Newton-Girard-Vieta and Waring-Lagrange theorems for two non-commuting variables

Nicholas Young

Leeds and Newcastle Universities

Joint work with Jim Agler, UCSD and John McCarthy, Washington University

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Power sums

In 1629 Albert Girard gave formulae for the sums of powers of the roots of a polynomial equation in terms of the coefficients of the polynomial. In 1593 François Viète had given the case of polynomials with positive roots.

The formulae were subsequently often attributed to Newton (Algebra Universalis, 1707).

Consider two commuting variables $x, y$. For any integer $n$ let

\[ p_n(x, y) = x^n + y^n \]
\[ \alpha = x + y \]
\[ \beta = xy. \]

$x, y$ are the roots of the equation $\lambda^2 - \alpha \lambda + \beta = 0$. 
François Viète
The first few Girard-Viète formulae

\[ p_1 = \alpha \]
\[ p_2 = \alpha^2 - 2\beta \]
\[ p_3 = \alpha^3 - 3\alpha\beta \]
\[ p_4 = \alpha^4 - 4\alpha^2\beta + 2\beta^2. \]

Further formulae are obtained from the recursion

\[ p_{n+2} = \alpha p_{n+1} - \beta p_n. \]
Symmetric polynomials

A polynomial $p$ is symmetric if it is unchanged by a permutation of the variables.

The Waring-Lagrange theorem

Every symmetric polynomial is expressible as a polynomial in the elementary symmetric polynomials.

Thus any symmetric polynomial in $x$ and $y$ can be written as a polynomial in $x + y$ and $xy$.

1762: Meditationes Algebraicae, by Edward Waring, Lucasian Professor at Cambridge.
1798: Traité de la Résolution des Équations Numériques de tous les Degrés, by Joseph Louis Lagrange.
Meditationes Algebraicae

An English Translation of the Work of Edward Waring

American Mathematical Society
Edward Waring
What if $x$ and $y$ do not commute?

A **free polynomial** is a polynomial in finitely many non-commuting variables.

The symmetric free polynomial

$$xyx + yxy$$

**cannot** be written as a free polynomial in $x + y$ and $xy + yx$.

Show this by substituting $2 \times 2$ matrices for $x, y$ in such a way that $xy + yx = 0$. 
Theorem (Margarete Wolf, 1936)

*There is no finite basis for the algebra of free polynomials in d indeterminates over \( \mathbb{C} \) when \( d > 1 \).*

Thus there is no reason to expect that the free polynomials \( p_n = x^n + y^n \), for integer \( n \), can be written as free polynomials in some finite collection of ‘elementary symmetric functions’ of \( x \) and \( y \).

Nevertheless, we do find three free polynomials \( \alpha, \beta, \gamma \) in \( x \) and \( y \) such that every \( p_n \) can be written as a free polynomial in \( \alpha, \beta, \gamma \) and \( \beta^{-1} \).
A free Newton-Girard-Vieta formula

Let

$$u = \frac{1}{2}(x + y), \quad v = \frac{1}{2}(x - y)$$

and let

$$\alpha = u, \quad \beta = v^2, \quad \gamma = vuv.$$ 

Then $\alpha, \beta, \gamma$ are symmetric free polynomials in $x, y$, and, for every positive integer $n$, there exists a free rational function $P_n$ in three variables such that

$$p_n(x, y) = P_n(\alpha, \beta, \gamma).$$

Moreover $P_n$ can be written as a free polynomial in $\alpha, \beta, \gamma$ and $\beta^{-1}$. 
Proof

Let $q_n = x^n - y^n$. For any integer $n$,

\[ p_n = xx^{n-1} + yy^{n-1} = (u + v)x^{n-1} + (u - v)y^{n-1} \]
\[ = u(x^{n-1} + y^{n-1}) + v(x^{n-1} - y^{n-1}) \]
\[ = up_{n-1} + vq_{n-1}. \]

Similarly,

\[ q_n = vp_{n-1} + uq_{n-1}. \]

Thus

\[
\begin{bmatrix} p_n \\ q_n \end{bmatrix} = T \begin{bmatrix} p_{n-1} \\ q_{n-1} \end{bmatrix} \quad \text{where} \quad T = \begin{bmatrix} u & v \\ v & u \end{bmatrix}.
\]

Hence

\[
\begin{bmatrix} p_n \\ q_n \end{bmatrix} = T^n \begin{bmatrix} p_0 \\ q_0 \end{bmatrix} = T^n \begin{bmatrix} 2 \\ 0 \end{bmatrix}.
\]
Proof – continued

Define the free polynomials \( s_{\text{even}}^n, s_{\text{odd}}^n \) in \( u, v \) for \( n \geq 0 \) to be the sum of all monomials in \( u, v \) of total degree \( n \) and of even or odd degree respectively in \( v \).

\( s_{\text{even}}^n \) and \( s_{\text{odd}}^n \) are symmetric and antisymmetric respectively as polynomials in \( x, y \).

By induction, for \( n \geq 1 \),

\[
T^n = \begin{bmatrix}
  s_{\text{even}}^n & s_{\text{odd}}^n \\
  s_{\text{odd}}^n & s_{\text{even}}^n \\
\end{bmatrix}.
\]

Hence

\[
p_n = 2s_{\text{even}}^n.
\]
Proof – conclusion

Any monomial in $u$ and $v$, in which $v$ occurs with even degree, can be written as a monomial in $\alpha, \beta, \gamma$ and $\beta^{-1}$.

Starting at one end of the monomial, replace all the initial $u$’s by $\alpha$’s. The first $v$ must be followed by another (since the number of $v$’s is even). If it is immediately following, replace $v^2$ by $\beta$. If there are $k$ $u$’s between the first and second $v$’s, replace $vu^kv$ by $(\gamma \beta^{-1})^{k-1} \gamma$.

Continue until all $u$’s and $v$’s have been replaced.

Hence $p_n = 2s_{\text{even}}^n$ is a sum of monomials in $\alpha, \beta, \gamma$ and $\beta^{-1}$. 
The first few $P_n$

$$x^n + y^n = P_n(\alpha, \beta, \gamma)$$

where $\alpha = \frac{1}{2}(x+y)$, $\beta = \frac{1}{4}(x-y)^2$, $\gamma = \frac{1}{8}(x-y)(x+y)(x-y)$.

$$P_1 = 2\alpha$$
$$P_2 = 2(\alpha^2 + \beta)$$
$$P_3 = 2(\alpha^3 + \alpha\beta + \gamma + \beta\alpha)$$
$$P_4 = 2(\alpha^4 + \alpha^2\beta + \alpha\gamma + \gamma\beta^{-1}\gamma + \alpha\beta\alpha + \gamma\alpha + \beta\alpha^2 + \beta^2)$$
$$P_{-1} = 2(\alpha - \beta\gamma^{-1}\beta)^{-1}$$
$$P_{-2} = 2 \left( \alpha^2 + \beta - (\alpha\beta + \gamma)(\gamma\beta^{-1}\gamma + \beta^2)^{-1}(\beta\alpha + \gamma) \right)^{-1}$$
$$P_{-3} = 2 \left( \alpha^3 + \alpha\beta + \beta\alpha + \gamma - (\alpha^2\beta + \alpha\gamma + \gamma\beta^{-1}\gamma + \beta^2) \times (\gamma\beta^{-1}\gamma\beta^{-1}\gamma + \gamma\beta + \beta\gamma + \beta\alpha\beta)^{-1}(\beta\alpha^2 + \gamma\alpha + \gamma\beta^{-1}\gamma + \beta^2) \right)^{-1}.$$
A free Waring-Lagrange theorem

Let

\[ u = \frac{1}{2}(x + y), \quad v = \frac{1}{2}(x - y) \]

and let

\[ \alpha = u, \quad \beta = v^2, \quad \gamma = vuv. \]

Every free polynomial in \( x \) and \( y \) can be written as a free polynomial in \( \alpha, \beta, \gamma \) and \( \beta^{-1} \).

It’s also true when ‘polynomial’ is replaced by ‘rational function’.
Proof

For \( d \geq 1 \) let \( \text{Sym}_d \) be the space of symmetric homogeneous polynomials of degree \( d \). Then \( \dim \text{Sym}_d = 2^{d-1} \).

Let \( Q_d \subseteq \text{Sym}_d \) comprise the polynomials in

\[
\begin{align*}
u, v^2, vu^2, \ldots, vu^{d-2}v
\end{align*}
\]

that are homogeneous of degree \( d \) in \( u, v \), and hence also in \( x, y \).

Then \( Q_1 = \mathbb{C}u \) and \( Q_d \subseteq \text{Sym}_d \).

By induction \( \dim Q_d = 2^{d-1} = \dim \text{Sym}_d \), whence

\[
Q_d = \text{Sym}_d.
\]
Proof – conclusion

Observe that

\[ vu^2 v = vuv(v^2)^{-1}vuv = \gamma \beta^{-1} \gamma \]
\[ vu^3 v = vuv(v^2)^{-1}vuv(v^2)^{-1}vuv = \gamma \beta^{-1} \gamma \beta^{-1} \gamma \]

and so on.

Hence every symmetric homogeneous free polynomial in \(x, y\) is expressible as a polynomial in \(\alpha, \beta, \gamma\) and \(\beta^{-1}\).
Non-commutative analysis

We wish to prove an analogue of the Waring-Lagrange theorem for analytic functions of two non-commuting variables.

We use the framework of non-commutative (or nc-) analysis introduced by J. L. Taylor in the 1970s and intensively developed over the last 10 years by many analysts.
What is an analytic function of noncommuting variables?

The function

\[ f(z, w) = \exp(3zwz - izw) \]

looks like an analytic function of noncommuting variables \( z \) and \( w \). How should we interpret this statement?

J. L. Taylor, Functions of several non-commuting variables, *Bull. AMS* 79 (1973)

interpreted \( f \) as a map

\[ f : \bigcup_{n=1}^{\infty} \mathcal{M}_n^2 \to \bigcup_{n=1}^{\infty} \mathcal{M}_n \]

where \( \mathcal{M}_n \) denotes the algebra of \( n \times n \) matrices over \( \mathbb{C} \).
The nc universe

The nc analogue of $\mathbb{C}^d$ is

$$\mathcal{M}^d \overset{\text{def}}{=} \bigcup_{n=1}^{\infty} (\mathcal{M}_n)^d.$$ 

$\oplus$ defines a binary operation on $\mathcal{M}^d$: if $x \in \mathcal{M}_n$ and $y \in \mathcal{M}_m$
then $x \oplus y \overset{\text{def}}{=} \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \in \mathcal{M}_{n+m}$.

If $x = (x^1, \ldots, x^d)$ and $y = (y^1, \ldots, y^d)$ are in $\mathcal{M}^d$ then

$$x \oplus y \overset{\text{def}}{=} (x^1 \oplus y^1, \ldots, x^d \oplus y^d) \in \mathcal{M}^d.$$ 

Similarities: if $s \in GL_n(\mathbb{C})$ and $x \in \mathcal{M}_n^d$ then

$$s^{-1}xs \overset{\text{def}}{=} (s^{-1}x^1s, \ldots, s^{-1}x^ds) \in \mathcal{M}_n^d.$$
Properties of the function

\[ f(x^1, x^2) = \exp(3x^1x^2x^1 - ix^1x^1x^2) \]

The function \( f : \mathcal{M}^2 \rightarrow \mathcal{M}^1 \) has three important properties.

1. \( f \) is **graded**: if \( x \in \mathcal{M}^2_n \) then \( f(x) \in \mathcal{M}_n \).

2. \( f \) preserves **direct sums**: \( f(x \oplus y) = f(x) \oplus f(y) \) for all \( x, y \in \mathcal{M}^2 \).

3. \( f \) preserves **similarities**: if \( s \in GL_n(\mathbb{C}) \) and \( x \in \mathcal{M}^2_n \) then
   \[
   f(s^{-1}xs) = s^{-1}f(x)s.
   \]
nc functions

An *nc set* is a subset of $\mathcal{M}^d$ that is closed under $\oplus$.

An *nc function* is a function $f$ defined on an nc set $D \subset \mathcal{M}^d$ which is graded and preserves direct sums and similarities.

Thus, if $x \in D \cap \mathcal{M}^d_n$, $s \in GL_n(\mathbb{C})$ and $s^{-1}xs \in D$ then

$$f(s^{-1}xs) = s^{-1}f(x)s.$$ 

Every free polynomial (that is, polynomial over $\mathbb{C}$ in $d$ non-commuting indeterminates) defines an nc function on $\mathcal{M}^d$.

An nc function $f$ on $D$ is *analytic* if $D$ is open in the disjoint union topology on $\mathcal{M}^d$ and $f|D \cap \mathcal{M}^d_n$ is analytic for every $n$.

Try to extend classical function theory to nc functions.
The free topology on $\mathcal{M}^d$

For any $I \times J$ matrix $\delta = [\delta_{ij}]$ of free polynomials in $d$ non-commuting variables define

$$B_\delta = \{ x \in \mathcal{M}^d : \| \delta(x) \| < 1 \}.$$  

The **free topology** on $\mathcal{M}^d$ is the topology for which a base consists of the sets $B_\delta$.

The free topology is not Hausdorff. It does not distinguish between $x$ and $x \oplus x$.

$\mathcal{M}^d$ is connected in the free topology.
Free holomorphy

A function $f$ on a set $D \subset \mathcal{M}^d$ is freely holomorphic if

1. $D$ is a freely open set in $\mathcal{M}^d$

2. $f$ is a freely locally nc function $D \to \mathcal{M}^1$

3. $f$ is freely locally bounded on $D$.

Surprising theorem A freely holomorphic function is analytic.
**nc manifolds**

Let $X$ be a set. A *d-dimensional nc chart* on $X$ is a bijective map $\alpha$ from a subset $U_\alpha$ of $X$ to a set $D_\alpha \subset \mathcal{M}^d$.

For charts $\alpha, \beta$ the *transition map* $T_{\alpha\beta} : \alpha(U_\alpha \cap U_\beta) \to \beta(U_\alpha \cap U_\beta)$ is

$$T_{\alpha\beta} = \beta \circ \alpha^{-1}.$$

$\mathcal{A}$ is a *d-dimensional nc atlas* for $X$ if $\{U_\alpha : \alpha \in \mathcal{A}\}$ covers $X$ and, for all $\alpha, \beta \in \mathcal{A}$,

1. $\alpha(U_\alpha \cap U_\beta)$ is a union of nc sets and
2. the restriction of $T_{\alpha\beta}$ to any nc subset of $\alpha(U_\alpha \cap U_\beta)$ is an nc map.

$(X, \mathcal{A})$ is a *d-dimensional nc manifold* if $\mathcal{A}$ is a $d$-dimensional nc atlas for $X$. 
Free manifolds

Let \((X, A)\) be a \(d\)-dimensional nc manifold and let \(\mathcal{T}\) be a topology on \(X\).

\((X, \mathcal{T}, A)\) is a \textit{\(d\)-dimensional free manifold} if the range of every chart \(\alpha \in A\) is freely open in \(\mathcal{M}^d\) and the transition maps \(T_{\alpha\beta}\) are freely holomorphic for every \(\alpha, \beta \in A\).

A map \(f : X \to \mathcal{M}^1\) is a \textit{freely holomorphic function} on the free manifold \((X, \mathcal{T}, A)\) if \(f \circ \alpha^{-1}\) is a freely holomorphic function on \(D_\alpha\) for every \(\alpha \in A\).
Waring and Lagrange again

In the commutative case, let $\pi : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ be given by

$$\pi(z, w) = (z + w, zw).$$

If $f : \mathbb{C}^2 \rightarrow \mathbb{C}$ is a symmetric analytic function, then there exists an analytic function $F$ on $\mathbb{C}^2$ such that $f = F \circ \pi$. 
An analytic free Waring-Lagrange theorem

There exists a two-dimensional Zariski-free manifold $\mathcal{G}$ and a holomorphic map $\pi : \mathbb{M}^2 \rightarrow \mathcal{G}$ with the following property. There is a canonical bijection between the classes of

(i) freely holomorphic symmetric nc functions $f$ on $\mathbb{M}^2$, and

(ii) holomorphic functions $F$ defined on the manifold $\mathcal{G}$ that are conditionally nc and are such that, for every $w \in \mathbb{M}^2$, there is a free neighborhood $U$ of $w$ such that $F$ is bounded on $\pi(U) \cap \mathcal{G}$. 
Reference

Jim Agler, John E. McCarthy and N. J. Young,

Non-commutative manifolds, the free square root and symmetric functions in two non-commuting variables,


The end

www1.maths.leeds.ac.uk/~nicholas/slides/2019/NewtonGirard.pdf