The unifying notion of Hopf monad

Gabriella Böhm

Wigner Research Centre for Physics, Budapest

Modelling Topological Phases of Matter

Leeds, 5-8 July 2016

Motivation

- groupoids
- usual Hopf algebras
- Hopf algebras in braided monoidal categories [say, Majid]
- weak Hopf algebras [GB-Nill-Szlachányi]
- x_R-Hopf algebras [Schauenburg]
  (so in particular Hopf algebroids [GB-Szlachányi])
- Hopf monads on monoidal categories [Bruguiéres-Lack-Virelizier]
- Hopf group algebras [Turaev]
- Hopf categories [Batista-Caenepeel-Vercruysse]
- Hopf polyads [Bruguiéres]

share certain properties

¿ are they instances of some common unifying structure ?

yes

they are all Hopf monads in suitable monoidal bicategories

Content

A lecture on Hopf monads in monoidal bicategories

—

illustrated by the above list of examples.

Category

A **(locally small) category** $\mathbf{C}$ consists of

- a class $\{X, Y, \dots\}$ of objects
- for each pair (X, Y) a **set** $\mathbf{C}(X, Y)$ of morphisms $X \to Y$
- for each triple (X, Y, Z) a composition **map** $\mathbf{C}(Y, Z) \times \mathbf{C}(X, Y) \to \mathbf{C}(X, Z)$
- for each X a distinguished unit **element** of $\mathbf{C}(X, X)$

such that the composition is **associative** and **unital**.

2-Category

A **2-category** $\mathbf{C}$ consists of

- a class $\{X, Y, \dots\}$ of objects
- for each pair (X, Y) a **category** $\mathbf{C}(X, Y)$ of morphisms $X \to Y$
- for each (X, Y, Z) a composition **functor** $\mathbf{C}(Y, Z) \times \mathbf{C}(X, Y) \to \mathbf{C}(X, Z)$
- for each X a distinguished unit **object** of $\mathbf{C}(X, X)$

such that the composition is **associative** and **unital**.

Examples

A **strict monoidal category** $(\mathcal{M}, \otimes, I)$ can be regarded as a 2-category $\mathbf{M}$

- of a single object $*$
- $\mathbf{M}(*, *) = \mathcal{M}$
- composition $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$
- unit object I of $\mathcal{M}$.

There is a 2-category **Cat** which has

- categories as objects
- the category of functors $\mathcal{C} \to \mathcal{D}$ and their natural transformations as $\mathbf{Cat}(\mathcal{C}, \mathcal{D})$
- composition provided by the composition of functors
- the identity functors as the unit objects.

2-Category

A **2-category** $\mathbf{C}$ consists of

- a class $\{X, Y, \dots\}$ of objects
- for each pair (X, Y) a category $\mathbf{C}(X, Y)$ of morphisms $X \to Y$
- for each (X, Y, Z) a composition functor $\mathbf{C}(Y, Z) \times \mathbf{C}(X, Y) \to \mathbf{C}(X, Z)$
- for each X a distinguished unit object of $\mathbf{C}(X, X)$

such that the composition is **associative** and **unital**.

¡ There is room to require associativity and unitality only **up-to** some coherent natural isomorphisms !

Bicategory

A **bicategory** $\mathbf{C}$ consists of

- a class $\{X, Y, \dots\}$ of objects
- for each pair (X, Y) a category $\mathbf{C}(X, Y)$ of morphisms $X \to Y$
- for each (X, Y, Z) a composition functor $\mathbf{C}(Y, Z) \times \mathbf{C}(X, Y) \to \mathbf{C}(X, Z)$
- for each X a distinguished unit object of $\mathbf{C}(X, X)$

such that the composition is associative and unital **up-to** some coherent natural isomorphisms (not to be explicitly denoted in this talk).

Notation & terminology

In a **bicategory** **C** the

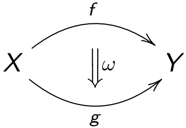
- objects are also called **0-cells**
- the objects of the hom categories are also called **1-cells**
- the morphisms of the hom categories are also called **2-cells**.

We denote

- the composition in the hom categories (=**vertical composition**) by $\bullet$ and their unit morphism by $\mathbb{1}$
- the composition in the bicategory (=**horizontal composition**) by $\circ$ and its unit by 1.

The coherence isomorphisms are not explicitly denoted.

Explanation of the terminology: a 2-cell is depicted as



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Examples

Any monoidal category can be regarded as a bicategory; just as before.

There is a bicategory **Bim** which has

- algebras (over a given field or ring) as objects
- the category of S-R bimodules as $\mathbf{Bim}(R, S)$
- the relative tensor product $\otimes_S : \mathbf{Bim}(S, T) \times \mathbf{Bim}(R, S) \to \mathbf{Bim}(R, T)$ as $\circ$
- the regular bimodule R as the unit 1-cell at R.

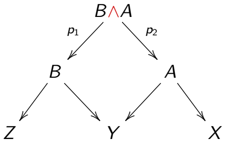
There is a bicategory **Prof** which has

- categories as objects
- the category of functors $\mathcal{D}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{set}$ as $\mathbf{Prof}(\mathcal{C}, \mathcal{D})$
- the coend $\int^{d \in \mathcal{D}} g(-, d) \times f(d, -)$ as $g \circ f$
- the hom functor $\mathcal{C}(-, -)$ as the unit 1-cell at $\mathcal{C}$.

There is a homomorphism $\mathbf{Cat} \to \mathbf{Prof}$, sending $\mathcal{C} \xrightarrow{f} \mathcal{D}$ to $\mathcal{D}(-, f(-))$.

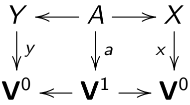
There is a bicategory **Span** which has

- sets as objects
- the category **Span**(X, Y) of spans $Y \leftarrow A \rightarrow X$
- $\circ$-composition provided by the pullback $\wedge$
- the trivial span $X = X = X$ as the unit 1-cell at X.



For any bicategory $\mathbf{V}$ there is an associated bicategory **Span**|**V** [GB] which has

- pairs of a set X and a map $X \xrightarrow{x} \mathbf{V}^0$ (to the set of 0-cells) as objects
- the categories **Span**|$\mathbf{V}((X, x), (Y, y))$ whose **objects** are pairs of a span $Y \leftarrow A \rightarrow X$ and a map $A \xrightarrow{a} \mathbf{V}^1$ (to the set of 1-cells) such that this commutes:

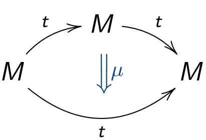


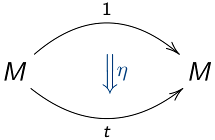
  morphisms are pairs of a map of spans $A \xrightarrow{f} A'$ and a natural transformation $a \overset{\varphi}{\Rightarrow} a' \circ f$ (where A and A' are regarded as discrete categories)
- horizontal composition $(B \wedge A, (b, a) \mapsto b(-) \circ a(-))$
- the trivial span $X = X = X$ and the map $X \xrightarrow{x} \mathbf{V}^0 \xrightarrow{1_{(-)}} \mathbf{V}^1$ as the unit 1-cell at (X, x).

If V is the trivial bicategory with one object then Span|V reduces to Span.

Monad

A **monad** in a bicategory consists of

- a 0-cell M
- a 1-cell $M \xrightarrow{t} M$
- 2-cells 

and 

such that μ is associative with unit η.

Examples

In a **monoidal category**, regarded as a bicategory, a monad is a monoid.

In **Cat** the usual notion of monad is re-obtained.

In **Bim** a monad ($R \xrightarrow{t} R$, μ, η) is equivalent to an algebra map $\eta : R \to t$.

In **Span** a monad is precisely a small category.

In **Span**|**Cat** a monad is precisely a polyad [Bruguiéres].

Graded monoids in a braided monoidal category $\mathcal{V}$ are monads in **Span**|$\mathcal{V}$ (on objects involving the singleton set $\{*\}$).

Categories enriched in a braided monoidal category $\mathcal{V}$ are monads in **Span**|$\mathcal{V}$ (with 1-cell parts involving a product span $X \leftarrow X \times X \rightarrow X$).

Monoidal bicategory

A **monoidal bicategory** $\mathbf{C}$ is a bicategory equipped with

- a distinguished monoidal unit object I
- a monoidal product operation $\mathbf{C} \times \mathbf{C} \to \mathbf{C}$ which is
  - compatible with $\bullet$ and $\mathbb{1}$
  - compatible with $\circ$ and 1
  - associative with unit I

  up-to coherent invertible 2-cells.

We denote the monoidal product by juxtaposition.

Examples

A **braided monoidal category** $(\mathcal{M}, \otimes, I)$ regarded as a bicategory is monoidal via the same $\otimes$ (braiding needed for compatibility with $\circ = \otimes$).

The 2-category **Cat** is monoidal via the Cartesian product $\times$.

The bicategory **Prof** is monoidal via the Cartesian product $\times$.

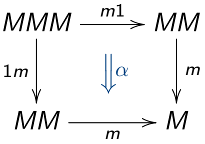
The bicategory **Bim** is monoidal via $\otimes$ over the base field or ring.

If **V** is a monoidal bicategory so is **Span**|**V** via the Cartesian product $\times$ and the monoidal product of **V**.
In particular, **Span**|**Cat**, **Span**|$\mathcal{V}$ for any braided monoidal category $\mathcal{V}$, so **Span** are monoidal bicategories.

Monoidale

A **monoidale** (or **pseudo-monoid**) in a monoidal bicategory consists of

- a 0-cell M
- 1-cells $MM \xrightarrow{m} M \xleftarrow{u} I$
- invertible 2-cells



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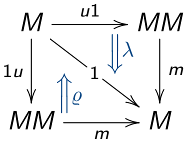


subject to MacLane's pentagon and triangle coherence conditions.

Examples

In a **braided monoidal category**, regarded as a monoidal bicategory, there is only one 0-cell; it is the monoidal unit hence trivially a monoidale.

In **Cat** a monoidale is precisely a monoidal category.

In **Bim** any commutative algebra R is a monoidale via the regular actions.

For any algebra R, there is a monoidale in **Bim** with

- 0-cell $R^{\mathrm{op}} \otimes R$
- multiplication 1-cell; i.e. $(R^{\mathrm{op}} \otimes R)$-$(R^{\mathrm{op}} \otimes R \otimes R^{\mathrm{op}} \otimes R)$ bimodule $R \otimes R \otimes R$
- unit 1-cell; i.e. left $R^{\mathrm{op}} \otimes R$ bimodule R

(where all actions are given by multiplication on the appropriate side).

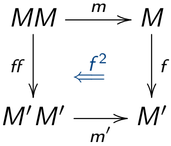
For any set of monoidales $\{(\mathrm{M}_p, m_p, u_p) | p \in X\}$ in a monoidal bicategory $\mathbf{V}$ there is an **induced monoidale** in $\mathbf{Span}|\mathbf{V}$ with

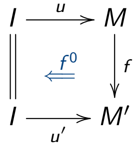
- 0-cell $(X, X \ni p \mapsto \mathrm{M}_p)$
- multiplication 1-cell $(X = X \xrightarrow{p \mapsto (p,p)} X \times X , X \ni p \mapsto m_p)$
- unit 1-cell $(X = X \xrightarrow{!} \{*\} , X \ni p \mapsto u_p)$.

It works in particular for $\mathbf{Span}|\mathbf{Cat}$, $\mathbf{Span}|\mathcal{V}$ for braided monoidal categories $\mathcal{V}$, hence for $\mathbf{Span}$.
In $\mathbf{Span}|\mathcal{V}$ (so in $\mathbf{Span}$) it makes any 0-cell an induced monoidale.

Opmonoidality

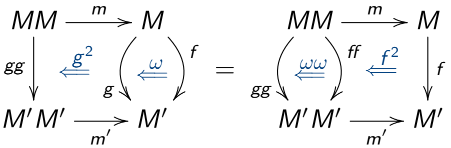
Between monoidales (M, m, u), (M', m', u'), an **opmonoidal** 1-cell consists of

- a 1-cell $M \xrightarrow{f} M'$
- 2-cells 

and 

such that f^2 is **coassociative** with **counit** f^0.

A 2-cell $f \xRightarrow{\omega} g$ between opmonoidal 1-cells is **opmonoidal** if



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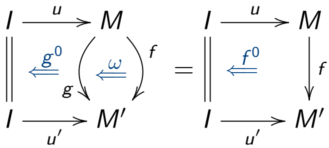


[Chikhladze-Lack-Street]: $\exists$ bicategory of monoidales, opmonoidal 1- and 2-cells.
An **opmonoidal monad** is a monad therein.

Examples

Bimonoids in a **braided monoidal category** are the opmonoidal monads in the corresponding monoidal bicategory.

Usual opmonoidal monads (or **bimonads**) are the opmonoidal monads in **Cat**.

Bialgebroids [Takeuchi] (so in particular **weak bialgebras** [GB-Nill-Szlachányi]) are opmonoidal monads in **Bim** on monoidales of the form $R^{\mathrm{op}} \otimes R$ seen before.

Bialgebroids over a commutative algebra R; with source and target maps landing in the center, are opmonoidal monads in **Bim** also on the monoidale R.

Opmonoidal polyads [Bruguiéres] are opmonoidal monads in **Span**|**Cat** on induced monoidales.

For any monoid G **semi-Hopf** G**-monoids** [Turaev] in a braided monoidal category $\mathcal{V}$ are opmonoidal monads in **Span**|$\mathcal{V}$ on $\{*\}$ (a trivial monoidale).

Categories enriched in the category of comonoids in a braided monoidal category $\mathcal{V}$ are are opmonoidal monads in **Span**|$\mathcal{V}$ on induced monoidales with 1-cell parts involving a product span.
In particular **categories** are opmonoidal monads in **Span** on induced monoidales.

Hopf monad [Chikhladze-Lack-Street]

A **Hopf monad** is an opmonoidal monad $M \xrightarrow{f} M$ s.t.

$$\begin{array}{ccccc} MM & \xrightarrow{f1} & MM & \xrightarrow{m} & M \\ \Vert & \overset{\mu 1}{\Leftarrow} & \downarrow{\scriptstyle ff} & \overset{f^2}{\Leftarrow} & \downarrow{\scriptstyle f} \\ MM & \xrightarrow[ff]{} & MM & \xrightarrow[m]{} & M \end{array}$$

is invertible.

¡ no antipode at this level of generality !

Examples

A bimonoid in a braided monoidal category is a **Hopf monoid**. . .
In particular a bialgebra is a **Hopf algebra**. . .

A bimonad is a **Hopf monad** [Bruguiéres-Lack-Virelizier]. . .

A bialgebroid is a $\times_R$-**Hopf algebra** [Schauenburg]. . .
In particular a weak bialgebra is a **weak Hopf algebra** [GB-Nill-Szlachányi]. . .

A bialgebroid over a commutative base algebra with source and target maps landing in the center is a **Hopf algeboid** [Ravenel]. . .

An opmonoidal polyad is a **Hopf polyad** [Bruguiéres]. . .

A semi-Hopf group monoid is a **Hopf group monoid** [Turaev]. . .

A category enriched in the category of comonoids in a braided monoidal category is a **Hopf category** [Batista-Caenepeel-Vercruysse]. . .
In particular a category is a **groupoid**. . .

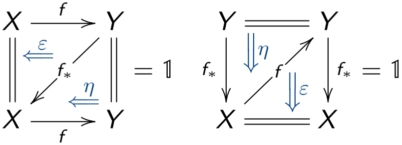
. . . iff as an opmonoidal monad it is Hopf.

(Op)map monoidale

A **map** in a bicategory is a 1-cell $X \xrightarrow{f} Y$ possessing a right adjoint.

An **opmap** is a map in the $\bullet$-opposite bicategory = a 1-cell f with left adjoint f_*:

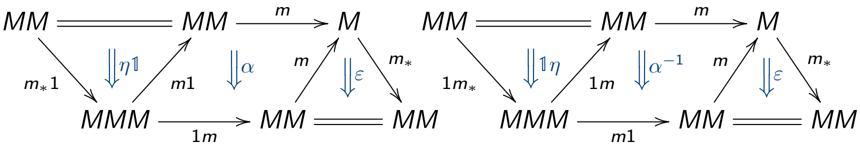
$\exists$ 2-cells $1 \overset{\eta}{\Rightarrow} f \circ f_*$ and $f_* \circ f \overset{\varepsilon}{\Rightarrow} 1$ s.t.



An **opmap monoidale** is a monoidale whose multiplication and unit are opmaps.

Naturally Frobenius opmap monoidale

An opmap monoidale (M, m, u) is **naturally Frobenius** if these are invertible:



Examples

Regarding a **braided monoidal category** as a monoidal bicategory, the only, trivial monoidale is a naturally Frobenius opmap monoidale.

A monoidal category $\mathcal{M}$ is a map monoidale in **Prof**, hence an opmap monoidale in its opposite. It is naturally Frobenius iff $\mathcal{M}$ has left and right duals.

A commutative algebra R, seen as a monoidale in **Bim** as before, is a naturally Frobenius opmap monoidale.

The monoidale $R^{\mathsf{op}} \otimes R$ in **Bim** seen before is a naturally Frobenius opmap monoidale whenever R is a separable Frobenius algebra (i.e. its multiplication has a bimodule section which is in addition a counital comultiplication).

A monoidale in **Span**|**V**, induced by some set of monoidales in a monoidal bicategory **V**, is a naturally Frobenius opmap monoidale whenever each member of the set is so.
In particular, any set seen as an induced monoidale in **Span**|$\mathcal{V}$, for a braided monoidal category $\mathcal{V}$, so in particular in **Span**, is a naturally Frobenius opmap monoidale.

Hopf monads on naturally Frobenius opmap monoidales

[GB-Lack]

For an opmonoidal monad f on a naturally Frobenius map monoidale (M, m, u), TFAE.

- It is a Hopf monad.
- It has an **antipode** — which is a 2-cell from f to

  $$M \xrightarrow{u1} MM \xrightarrow{m_*1} MMM \xrightarrow{1f1} MMM \xrightarrow{1m} MM \xrightarrow{1u_*} M$$

  and the convolution inverse of $\mathbb{1} : f \to f$ in a suitable sense.

Examples

Any **bimonoid** in a **braided monoidal category** is an opmonoidal monad on the naturally Frobenius opmap monoidale in the corresponding monoidal bicategory.

A **bimonad** on a monoidal category with left and right duals is an opmonoidal monad on a naturally Frobenius opmap monoidale in $\mathbf{Prof}^{\mathrm{op}}$.

A **bialgebroid** over a commutative algebra with source and target maps landing in the center is an opmonoidal monad on a naturally Frobenius opmap monoidale in **Bim**.

weak bialgebra = bialgebroid + separable Frobenius str on the base algebra ⇒ Any **weak bialgebra** can be seen as an opmonoidal monad on a naturally Frobenius opmap monoidale in **Bim**.

For any monoid G any **semi-Hopf G-monoid** in a braided monoidal category $\mathcal{V}$ is an opmonoidal monad on a naturally Frobenius opmap monoidale in **Span**$|\mathcal{V}$.

Any **category enriched in the category of comonoids in a braided monoidal category** $\mathcal{V}$ is an opmonoidal monad on a naturally Frobenius opmap monoidale in **Span**$|\mathcal{V}$.
In particular, any **category** is an opmonoidal monad on a naturally Frobenius opmap monoidale in **Span**.

This explains why

- Hopf monoids in braided monoidal categories (thus Hopf algebras)
- Hopf monads on monoidal categories with left and right duals
- Hopf algebroids over commutative algebras
- weak Hopf algebras (thus Hopf algebras)
- groupoids
- Hopf group monoids
- Hopf categories

have **antipodes**.

Thank you!