Notes on de Rham cohomology

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0.1 Introduction

These notes are based upon the later half of the course "Differentiable Manifolds" given by Dr. D. Barden in Lent 2000, at the University of Cambridge. I have also based the preliminary discusion about differentiable manifolds on this course.

I plan to expand these notes in the future, but for the moment they cover a definite subset of what is needed for said lecture course, in the part 2B exam.

I would very much like to hear any comments (especially corrections), directed to `matt.daws@cantab.net`.

0.2 Disclaimer

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Chapter 1

Differentiable Manifold Basics

In this chapter we shall, very briefly, develop the theory of differentiable manifolds, and forms upon them.

1.1 Differentiable Manifolds

Definition 1.1.1. A *topological manifold* is a topological space M which is Hausdorff and second countable, together with an open covering $\mathcal{U}$ and a set of homeomorphism between members of $\mathcal{U}$ and open subsets of $\mathbb{R}^m$. Such homeomorphisms are called *charts*, and the collection of charts is called an *atlas*. It is quite possible to given M more than one *atlas*, these being considered different manifolds, even though they have the same underlying topological space.

Recall that a Hausdorff topological space M is one such that if $x, y \in M$ then $x \neq y$ implies we can find disjoint open sets U_x and U_y such that $x \in U_x$ and $y \in U_y$. This is a technical condition that is implies by the manifold structure. A second countable toplogical space M is one such that there is a countable base $\mathcal{B}$, the topology being constructable by taking arbitrary unions and finite intersections of members of $\mathcal{B}$ (essentially finding the smallest topology to contain $\mathcal{B}$).

Definition 1.1.2. A *differentiable manifold* is a toplogical manifold $(M, \mathcal{A})$ such that the *transformation maps* $\phi_2 \circ \phi_1^{-1} : \mathbb{R}^m \to \mathbb{R}^m$, where ϕ_i is a chart, are differentiable. In these notes, we assume infinitely differentiable, smooth, although it is seldom neccesary to require more than the 2nd derivative to exist and be continuous.

Note that $\theta_{21} := \phi_2 \circ \phi_1^{-1}$ is a homemorphism, and by swapping ϕ_2 and ϕ_1 we see that it is differentiable in both directions, hence is a diffeomorphism. It is not enough that θ_{21} is differentiable, we need θ_{12} to be as well.

Definition 1.1.3. An atlas $\mathcal{A}$ is *maximal* if any chart compatible will all charts in $\mathcal{A}$, is already in $\mathcal{A}$.

Proposition 1.1.4. *Every atlas is contained in a unique maximal atlas.*

Proof. Setup an equivalence relation on the space of all possible atlases (ϕ, U), $(\phi, U) \sim (\psi, V)$ iff they are compatible. Then clearly $(\phi, U) \sim (\phi, U)$ and $(\phi, U) \sim (\psi, V) \Leftrightarrow (\phi, V) \sim (\psi, U)$. Also, $(\phi_1, U_1) \sim (\phi_2, U_2)$ and $(\phi_2, U_2) \sim (\phi_3, U_3)$ implies θ_{21} and θ_{32} are smooth, so $\theta_{31} = \theta_{32} \circ \theta_{21}$ is smooth, so $(\phi_1, U_1) \sim (\phi_3, U_3)$. Hence this is an equivalence relation.

So, partition the space of all possible charts into equivalence classes. Then each class is an atlas (as all charts are compatible) and is maximal. As $\mathcal{A}$ is a chart, it must be contained in a class, this class being the unique maximal atlas containing $\mathcal{A}$. $\square$

We say that $\mathcal{M}$ gives M a *differential structure*; it is possible to have different differential structures on the same manifold. We will usually work by defining an atlas $\mathcal{A}$, and the implicitly moving to the maximal atlas which contains it. Working with a maximal atlas simplifies many problems: for example, given $a \in M$ we can always find a chart $\phi : U \to V$ with $a \in U$ such that $\phi(a) = 0$, this chart is said to be *about* a.

1.2 Submanifolds etc.

Definition 1.2.1. A subset $M \subset N^{m+p}$ is a *m-submanifold* or *codimension-p-submanifold* if for all $x \in M$, there is a chart $\phi : U \to \mathbb{R}^{m+p}$ for N about x such that $\phi(U \cap M) = \phi(U) \cap (\mathbb{R}^m \times \{0\})$.

Corollary 1.2.2. *An m-submanifold is an m-dimensional manifold.*

Proof. For x, U, ϕ as in definition, $\phi(U) \cap (\mathbb{R}^m \times \{0\})$ is open, so $U \cap M$ is open. Let $\pi_m : \mathbb{R}^{m+p} \to \mathbb{R}^m$ to be the projection onto the first m coordinates, and define $\tilde{U} = U \cap M$, $\tilde{\phi} = \pi_m \circ \phi|_{\tilde{U}}$. Then $\tilde{\phi}$ is a bijection from $\tilde{U}$ to $\tilde{\phi}(\tilde{U}) = \pi_m \circ \phi|_{\tilde{U}}(U \cap M) = \pi_m(\phi(U \cap M)) = \pi_m(\phi(U) \cap (\mathbb{R}^m \times \{0\})) = \pi_m(\phi(U))$, which is clear from this expansion. It is also clear that $\tilde{\phi}$ is continuous. $\tilde{\phi}^{-1} = \phi^{-1} \circ \pi_m^{-1}$, showing that $\tilde{\phi}^{-1}$ is also continuous.

In this way, we can construct charts for M, so M is a manifold. Then the transition maps are

$$\tilde{\theta}_{21} = \pi_m \circ \phi_2 \circ \phi_1^{-1} \circ \pi_m^{-1}$$

where $\pi_m^{-1}(x) = (x, 0)$. So $\tilde{\theta}_{21}$ is θ_{21} restricted to the first m coordinates, and so is smooth. Hence M is indeed an manifold. □

Definition 1.2.3. For a map $f : M^m \to N^n$ a point $y \in N$ such that f has rank n at all points $x \in f^{-1}(y)$ is called a *regular value* of f.

Corollary 1.2.4. *If y is a regular value of $f : M^m \to N^n$ then $f^{-1}(y)$ is an n-dimensional submanifold of M.*

Proof. □

1.3 Maps between manifolds

Definition 1.3.1. A continous map $f : M \to N$ between two manifolds is said to be *differentiable* if given any two charts (U, ϕ) and (V, ψ) in M and N respectively, we have

$$\psi \circ f \circ \phi^{-1} : \phi(U \cap f^{-1}(V)) \to \psi(V \cap f(U))$$

is differentiable.

Note that f continuous implies $\phi(U \cap f^{-1}(V))$ is an open subset of $\mathbb{R}^m$, so it is valid to talk about $\psi \circ f \circ \phi^{-1}$ being differentiable. We readily generalise this to diffeomorphisms between manifolds.

1.4 Tangents on manifolds

If a function $f : U \to V$, with $U, V \in \mathbb{R}^m$, is differentiable at $a \in \mathbb{R}^m$ then it has a derivative $Df(a)$ which is a linear map of $\mathbb{R}^m$ into itself. If f is differentiable in U, then we can define

$$Df : U \times \mathbb{R}^m \to U \times \mathbb{R}^m; (x, v) \to (x, Df(x)(v))$$

We want to generalise this to a manifold. We say γ is a curve passing through $a \in \mathbb{R}^m$ if γ is a smooth map $\mathbb{R} \supset U \to \mathbb{R}^m$ such that $\gamma(0) = a$. Then γ has a tangent at a, namely $\gamma'(0)$, and this tangent is a member of $\mathbb{R}^m$. This is precisely what we want a derivative to act upon, and we can generalise this to a manifold!

For a point a in a manifold M, with chart (ϕ, U), $a \in U$, we define an equivalence relation of smooth curves passing through a by $\gamma_1 \sim \gamma_2$ iff $(\phi \circ \gamma_1)'(0) = (\phi \circ \gamma_2)'(0)$. This is independent of the choice of chart, and smooth is interpretted in the sense that $\phi \circ \gamma_i$ is a smooth map $\mathbb{R} \to \mathbb{R}^m$. We write an equivalence class as $[\gamma]_a$, and refer to this as a tangent at a.

Then the derivative of a differentiable map $f : M \to N$ is f_* where $f_*(a)[\gamma]_a = [f \circ \gamma]_{f(a)}$. Note that $(\mathrm{id})_* = \mathrm{id}$ and if $g : N \to P$ is a further differentiable map, then $(g \circ f)_*(a) = g_*(f(a)) \circ f_*(a)$, the analogue of the chain rule. It is a simple check to see that this is well-defined, i.e. independent of choice of curve defining the tangent.

We refer to the set of tangents at a point p as $\tau_p(M)$, the *tangent space*. Given a chart (ϕ, U) at p, we have a bijection $\tau_p(M) \leftrightarrow \mathbb{R}^m$, given by $[\gamma]_p \mapsto (\phi \circ \gamma)'(0)$. This is clearly injective by the way we defined $\tau_p(M)$, and it has inverse $v \mapsto \{t \to \phi^{-1}(vt)\}$.

However, the space $\mathbb{R}^m$ has a natural vector space structure, and we can use the above bijection to transfer this to $\tau_p(M)$. So we define $\lambda[\gamma]_p$ to be the class defined by the curve $t \to \phi^{-1}(\lambda(\phi \circ \gamma)'(0)t)$, and $[\gamma]_p + [\delta]_p$ to be the class defined by $t \to \phi^{-1}(((\phi \circ \gamma)'(0) + (\phi \circ \delta)'(0))t)$. This is again independent of the choice of chart.

Proposition 1.4.1. *If $f : M \to N$ is a smooth map, then the derivative $f_*(p) : \tau_p(M) \to \tau_{f(p)}(N)$ is linear.*

Proof. For charts (ϕ, U) and (ψ, V) at p and $f(p)$ respectively we get

$$\begin{aligned} f_*(p)(\lambda[\gamma]_p + \mu[\delta]_p) &= f_*(p)[t \to \phi^{-1}((\lambda(\phi \circ \gamma)'(0) + \mu(\phi \circ \delta)'(0))t)]_p \\ &= [t \to f \circ \phi^{-1}(\lambda t(\phi \circ \gamma)'(0) + \mu t(\phi \circ \delta)'(0))]_{f(p)} \end{aligned}$$

while

$$\begin{aligned} &f_*(p)(\lambda[\gamma]_p) + f_*(p)(\mu[\delta]_p) \\ &\quad = f_*(p)[t \to \phi^{-1}(t\lambda(\phi \circ \gamma)'(0))]_p + f_*(p)[t \to \phi^{-1}(t\mu(\phi \circ \delta)'(0))]_p \\ &\quad = [t \to f \circ \phi^{-1}(t\lambda(\phi \circ \gamma)'(0))]_{f(p)} + [t \to f \circ \phi^{-1}(t\mu(\phi \circ \delta)'(0))]_{f(p)} \\ &\quad = [t \to \psi^{-1}(\lambda t D(\psi \circ f \circ \phi^{-1})(0)(\phi \circ \gamma)'(0) + \mu t D(\psi \circ f \circ \phi^{-1})(0)(\phi \circ \delta)'(0))]_{f(p)} \\ &\quad = [t \to \psi^{-1}(t D(\psi \circ f \circ \phi^{-1})(0)(\lambda(\phi \circ \gamma)'(0) + \mu(\phi \circ \delta)'(0)))]_{f(p)} \end{aligned}$$

So, setting $h = D(\psi \circ f \circ \phi)$ (a linear map) and $v = \lambda(\phi \circ \gamma)'(0) + \mu(\phi \circ \delta)'(0)$ (a vector) we need

$$[t \to f \circ \phi^{-1}(tv)]_{f(p)} = [t \to \psi^{-1}(th(v))]_{f(p)}$$

which is iff $h(v) = h(v)$, so done. □

On $\mathbb{R}^m$, each tangent space $\tau_p(\mathbb{R}^m)$ is simply $\mathbb{R}^m$ itself, the identification is refered to as *parallel translation*. For a smooth map $f : M \to \mathbb{R}$ we have

$$\tau_p(M) \to \tau_{f(p)}(\mathbb{R}) \cong \mathbb{R}$$

the composition being called $df(p)$, which lies in the dual space $L(\tau_p(M), \mathbb{R})$ denoted $\tau_p^*(M)$. Such elements are called *cotangent vectors* and $\tau_p^*(M)$ is called the *cotangent space*.

1.5 The tangent and cotangent bundles

For any open subset $U \subset M$ we write $\tau(U)$ for the disjoint union $\bigcup_{p \in U} \tau_p(M)$. We then introduce a topology and differential structure onto $\tau(U)$:

1. For $V \subset \mathbb{R}^m$ we have $\tau(V) \cong V \times \mathbb{R}^m$ via parallel translation. We give this a topology and differential structure by regarding it as an open subset of $\mathbb{R}^{2m}$.

2. For a chart (ϕ, U) on M we have the bijection
$$\Phi : \tau(U) \to \tau(\phi(U)); \quad \phi_*(x) : [\gamma]_x \mapsto [\phi \circ \gamma]_{\phi(x)}$$
which is linear on each tangent space $\tau_x(U)$. This bijection induces a topological and differential structure onto $\tau(U)$.

3. If we have charts (ϕ_1, U_1) and (ϕ_2, U_2) such that $U = U_1 \cap U_2$ then our charts on $\tau(U)$ differ by
$$\Phi_2 \circ \Phi_1^{-1} : V_1 \times \mathbb{R}^m \to V_2 \times \mathbb{R}^m$$
where $\Phi_1^{-1}(x, v) = [t \mapsto \phi_1^{-1}(x + vt)]_{\phi^{-1}(x)}$ and $\Phi_2([\gamma]_x) = [\phi_2 \circ \gamma]_{\phi_2(x)}$. So
$$\Phi_2 \circ \Phi_1^{-1}(x, v) = [t \mapsto \theta_{21}(x + vt)]_{\theta_{21}(x)} = (\theta_{21}(x), D\theta_{21}(x)v)$$
where $\theta_{21} = \phi_2 \circ \phi_1^{-1}$. So as θ_{21} is smooth, so is $D\theta_{21}$, so the charts Φ_i are compatible, and the topologies induces the same.

   A formal check shows second countable (cover M by a countable number of opens, push down to coordinate spaces, then the tangent bundles here are second countable, giving a countable union of second countable sets so $\tau(M)$ is second countable) and Hausdorff (again, push down to coordinate spaces).

We note that the projection $\pi : \tau(M) \to M; v_p \to p$ is a differentiable submersion.

We call $\tau(M)$ the *tangent bundle* of M. This forms the natural domain for the derivative of a differentiable map, for if $f : M \to N$ then
$$f_* : \tau(M) \to \tau(N); \quad [\gamma]_x \mapsto [f \circ \gamma]_{f(x)}$$
is also differentiable.

Proof. For charts (ϕ, U) and (ψ, V) in M and N we need
$$\Psi \circ f_* \circ \Phi^{-1} : \phi(U) \times \mathbb{R}^m \to \psi(V)\mathbb{R}^n; \quad (x, v) \mapsto (\psi \circ f \circ \psi^{-1}(x), D(\psi \circ f \circ \phi^{-1})(x)v)$$
to be smooth, while is implied by f being smooth. □

For a chart (ϕ, U) we have a dual isomorphism
$$(\phi^{-1})^* : \tau^*(U) \to \tau^*(\phi(U))$$
which allows us to put a topological and differential structure on $\tau^*(U)$ exactly as above. We call the resulting manifold $\tau^*(M)$ the *cotangent bundle.*

Definition 1.5.1. A smooth map $X : M \to \tau(M)$ such that $\pi \circ X = \mathrm{id}_M$ is called a *vector field on M*. A map $\omega : M \to \tau^*(M)$ such that $\pi \circ w = \mathrm{id}_M$ is called a *cotangent field* or *1-form on M*.

Chapter 2

Submanifolds

2.1 The Inverse Function Theorem

Theorem 2.1.1. ***Inverse Function Theorem***
Let $U \subset \mathbb{R}^m$ be open, and $f : U \to \mathbb{R}^n$ be differentiable such that at $x \in U$, $Df(x)$ is a linear isomorphism. Then there is a neighbourhood V of x such that f is a diffeomorphism from V to $f(V)$ (and $n = m$).

Proof. Without loss of generatily, $x = 0$, $f(0) = 0$ and $Df(0) = \mathrm{id}$, applying translation and linear diffeomorphisms to get here from the general case. Then apply contractive mapping theorem. □

Corollary 2.1.2. ***Submersion theorem***
With notation $\iota_2 : \mathbb{R}^p \to \mathbb{R}^m \times \mathbb{R}^q; y \mapsto (0, y)$ and $\pi_2 : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^p; (x, y) \mapsto y$. Then if $0 \in U \subset \mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^p$ and if $f : U \to \mathbb{R}^q$ is smooth with the second partial derivative $D_2 f(0) = Df(0) \circ \iota_2 : \mathbb{R}^p \to \mathbb{R}^q$ a linear isomorphism, then ($p = q$ and) there is a local diffeomorphism ϕ at 0 in $\mathbb{R}^n$ such that $f \circ \phi = \pi_2$.

This means that if f is mapping from a higher dimensional space, but is in some sense an isomorphism (i.e. the 2nd partial derivative condition), then we can compose f with a diffeomorpism so that f now maps as projection onto the last p coordinates.

Proof. Let $\psi(x, y) = (x, f(x, y))$, then $D\psi(0, 0)$ has jacobian matrix

$$J\psi(0,0) = \begin{pmatrix} I_m & 0 \\ J_1 f(0,0) & J_2 f(0,0) \end{pmatrix}$$

where $J_i f(0, 0)$ is the jacobian matrix of $D_i f(0, 0)$. Then $J\psi(0, 0)$ is an isomorphism, and so the inverse function theorem gives that ψ is a local diffeomorphism as $(0, 0)$. If ψ is the local inverse from $W \subset \mathbb{R}^m \times \mathbb{R}^p \to V \subset U$, then every point $w \in W$ is $w = \psi(x, y) = (x, f(x, y))$ for some $(x, y) \in V$. Then $(f \circ \phi)(w) = (f \circ (\phi \circ \psi))(x, y) = f(x, y) = \pi_2(w)$. □

Corollary 2.1.3. ***Implicit function theorem***
Data as in the statement and proof of the above, if $(0, 0) \in W_1 \times W_2 \subset W$ and $f(0, 0) = 0$ then there is a unique function $g : W_1 \to \mathbb{R}^p$ such that $f(x, g(x)) = 0$ for all $x \in W_1$.

Proof. Uniqueness follows from $\psi(x, g(x)) = (x, f(x, g(x))) = (x, 0)$ and the fact that ψ bijects locally.

For existence we define $g(x) = \pi_2 \circ \phi(x, 0)$ and then check that

$$f(x, g(x)) = f(x, \pi_2 \circ \phi(x, 0)) = f \circ \phi(x, 0) = \pi_2(x, 0) = 0$$

where ϕ preserves the first coordinate, as it's inverse ψ does. □

Corollary 2.1.4. *The map $\alpha : W_1 \to f^{-1}(0); x \to (x, g(x))$ is a local homeomorphism onto its image, an open neighbourhood of $(0,0)$ in $f^{-1}(0)$. Moreover, any two such are compatible charts of $f^{-1}(0)$.*

Proof. We have $\alpha(x) = (x, \pi_2 \circ \phi(x,0)) = \phi(x,0) = \phi \circ \iota_1(x)$. So $f \circ \alpha(x) = 0$, so $\alpha(x) \in f^{-1}(0)$. Also,

$$\begin{aligned}\alpha(W_1) &= \phi(W_1 \times \{0\}) = \psi^{-1}(W_1 \times \{0\}) \\ &= \psi^{-1}(W_1 \times \mathbb{R}^p) \cap \psi^{-1}(\mathbb{R}^m \times \{0\}) = \psi^{-1}(W_1 \times \mathbb{R}^p) \cap f^{-1}(0)\end{aligned}$$

So $\alpha(W_1)$ is open in $f^{-1}(0)$. Then

$$\pi_1 \circ \psi \circ \alpha = \pi_1 \circ \psi \circ \phi \circ \iota_1 = \pi_1 \circ \iota_1 = \mathrm{id}$$

Thus α is injective and so bijects onto its image $\alpha(W_1)$, and both $\alpha = \psi \circ \iota_1$ and it inverse $\pi_1 \circ \psi$ are continuous.

For any two such charts, $\alpha_2^{-1} \circ \alpha_1 = \pi_1 \circ \psi_2 \circ \phi_1 \circ \iota_1$ is a composite of smooth maps, and so smooth. Thus any two charts arecompatible. □

So we have shown that if $f : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^p$ is such that f has maximal rank p at each point of $f^{-1}(0)$, then $f^{-1}(0)$ is a smooth manifold.

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