law valid for lower Froude numbers, empirical friction law

\[
\mu = \mu(h, F) = \tan(\delta_1) + \frac{\tan(\delta_2) - \tan(\delta_1)}{\beta h / (Ad(F + \gamma)) + 1}. \quad (2.7)
\]

As \( \delta_1 \rightarrow \delta_2 \), the Coulomb’s model is recovered, see Grigorian et al. (1967).

### 2.2. Steady state solutions

By utilising the improved macro-scale constitutive friction law (2.7), the steady flow states in a granular flow – through a contraction – are predicted through the shallow-layer granular model (2.4).

We begin by defining a non-dimensional Froude variable as

\[
F(x) = F_1 \frac{u(x)}{\sqrt{h(x)}}, \quad (2.8)
\]
where \( F_l \) takes \( F_l = F_0 \) for values \( u_0, W_0 \) and \( h_0 \) at \( x = x_0 \) near the sluice gate (channel upstream) or \( F_l = F_m \) for values \( u_m, W_0 \) and \( h_m \) at the contraction entrance \( x = x_m \), see Fig. 1. Note that, even after scaling (2.2) and the other variables, we still retain the parameters \( F_l, B_c = W_c/W_0 \), as well as the scaled variable \( \tan \theta - \mu(h, F) \) and dimensionless \( x_c \) and \( x_l \).

For flows in steady state, from the mass balance equation \((huW)_x = 0\), we obtain a constant mass flux \( huW = \) constant. We define the integration constant as \( Q \) and for our scaling we consider \( Q = 1\). Similarly, in steady state, the momentum balance equation is stated in conservative form as

\[
F_l^2 \left( \frac{u^2}{2} \right)_x + h_x = \tan \theta - \mu(h, F). \tag{2.9}
\]

Using \( u^2 = \frac{F^2 h}{F_l^2} \), we obtain

\[
\frac{d}{dx} \left[ \left( 1 + \frac{F^2}{2} \right) h \right] = \tan \theta - \mu(h, F), \tag{2.10}
\]

and derive expressions for the flow height \( h(x) \) and its derivative, see Appendix A.1,

\[
h = \left( \frac{Q F_l}{W F} \right)^{2/3} \quad \text{and} \quad \frac{dh}{dx} = -\frac{2}{3} \left[ h F F_x + h W W_x \right]. \tag{2.11}
\]

Combining (2.10) and (2.11), see Appendix. A.2, we have

\[
\frac{dF}{dx} = \frac{1}{2} \left( \frac{F^2 + 2}{F^2 - 1} \right) F d \ln(W) - \frac{3}{2} \left( \frac{Q F_l}{2/3} \right)^{2/3} \left( F^2 - 1 \right), \tag{2.12}
\]

with \( C_d(h, F) = -(\tan \theta - \mu(h, F)) \). Note that (2.12) is analogous to the equation (7) \( C_d(h, F) = C_d \) a constant; the coefficient \( C_d \) is also known as frictional drag in hydraulics. An analytical solution is found for a special case where for a given steady flow, \( \mu = \tan \theta \) (inviscid flow) holds (approximately). For this inviscid case, (2.12) can be written as

\[
\frac{dF}{dx} = \frac{1}{2} \left( \frac{F^2 + 2}{F^2 - 1} \right) F dW/W dx, \tag{2.13}
\]

which when analytically integrated with respect to \( x \), from the channel upstream position to some point downstream of the channel yields, see Appendix A.3,

\[
\frac{F_l}{F} \left( \frac{2 + F^2}{2 + F_l^2} \right)^{3/2} = \frac{W}{W_0}. \tag{2.14}
\]

For the given closed system (2.12), the well known critical channel exit condition from Houghton & Kasahara (1968) where the Froude number \( F = 1 \), plays the role of a boundary condition at the channel exit. At this condition the flow at the channel exit is "sonic" or "critical", such that the flow speed \( u \) equals the gravity wave speed \( \sqrt{gh}/F_l \), which in terms of dimensional quantities means that the flow speed \( u \) equals \( \sqrt{g h_0} \). On utilising this critical condition for the inviscid case, the solution with \( F(x_c) = 1 \) and for \( x < x_m \), where \( F_l = F_0 \), is

\[
F_0 \left( \frac{3}{2 + F_0^2} \right)^{3/2} = F_l. \tag{2.15}
\]

Note that in the above equation \( B_c := W_c/W_0 \). Thereby, the average solutions obtained are smooth as long as the flow is critical with \( F_0 < 1 \) or supercritical with \( F_0 > 1 \).
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**Parameter Plane**

The curves above demarcate a region in the $F_0/m - B_c$ plane where either smooth or shock solutions exist for inviscid flows. The solid line indicates the solution of (2.15), whereas the dashed line corresponds to the solution of (2.25). Note that $F_m$ corresponds to the value at the entrance of the contraction and $F_0$ corresponds to the value at the entrance of the channel, which for inviscid flows are the same, i.e. $F_0 = F_m$. Hence, the notation $F_0/m$.

Furthermore, the inviscid solution (2.15) divides the $F_0 - B_c$ parameter plane into regions characterising the smooth sub- and supercritical flows, see Fig. 2.

In order to obtain the demarcating curves for granular flows with frictional effects, we integrate the ordinary differential equation (ODE) (2.12) using the fourth-order Runge-Kutta scheme, known as RK4, starting from the contraction exit $x = x_c$ given the critical channel exit condition $F(x_c) = 1$, $B_c$ and the width $W = W(x)$. The location of the contraction exit, $x = x_c$, is given by the relation $x_c = L \cos \theta_c$, with $\theta_c = \sin^{-1}((W_0 - W_c)/2L)$, where $L$ is the length of the contraction paddle. The Froude number $F_l$ and depth $h_l$ are prescribed upstream of the channel at $x = x_l$. Either $F_l = F_0$ upstream of the channel or $F_l = F_m$ at the contraction entrance. Given the critical condition is at the contraction exit, where $F = 1$ by definition, a closer look at Eq. (2.12) indicates,

\[
\frac{2}{3} \left( \frac{F^2 - 1}{F^2} \right) \frac{dF}{dx} = \left( \frac{F^2 + 2}{3F} \right) \frac{d(\ln W)}{dx} + \frac{F^{-1/3}W^{2/3}}{(QF_l)^{2/3}}(\tan \theta - \mu(F)),
\]

\[
0 \frac{dF}{dx} \neq \left( \frac{3}{2} \frac{dW}{dx} \right) \left( \frac{1}{2} \frac{dW}{dx} \right)^{2/3} \frac{W^{2/3}}{W_c^{1/3}}(\tan \theta - \mu(1)),
\]

(2.16)

The LHS is singular at the contraction exit $x_c$ implying the ODE does not hold (2.16) at the channel exit, i.e. LHS $\neq$ RHS when $F = 1$, from a mathematical point of view. In Eqn. (2.16), $\alpha = dW/dx$ at the contraction exit. Regularisation we intend to determine the finite value for $dF/dx$ at the contraction exit. The first step here is to construct more slopes, i.e. compute more values of $\alpha$ and find the new contraction exit $x = x_{\text{new}}$ at which LHS $=\text{RHS}$. This is achieved by fitting a circle at the exit of the channel as shown.
in Fig. 3. Choosing a circle is convenient, as it has an infinite number of slopes and can be easily fitted at the contraction exit \( x_c \). Moreover, as the radius \( R \to 0 \), we return back to the initial sidewall geometry (approximately). Once the new contraction exit, \( x_{c_{\text{new}}} \), is determined, such that LHS=RHS, we arrive at a classic limit problem, stated as

\[
\lim_{x \to x_{\text{new}}, F(x_{\text{new}}) \to 1} \frac{dF}{dx} = 0
\]

(2.17)

The limit (2.17), is solved by using the Taylor series expansion by which the finite slope at the new contraction exit \( x = x_{c_{\text{new}}} \) is determined for both the inviscid and viscous cases. 

For a given sufficiently large \( F_l > 1 \) or sufficiently small \( F_l < 1 \); in order to produce the critical (demarcating) curves – for viscous flows – that allows one to distinguish between the smooth supercritical and subcritical flows and flows with jump, the ODE (2.12) can be integrated in two ways:

(i) Non-regularised approach:

To avoid the singularity in Eqn. 2.12, the ODE is integrated starting from the initial contraction exit, \( x_c \), with Froude number \( F = \lim_{\varepsilon \to 0} 1 \pm \varepsilon \), respectively.

(ii) Regularised approach:

Given the singularity is resolved by determining the new contraction exit and the correct slope \( dF/dx \), as described in Appendix A.1.5 in Tunuguntla (2015), a regularised approach can now simply use the critical boundary condition \( F = 1 \) and integrates the ODE starting from the new nozzle exit \( x = x_{c_{\text{new}}} \).

To compare both approaches, regularised and non-regularised, the ODE (2.12) is integrated to a point channel upstream, \( x = x_m \), to find a new estimate for \( F_l \). Since, the scaling parameter \( F_l \) is unknown beforehand (as it is part of the solution) the correct value is found iteratively. As an educated guess, one can consider the initial value for \( F_l = F_{0/m}(B_c) \) as the one obtained from the solution for the inviscid case, where \( F_{0/m} \) is a function of \( B_c \). Given this we proceed iteratively until convergence is reached. As a result, Fig. 4 shows the demarcation curves obtained, for inviscid and frictional flows, using both the regularised and non-regularised approach. As illustrated, the constructed curves via both approaches are in an excellent agreement. Thus, by utilising the same regularisation approach as described above, the ODE (2.12) is further integrated to a
Figure 4. Non-regularised and regularised approach compared for both (a) inviscid flows (b) frictional flows. The dashed lines demarcate the extent of upstream moving/steady shocks.
point further channel upstream\(^\dagger\), \(x = x_0\). Fig. 5 illustrates the existence of different flow regimes by plotting the critical curves which demarcate the \(-B_c\) and \(F_0 - B_c\) plane into several regions.

2.3. Shock solutions

The closed system of equations, (2.4), is hyperbolic and thus can develop discontinuities in the flow field in finite-time. These discontinuities are nothing but jumps in the depth or velocity of the flow that propagate at a well-defined velocity.

For upstream moving shocks instead of matching the upstream conditions with the channel exit conditions, we relate the depth \(h_u\) and velocity \(u_u\) upstream of the shock to \(h_1\) and \(u_1\) downstream of the shock and also to the depth \(h_c\) and \(u_c\) at the channel exit. From the mass and momentum balance equations, for a shock moving at speed \(s\) (using (A 6) and (A 8) with \(h^+ = h_u\), \(u^+ = u_u\) and \(h^- = h_1\), \(u^- = u_1\)) the jump conditions are restated as

\[
(u_u + s)h_u = (u_1 + s)h_1, \\
(u_u + s)^2 = \frac{h_1}{2F_l^2} \left(1 + \frac{h_1}{h_u}\right). 
\]

(2.18)

Similarly in order to obtain the jump conditions for a shock in the contraction, we combine the above jump conditions with the Bernoulli equation and mass conservation in the contraction, thus leading to

\[
\frac{1}{2} u_1^2 + h_1/F_l^2 = \frac{1}{2} u_c^2 + h_c/F_c^2, \\
u_1h_1W_1 = u_ch_cW_c 
\]

(2.19)

and from (2.8) with \(F(x_c) = 1\), we obtain the critical condition

\[
u_c^2 = h_c/F_c^2. 
\]

(2.20)

Variables \(u_c\) and \(W_c\) are the downstream speed and width at the contraction exit. By introducing \(F_u = u_uF_l/\sqrt{h_u}\), \(S = sF_l/\sqrt{h_u}\), \(B_1 = W_c/W_1\) and \(H_1 = h_1/h_u\), below we substitute (2.18)-(2.20) to express the jump conditions in terms of the upstream Froude number \(F_u\), the scaled shock speed \(S\) and \(H_1\). On substituting these scalings into (2.18)\(_2\), we arrive at

\[
\left(\frac{\sqrt{h_u}}{F_l}(F_u + S)\right)^2 = \frac{h_1}{2F_l^2} (1 + H_1) \underbrace{\frac{\sqrt{h_u}}{F_l}}_{\text{Taking out}} \rightarrow (F_u + S)^2 = \frac{1}{2} H_1 (1 + H_1). 
\]

Condition 1

(2.21)

Similarly, below we substitute the above scalings in (2.18), to arrive at a few relations

\(\dagger\) Note that the curves in Fig. 4 were obtained by numerically integrating the ODE (2.12) to a point channel upstream, \(x = x_m\).
Figure 5. Illustrates the $F_m$, $B_c$ plane and the $F_0$, $B_c$ plane divided into regions of different flow regimes for frictional flows. Note that $F_m$ corresponds to the value at the entrance of the contraction and $F_0$ corresponds to the value at the entrance of the channel. In regions (i) and (ii) analysis predicts supercritical and subcritical smooth flows, whereas the flows in region (iii) are expected to have upstream moving shocks. Nevertheless, region (i/iv) suggest a possibility of multiple flow regimes such as steady shocks in the contraction (reservoir state) and oblique waves, i.e averaged smooth flows, see Fig. 6. The solid lines demarcate the existence of the region of supercritical and subcritical flows. Similar to the one in Fig. 2 for inviscid flows, the dashed line demarcates the extent of the moving/steady upstream shocks.
useful for further manipulation of (2.19)_1,

\[(u_u + s) = (u_1 + s)H_1 \rightarrow \sqrt{h_u/F_1} (F_u + (1 - H_1)S) = u_1H_1,\]

\[\rightarrow u_1^2 = \frac{h_u}{H_1^2F_1^2} (F_u + (1 - H_1)S)^2,\]  
(2.22)

\[\text{Relation } \#1\]

\[(2.18)_2 \rightarrow h_1 = \frac{2F_1^2(u_u + s)^2}{1 + H_1} \rightarrow h_1 = 2h_u \frac{(F_u + S)^2}{1 + H_1} \quad \text{jump cond. } \#1 \rightarrow h_1 = h_uH_1.\]  
(2.23)

Furthermore, by using the critical condition (2.20) in (2.19)_2, we obtain

\[u_1h_1 = u_c h_c B_1 \frac{u^2 = h_c/F_1^2}{u^2} \rightarrow u^2 = \left( \frac{u_1h_1}{B_1F_1^2} \right)^{2/3}.\]  
(2.24)

On substituting the above relations (2.22)-(2.23) in (2.19)_1, we systematically proceed to arrive at

\[\frac{u^2 = h_c/F_1^2}{2} \rightarrow \frac{1}{2} u^2 + \frac{h_1}{F_1^2} = \frac{3}{2} \frac{u^2}{F_1^2} \quad \text{Relation } \#2 \rightarrow \frac{1}{2} u_1^2 + \frac{h_1}{F_1^2} = \frac{3}{2} \left( \frac{u_1h_1}{B_1F_1^2} \right)^{2/3},\]

\[\text{Relation } \#1 \& \#2 \rightarrow \frac{h_u}{2H_1^2F_1^2} (F_u + (1 - H_1)S)^2 + \frac{h_u}{F_1^2} h_1 = \frac{3}{2} \left( \frac{h_u^2}{F_1^2} (F_u + (1 - H_1)S) \right)^{2/3},\]

\[\text{Factoring out } \frac{h_u}{F_1^2} \rightarrow \frac{1}{2} [F_u + (1 - H_1)S]^2 = \frac{3}{2} H_1^2 \left[ \frac{F_u + (1 - H_1)S}{B_1} \right]^{2/3} - H_1^3.\]  
(2.25)

Thus, through these above manipulations, we arrive at the jump conditions in terms of the upstream Froude number, shock speed and flow depth ratio

\[(F_u + S)^2 = \frac{1}{2} H_1 (1 + H_1),\]

\[\frac{1}{2} [F_u + (1 - H_1)S]^2 = \frac{3}{2} H_1^2 \left[ \frac{F_u + (1 - H_1)S}{B_1} \right]^{2/3} - H_1^3.\]

As we can see when \(H_1 = 1\), i.e., when there exists no jump in depth/height across the shock, the above equation (2.25)_2 reduces to the purely inviscid case (2.15) for \(F_u = F_0 \leq 1\) and \(B_1 = B_c\). For \(F_0 > 0\) and \(B_1 = B_c\), we get the dashed line in Fig. 5. In the same illustration, Fig. 5, the lower solid line corresponds to \(F_0 < 1\) and the upper solid line concerns \(F_0 > 1\). The curves divide the \(F_0 - B_c\) plane into regions where moving shocks and smooth solutions co-exist when the viscous term is included, i.e. \(\tan \theta \neq \mu\), the shocks become steady with time. Hence, we consider the shock speed \(s = 0\). The Bernoulli equations stated for the inviscid case are no longer valid when we include frictional effects. They are replaced by eq. (2.12) from the shock position to the contraction exit. Similar to the inviscid case, we calculate the arrested shock at the contraction entrance by integrating (2.12) from the channel exit to the contraction entrance with the critical channel exit condition \(F(x_c) = 1\), to the point where the
granular jump occurs, i.e. at the contraction entrance \( x_m \). Given that we denote the Froude number just downstream of the contraction entrance as \( F = F_1 \) and at the upstream as \( F = F_0 \) with \( s = 0 \), using the shock relation (2.25) with \( h_u = h_m \), \( u_u = u_m \), \( F_u = F_m \), we arrive at the quadratic equation stated below,

\[
\frac{1}{2} \left( \frac{h_1}{h_m} \right)^2 + \frac{1}{2} \left( \frac{h_1}{h_m} \right) - F_m^2 = 0,
\]

\[
\text{solution } \frac{h_1}{h_m} = \frac{1}{2} \left[ -1 + \sqrt{1 + 8F_m^2} \right].
\]

From (2.18) we have \( h_1 u_1 = h_u u_u \), which, when rearranged results in, \( u_m / u_1 = h_1 / h_m \). Using (2.8), this is restated as \( F_m / F_1 = (h_1 / h_m)^{3/2} \). Substituting \( F_m = F_1 H_1^{3/2} \) in (2.25), results in a quadratic equation stated below

\[
F_1^2 = \frac{1}{2H_1} + \frac{1}{2H_1} \rightarrow \frac{1}{2} \left( \frac{h_m}{h_1} \right)^2 + \frac{1}{2} \left( \frac{h_m}{h_1} \right) - F_1^2 = 0 \quad \text{solution } \frac{h_m}{h_1} = \frac{1}{2} \left[ -1 + \sqrt{1 + 8F_1^2} \right].
\]

\[
\frac{F_m}{F_1} = \left( \frac{h_1}{h_m} \right)^{3/2} \rightarrow F_m = \sqrt{8F_1} / \left[ -1 + \sqrt{1 + 8F_1^2} \right]^{3/2} > 1.
\]

The ODE (2.12) is further integrated upstream from \( F = F_m > 1 \) at \( x = x_m \) to find our next estimate of \( F_1^* \) at \( x = x_0 \). Generally, \( F_1^* \neq F_1 \), where \( F_1 \) is the scaling used in Eqn. (2.12). Thereby, one begins with the inviscid result \( F_1 = F_0(B_c) \). In the inviscid case one can use (2.14) with \( F_1 = F_1 \) at the entrance of the contraction and \( F_m = F_0 > 1 \) to find \( F_1 = \sqrt{8} \left[ -1 + \sqrt{1 + 8F_1^2} \right]^{3/2} \) from (2.27) to give

\[
F_1 \left( \frac{3}{2 + F_1^2} \right) = B_c
\]

For comparison purposes, the demarcating curves, obtained either by regularisation or non-regularisation, are almost identical, see Fig. 4. Nevertheless, the regularisation is necessary to make the numerical analysis mathematically sound, i.e., without a singularity at \( F = 1 \).

The supercritical \( (F > 1) \) and subcritical \( (F < 1) \) flow profiles are illustrated via \( F \) and \( h \) versus \( x \) plots in Fig. 6. These profiles are obtained by integrating from a point upstream of the channel into the downstream direction to the contraction exit. For flows with granular jumps (discontinuities in flow quantities), the critical condition at the channel exit is \( F(x_c) = 1 \). We commence then at the channel exit and move into the upstream direction. The jumps in the flow quantities are computed by applying the jump condition, and then finding a point where the downstream and upstream profiles match. Thereby, the flow profiles are efficiently computed from the novel granular hydraulic theory. In the following section we verify two of the above flow profiles, super- and sub-critical i.e. Fig. 6(i) and Fig. 6(ii), by solving the depth-averaged model (2.1) using DGFEM.

3. Verification of one-dimensional theory via two-dimensional DGFEM

The asymptotic theory, presented above, helps to approximately predict the flow regimes in \( F_0 - B_c \) plane. In order to verify the one-dimensional results, two-dimensional numerical solutions of the shallow granular equations (2.1) are compared with the former. The additional degree of freedom helps us illustrate the results from the one-dimensional
Figure 6. Profiles of Froude number $F = F(x)$ and height $h = h(x)$, as a function of downstream coordinate $x$ for the three flow states: (i) Supercritical flow with $F > 1$ and $B_c = 0.5$, (ii) subcritical flow for $F < 1$ and $B_c = 0.7$ and (iv) reservoir with shock in the contraction with $B_c = 0.5$. Profiles (i) and (iv) corresponds to $*$ in the multiple steady states region, i/iv, in the $F_0 - B_c$ plane, see Fig. 5. The extent of the contraction is indicated by the thick line on the $x$-axis from $x_m = 6$ to $x_c = 11$.

analysis in two-dimensional space. Moreover, the two-dimensional solutions also allow us to verify the validity of the constitutive friction law, the closure relation, in two-dimensional space.

3.1. Two-dimensional DGFEM solutions

Two-dimensional solutions are obtained by solving the SGE, Eqns (2.1), along with the closure relation $\mu(h, F)$ (2.7), using a discontinuous Galerkin finite element method (DGFEM). The method is a blend of high resolution finite element and finite volume method, which is often utilised for solving systems of partial differential equations, see for example (Di Pietro & Ern 2011). For solving the system of shallow granular equations (2.1), we adopt the second order space-discontinuous Galerkin method of Tassi et al. (2007) combined with the HLL numerical flux (Harten et al. 1997).

3.1.1. Domain

Similar to the schematic illustrated in Fig. 1, we consider an inclined channel with contracting sidewalls with $(x, y) \in [0, 11] \times [0, 1]$. Moreover, for our verification purposes, we consider a channel configuration that converges from $x = x_m = 6$ to $x = x_c = 11$ and has a channel exit ratio of $B_c = 0.5$ (for supercritical flows) and $B_c = 0.7$ (for subcritical flows). Note that this is the same channel configuration we considered for computing the flow profiles illustrated in Fig. 6.

The channel domain is discretised with a uniform quadrilateral mesh, where the channel sidewalls are prescribed with a solid wall boundary condition, as proposed in Ambati &
Figure 7. The top panels of (a) and (b) illustrate a contour plot of the flow height \( h = h(x, y) \) as a function of downstream and cross-slope coordinate \( x \) and \( y \) for a prescribed upstream inflow Froude number (a) supercritical flow \( F = 4 \) and (b) subcritical flow \( F = 0.1 \), obtained by solving the two-dimensional shallow granular model using a discontinuous Galerkin finite element method in an open-source code \( \text{hpgem.org} \). On the contrary, the bottom panels of (a) and (b) illustrate the profiles of Froude number \( F = F(x) \) and height \( h = h(x) \) as a function of downstream coordinate \( x \) for super- and subcritical flow (a) \( F = 4 \) and (b) \( F = 0.1 \), obtained from the one- and two-dimensional shallow granular model. The circles indicate the averaged DGFEM solution and the solid line represents the solution obtained using our one-dimensional theory.

Bokhove (2007). The boundaries at \( x = 0 \) and \( x = 11 \) are prescribed with inflow and outflow boundary conditions, respectively.
3.1.2. Results

For supercritical flows, i.e. $F > 1$, we prescribe the inflow height and downstream velocity at the inflow boundary and a free-flow boundary condition at the outflow boundary. For comparison with our one-dimensional solution, as shown in Fig. 6(i), we specify the inflow height $h = 1$ and the downstream velocity $u$, such that the inflow Froude number is $F = 4$. Moreover, this prescribed value also corresponds to a point in the multiple steady state region, see the asterisk symbol in Fig. 5. When solved, the resulting two-dimensional profile illustrates weak oblique jumps in the converging region of the channel, see Fig. 7(a) top panel.

On the other hand, for the subcritical flows with Froude number $F < 1$, prescribing certain inflow height and downstream velocity at the inflow boundary and a free-flow boundary condition at the outflow boundary is not sufficient. To overcome this difficulty, we treat the boundary conditions using a Riemann invariant (Bristeau & Cousson 2001) and specify the height $h^{in} = 1$ and $h^{out} = 0.75$ at either end. As a result, the resulting two-dimensional flow height profile is illustrated in the top panel of Fig. 7(b).

3.2. Two-dimensional DGFEM vs. one-dimensional asymptotic theory

The two-dimensional solutions obtained from the DG-formulation are averaged in the cross-slope direction and compared with the solution profile constructed using the novel one-dimensional granular theory, cases (i) and (ii) in Fig. 6. As a result, from Fig. 7, one can conclude that the one-dimensional asymptotic theory is in an excellent agreement with the two-dimensional solutions for sub- and supercritical flows.

4. Conclusions

4.1. Summary

As a stepping stone towards analysing the monodisperse granular mixture flowing over rough inclined channels – with constrictions – we use the extensively utilised depth-averaged shallow granular equations. On further exploiting the shallowness aspect in the cross-slope direction, which is often associated with the flows considered, we width-averaged the depth-averaged shallow granular equations, and arrived at a novel one-dimensional granular hydraulic theory. For closure, we used an empirically determined constitutive friction law, which was validated using discrete particle simulations by Weinhart et al. (2012a). By using the one-dimensional theory, we can compute the flow profiles for any channel opening and inclination in a quick and efficient manner. Besides this ability, we also illustrated – for steady flows – the existence of multiple flow regimes. As a verification step, for supercritical and subcritical flows, the one-dimensional theory is compared to solutions obtained via numerically solving an equivalent depth-averaged shallow granular system of equations using DGFEM. On comparison, we found an excellent agreement between the one- and two-dimensional theories.

4.2. Future work

Although, the one-dimensional model is verified for supercritical and subcritical flows, a complete verification of the one-dimensional theory has to be carried out for flows with jumps or shocks. Besides verifying the theory, a thorough validation needs to be performed to illustrate the underlying strength of the one-dimensional model. To perform the validation, discrete particle simulations will be utilised as shown in Fig. 8. Although initial attempts have been made to validate the above one-dimensional theory using fully three dimensional discrete particle simulations, close agreement could not be reached,
Figure 8. Illustrates a discrete particle simulation of a monodisperse granular mixtures flowing through an inclined channel with a linear contraction at the outflow. The simulation was performed using our in-house open-source particle solver MercuryDPM (Thornton et al. 2013).

i.e. the flow profiles in the contraction did not completely match the one-dimensional predictions. However, the flow profiles for flows over an inclined channel without a contraction do agree with solutions obtained using the one-dimensional theory. This suggests for further investigation to understand the underlying dynamics in a channel with a linear contraction.

4.3. Acknowledgements

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Appendix A

A.1. Derived relations

By substituting the definition corresponding to the local Froude number $F = F_l \frac{u}{\sqrt{h}}$, into the mass balance equation (2.2)$_1$, we have for steady state,

$$huW = Q \Rightarrow u = \frac{Q}{hW}$$

$$h = F_l^2 \frac{u^2}{F} \rightarrow h = \frac{F_l^2 Q^2}{h^2 W^2 F^2} \rightarrow h^3 = \frac{F_l^2 Q^2}{W^2 F^2} \Rightarrow h = \left(\frac{Q F_l}{W F}\right)^{2/3}$$

(A 1)
From above relations,

\[
\frac{dh}{dx} = 2 \left( \frac{QF_l}{WF} \right) \frac{1}{3} \frac{d}{dx} \left( \frac{QF_l}{WF} \right),
\]

\[
\frac{dh}{dx} = 2 \left( \frac{QF_l}{WF} \right) \frac{1}{3} \left( \frac{QF_l}{WF} \right) \left( DF \frac{d}{dx} \left( \frac{1}{WF} \right) \right),
\]

\[
\frac{dh}{dx} = \frac{2}{3} \left( \frac{QF_l}{WF} \right) \frac{2}{3} \left[ \frac{-WF_x - W_x F}{WF} \right],
\]

\[
\frac{dh}{dx} = \frac{2}{3} \left( \frac{QF_l}{WF} \right) \frac{2}{3} \left[ \frac{hF_x + hW_x}{W} \right].
\]

(A 2)

A.2. Froude function

For \( K = 1 \) and substituting (A 1) and (A 2) in the one-dimensional momentum equation (2.2)_2, we have using the following steps

\[
\frac{d}{dx} \left[ \left( \frac{1}{2} F^2 + 1 \right) h \right] = (\tan \theta - \mu(h, F)),
\]

\[
\frac{dh}{dx} + hF \frac{dF}{dx} + \frac{F^2}{2} \frac{dh}{dx} = (\tan \theta - \mu(h, F)),
\]

\[
hF \frac{dF}{dx} = -\frac{F^2}{2} \frac{dh}{dx} - \frac{dh}{dx} + (\tan \theta - \mu(h, F)),
\]

\[
hF \frac{dF}{dx} = \frac{2}{3} \left[ h \left( \frac{F^2}{2} + 1 \right) \right] \left[ \frac{1}{F} \frac{dF}{dx} + \frac{1}{W} \frac{dW}{dx} \right] + (\tan \theta - \mu(h, F)),
\]

\[
\frac{dF}{dx} = \left( \frac{F^2 + 2}{3F^2} \right) \frac{1}{W} \frac{dW}{dx} + \left( \frac{F^2 + 2}{3F^2} \right) \frac{dF}{dx} + \frac{1}{hF} (\tan \theta - \mu(h, F)),
\]

\[
\left( 1 - \frac{F^2 + 2}{3F^2} \right) \frac{dF}{dx} = \left( \frac{F^2 + 2}{3F^2} \right) \frac{1}{W} \frac{dW}{dx} + \frac{1}{hF} (\tan \theta - \mu(h, F)),
\]

\[
\frac{dF}{dx} = \left( \frac{3F^2 - 2 - F^2}{3F^2} \right) \frac{dF}{dx} = \frac{F^2 + 2}{3FW} \frac{dW}{dx} + \frac{1}{hF} (\tan \theta - \mu(h, F)),
\]

\[
\frac{dF}{dx} = \frac{2}{3} \left( \frac{F^2 - 1}{F^2 - 1} \right) \frac{dF}{dx} = \left( \frac{F^2 + 2}{3F} \right) \frac{dW}{dx} + \frac{1}{hF} (\tan \theta - \mu(h, F)),
\]

\[
\frac{dF}{dx} = \frac{1}{2} \left( \frac{F^2 + 2}{F^2 - 1} \right) F \frac{dW}{dx} + \frac{1}{hF} (\tan \theta - \mu(h, F)).
\]

Substituting \( h = \left( \frac{QF_l}{WF} \right)^{2/3} \), results in

\[
\frac{dF}{dx} = \frac{1}{2} \left( \frac{F^2 + 2}{F^2 - 1} \right) F \frac{dW}{dx} + \frac{3}{2} \left( \frac{F^{5/3}}{WF^{2/3}} \right) \left( \frac{F^{5/3}}{F^{2/3}} \right) (\tan \theta - \mu(h, F)).
\]

(A 3)
A.3. Analytic solution for inviscid flow

Eq. (2.12) can for the inviscid case, be written as

\[ \frac{dF}{dx} = \frac{1}{2} \left( \frac{F^2 + 2}{F^2 - 1} \right) \frac{F}{W} \frac{dW}{dx}, \]  

which when analytically integrated with respect to \( x \), from the channel upstream position \( x_l \) to some point channel downstream yields

\[
\int_{x_l}^{x} \frac{2(F^2 - 1)}{(F^2 + 2)F} \, dF = \int_{W_0}^{W} \frac{1}{W} \, dW \Rightarrow \int_{F_l}^{F} \frac{F - 1}{(F + 2)F} \, dF = \left[ \ln W \right]_{W_0}^{W},
\]

\[
\left[ \frac{3}{2} \ln (F + 2) - \frac{\ln (F)}{2} \right]_{F_l}^{F} = \left[ \ln \left( \frac{(F + 2)^{3/2}}{F^{1/2}} \right) \right]_{F_l}^{F} = \left[ \ln W \right]_{W_0}^{W}, \tag{A 5}
\]

\[
\frac{F_l}{F} \left( \frac{2 + F_l^2}{2 + F_l^2} \right)^{3/2} = \frac{W}{W_0}.
\]

A.4. Jump conditions

The jump conditions are derived as follows. We integrate both the non-dimensional depth- and width-averaged mass and momentum balance equation, from \( X(t) - \delta \) to \( X(t) + \delta \) and take the limit \( \delta \to 0 \). Both \( h \) and \( u \) are discontinuous at \( x = X(t) \). For simplicity, we define \( X^- \) as the limit position on the left side of the jump and \( X^+ \) the limit on the right side of the jump, and the shock speed \( s = -dX/dt \). Utilising these definitions, we start by integrating the continuity equation. On utilising Leibniz’s rule, we have

\[
\int_{X^-(t)}^{X^+(t)} (hW)_t + (huW)_x \, dx = 0,
\]

\[
\frac{d}{dt} \int_{X^-(t)}^{X^+(t)} hW \, dx + s \left[ hW \right]_{X^-}^{X^+} + \left[ huW \right]_{X^-}^{X^+} = 0, \tag{A 6}
\]

As \( \int_{X^-}^{X^+} hW \, dx = 0 \) and \( W^+ = W^- = W \Rightarrow \left[ (u + s) \right]_{X^-}^{X^+} = 0, \)

we obtain

\[ h^+ (u^+ + s) = h^- (u^- + s). \]

Similarly, we integrate the momentum equation as below

\[
\int_{X^-(t)}^{X^+(t)} \left( (huW)_t + (hu^2W)_x + \frac{W}{F_l^2} \left( \frac{h^2}{2} \right)_x - \frac{1}{F_l^2} hW (\tan \theta - \mu(F)) \right) \, dx = 0.
\]

As \( W^+ = W^- = W \), the above equation can be simplified into

\[
\int_{X^-(t)}^{X^+(t)} \left( (hu)_t + (hu^2)_x + \frac{1}{F_l^2} \left( \frac{h^2}{2} \right)_x - \frac{1}{F_l^2} h (\tan \theta - \mu(F)) \right) \, dx = 0.
\]
On utilising Leibniz’s rule, 
\[ \int_{X^{-}(t)}^{X^{+}(t)} hu \, dx = 0 \quad \text{and} \quad \int_{X^{-}(t)}^{X^{+}(t)} h (\tan \theta - \mu(F)) \, dx = 0, \]
we have
\[ s [hu]_{X^{-}}^{X^{+}} + [hu^2]_{X^{-}}^{X^{+}} + \frac{1}{2F_i^2} [h^2]_{X^{-}}^{X^{+}} = 0, \]

After some manipulations, the above equation can be restated as follows
\[ h^+ (\left((u^+)^2 + u^+ s\right) - h^- (\left((u^-)^2 + u^- s\right) + \frac{1}{2F_i^2} ((h^+)^2 - (h^-)^2) = 0, \]
which is rewritten as
\[ h^+ (\left((u^+)^2 + 2u^+ s + s^2 - u^+ s - s^2\right) - h^- (\left((u^-)^2 + 2u^- s + s^2 - u^- s - s^2\right) \]
\[ \ldots + \frac{1}{2F_i^2} ((h^+)^2 - (h^-)^2) = 0. \]

Using the jump condition obtained from the continuity equation (A 6), the above equation (A 7) is restated, using the following steps,
\[ h^+ (u^+ + s)^2 - h^- (u^- + s)^2 + \frac{1}{2F_i^2} ((h^+)^2 - (h^-)^2) = 0, \]
\[ h^+ (u^+ + s)^2 - h^- \left(\frac{h^+}{h^-}\right)^2 (u^+ + s)^2 + \frac{1}{2F_i^2} ((h^+)^2 - (h^-)^2) = 0, \]
\[ (u^+ + s)^2 \frac{h^+ (h^- - h^+)}{h^-} = \frac{1}{2F_i^2} ((h^-)^2 - (h^+)^2), \]
\[ (u^+ + s)^2 = \frac{1}{2F_i^2} \left(1 + \frac{h^-}{h^+}\right) h^- . \]

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