Menger’s theorem in $\Pi^1_1$–CA

Paul Shafer

Abstract. We prove Menger’s theorem for countable graphs in $\Pi^1_1$–CA. Our proof in fact proves a stronger statement, which we call extended Menger’s theorem, that is equivalent to $\Pi^1_1$–CA over RCA.

1. Introduction

König’s duality theorem for finite bipartite graphs is a classic theorem in graph theory and one of the pillars of matching theory. It expresses a duality between matchings and covers in bipartite graphs. Let $(X, Y, E)$ be a bipartite graph. A matching is a set of edges $M \subseteq E$ such that no two edges in $M$ share a vertex. A cover is a set of vertices $C \subseteq X \cup Y$ such that every edge in $E$ has a vertex in $C$. Finite König’s duality theorem says that the cardinalities of matchings and the cardinalities of covers meet in the middle.

Finite König’s Duality Theorem. In every finite bipartite graph, the maximum cardinality of a matching equals the minimum cardinality of a cover.

Finite Menger’s theorem generalizes finite König’s duality theorem from bipartite graphs to arbitrary graphs. Let $G$ be a graph with vertices $V(G)$ and edges $E(G)$. A web is a triple $(G, A, B)$ where $G$ is a graph and $A$ and $B$ are distinguished sets of vertices $A, B \subseteq V(G)$. The notion of a matching in a bipartite graph is generalized by the notion of a set of disjoint $A$–$B$ paths in a web. An $A$–$B$ path in a web $(G, A, B)$ is a path that starts in $A$ and ends in $B$. Two paths are disjoint if they have no vertices in common. The notion of a cover in a bipartite graph is generalized by the notion of an $A$–$B$ separator in a web. An $A$–$B$ separator in a web $(G, A, B)$ is a set of vertices $C \subseteq V(G)$ such that every $A$–$B$ path in $G$ contains a vertex of $C$ (so that removing $C$ from the graph separates $A$ from $B$).

Finite Menger’s Theorem. In every finite web $(G, A, B)$, the maximum cardinality of a set of disjoint $A$–$B$ paths equals the minimum cardinality of an $A$–$B$ separator.


The conclusions of finite König’s duality theorem and finite Menger’s theorem remain true for infinite bipartite graphs and infinite webs, but they are more an exercise in cardinal arithmetic than they are in combinatorics. To deepen the combinatorial content of these theorems, Erdős conjectured that there always exist a matching and a cover that simultaneously witness each other’s optimality. His reformulations are what we now call König’s duality theorem and Menger’s theorem.

König’s Duality Theorem. In every bipartite graph $(X, Y, E)$, there is a matching $M$ and a cover $C$ such that $C$ consists of exactly one vertex from each edge in $M$.

Menger’s Theorem. In every web $(G, A, B)$, there is a set of disjoint $A$–$B$ paths $M$ and an $A$–$B$ separator $C$ such that $C$ consists of exactly one vertex from each path in $M$.

1 This research was partially supported by NSF grants DMS-0554855 and DMS-0852811.

1 For us, “path” always means “simple path,” that is, no repeated vertices.
The most general case, Menger’s theorem for webs of arbitrary cardinality, is now known to be true. The proof took more than forty years to discover. The first progress was by Podewski and Steffens, who proved König’s duality theorem for countable bipartite graphs [7]. Aharoni next proved König’s duality theorem for arbitrary bipartite graphs [1]. He then proved Menger’s theorem for countable webs [2]. Finally, Aharoni and Berger proved Menger’s theorem for arbitrary webs [3].

The question motivating our work is the following.

**Question 1.1.** What is the axiomatic strength of Menger’s theorem for countable webs in the context of second-order arithmetic?

Aharoni, Magidor, and Shore [4] and Simpson [8] answered Question 1.1 for König’s duality theorem for countable bipartite graphs. Aharoni, Magidor, and Shore noticed that Aharoni’s proof of König’s duality theorem in [1] actually proves a stronger statement, which they call extended König’s duality theorem. They proved that extended König’s duality theorem is equivalent to $\Pi^1_1$-$\text{CA}_0$ over $\text{RCA}_0$, and they proved that König’s duality theorem implies $\text{ATR}_0$ over $\text{RCA}_0$ [4]. Simpson produced a new proof of König’s duality theorem in $\text{ATR}_0$ by exploiting the fact that $\text{ATR}_0$ proves the existence of models of $\Sigma^1_1$-$\text{AC}_0$ [8]. Therefore König’s duality theorem for countable bipartite graphs is equivalent to $\text{ATR}_0$ over $\text{RCA}_0$.

A priori, Menger’s theorem for countable webs implies $\text{ATR}_0$ over $\text{RCA}_0$ because it implies König’s duality theorem for countable bipartite graphs over $\text{RCA}_0$. Here we provide a proof Menger’s theorem for countable webs in $\Pi^1_1$-$\text{CA}_0$. The general plan for our proof is inspired by Aharoni’s proof in [2] and Diestel’s presentation of it in [5] Section 8.4. As with König’s duality theorem, we notice that this proof in fact proves a stronger statement, which we call extended Menger’s theorem, that is equivalent to $\Pi^1_1$-$\text{CA}_0$ over $\text{RCA}_0$. By general considerations, Menger’s theorem cannot imply $\Pi^1_1$-$\text{CA}_0$ over $\text{RCA}_0$. Menger’s theorem can be written as a $\Pi^1_2$ sentence in the language of second-order arithmetic, and no true $\Pi^1_2$ sentence implies $\Pi^1_1$-$\text{CA}_0$, even over $\text{ATR}_0$ (see [4] Proposition 4.17). Question 1.1 now becomes more specific.

**Question 1.2.** Is Menger’s theorem for countable webs provable in $\text{ATR}_0$?

This paper is organized as follows. Section 2 explains the background graph-theoretic primitives and subsystems of second-order arithmetic needed for this work. Section 3 develops in $\text{ACA}_0$ the tools needed to prove Menger’s theorem in $\Pi^1_1$-$\text{CA}_0$. Section 4 gives a proof of Menger’s theorem for countable webs in $\Pi^1_1$-$\text{CA}_0$. Section 5 introduces extended Menger’s theorem and proves that it is equivalent to $\Pi^1_1$-$\text{CA}_0$ over $\text{RCA}_0$. 

### 2. Background

#### 2.1. Graph theory basics and conventions.

All the graphs that we consider are countable because we are working in second-order arithmetic. All the graphs that we consider are directed. Menger’s theorem for undirected graphs follows from Menger’s theorem for directed graphs by the usual trick of replacing an undirected edge by two directed edges. Henceforth a “graph” is a countable directed graph.

As defined in the introduction, a web is a triple $(G, A, B)$ where $G$ is a graph and $A$ and $B$ are distinguished sets of vertices $A, B \subseteq V(G)$. We often abuse this notation by writing $G$ for $(G, A, B)$. For convenience, we always assume that there are no edges directed into $A$, that there are no edges directed out of $B$, and that $A \cap B = \emptyset$.

If $H$ and $H'$ are subgraphs of a graph $G$, then $H \cup H'$ is the subgraph of $G$ induced by $V(H) \cup V(H')$, and $G - H$ is the subgraph induced by $V(G) - V(H)$.

Let $G$ be a graph. If $P$ is a path in $G$, we write $\text{in}(P)$ for the first vertex of $P$ (if it exists) and $\text{ter}(P)$ for the last vertex of $P$ (if it exists). If $P$ is a path with $\text{in}(P) \in A$ and $\text{ter}(P) \in B$ for some $A, B \subseteq V(G)$, then we call $P$ an $A$-$B$ path. If $P$ is a path and $x \in V(P)$, then $Px$ denotes the subpath of $P$ consisting of all the vertices up to and including $x$, and $\overline{Px}$ denotes the subpath...
of $P$ consisting of all the vertices up to and not including $x$. Similarly, $xP$ denotes the subpath of $P$ consisting of all the vertices following $x$ and including $x$, and $xP$ denotes the subpath of $P$ consisting of all the vertices following $x$ and not including $x$. If $P$ and $Q$ are paths with $V(P) \cap V(Q) = \{x\}$, then $PxQ$ is the path obtained by concatenating the paths $Px$ and $xQ$. If $V(P) \cap V(Q) = \{\text{ter}(P)\} = \{\text{in}(Q)\}$, then $PQ$ denotes $P\text{ter}(P)Q$, the concatenation of the paths $P$ and $Q$.

For our purposes, a tree is a directed acyclic graph $T$ that has a distinguished root $r \in V(T)$ such that for any $x \in V(T)$ there is a unique path in $T$ from $r$ to $x$. The path in a tree $T$ from its root to an $x \in V(T)$ is denoted $Tx$. If $P$ is a finite path, a tree with trunk $P$ is a tree $T$ of the form $P \cup T'$ where $T'$ is a tree rooted at ter($P$). A tree with trunk $P$ has root $\text{in}(P)$. If $G = (G, A, B)$ is a web, an $A$-$B$ tree in $G$ is a subgraph of $G$ that is a tree with root in $A$ and exactly one vertex in $B$.

2.2. Reverse mathematics. Reverse mathematics, introduced by Friedman [6], is an analysis of the logical strength of the theorems of ordinary mathematics in the context of second-order arithmetic. A result in reverse mathematics typically has the form “$T$ is equivalent to strong system over weak system,” where strong system and weak system are subsystems of second-order arithmetic and $T$ is some theorem from ordinary mathematics. This means that $T$ is provable in strong system and that all the axioms of strong system are provable in weak system $\cup \{T\}$. The proof of strong system from weak system $\cup \{T\}$ is called a reversal.

We now describe the axiomatic systems that we will use to analyze Menger’s theorem. We follow [9], the standard reference for reverse mathematics. Also see [4] Section 2 for a thorough introduction to most of the systems we consider and for computability-theoretic interpretations of these systems.

Before we describe the systems, we need to know that the basic axioms are the sentences

\[
\begin{align*}
\forall m(m + 1 & \neq 0) \\
\forall m \forall n(m + 1 = n + 1 \rightarrow m = n) \\
\forall m(m + 0 & = m) \\
\forall m \forall n(m + (n + 1) = (m + n) + 1) \\
\forall m(m \times 0 & = 0) \\
\forall m \forall n(m \times (n + 1) = (m \times n) + m) \\
\forall m & \neg(m < 0) \\
\forall m \forall n(m < n + 1 & \leftrightarrow (m < n \land m = n)),
\end{align*}
\]

that the induction axiom is the sentence

\[
\forall X((0 \in X \land \forall n(n \in X \rightarrow n + 1 \in X)) \rightarrow \forall n(n \in X)),
\]

and that the comprehension scheme consists of all universal closures of formulas of the form

\[
\exists X\forall n(n \in X \leftrightarrow \varphi(n)),
\]

where $\varphi$ can be any formula in the language of second-order arithmetic in which $X$ does not occur freely. Full second-order arithmetic consists of the basic axioms, the induction axiom, and the comprehension scheme.

$\text{RCA}_0$ (for recursive comprehension axiom) consists of the basic axioms, the $\Sigma^0_1$ induction scheme, and the $\Delta^0_1$ comprehension scheme. The $\Sigma^0_1$ induction scheme consists of all universal closures of formulas of the form

\[
(\varphi(0) \land \forall n(\varphi(n) \rightarrow \varphi(n + 1))) \rightarrow \forall n \varphi(n)
\]
where \( \varphi \) is \( \Sigma^0_1 \). The \( \Delta^0_1 \) comprehension scheme consists of all universal closures of formulas of the form

\[
\forall n(\varphi(n) \leftrightarrow \psi(n)) \rightarrow \exists X \forall n(n \in X \leftrightarrow \varphi(n))
\]

where \( \varphi \) is \( \Sigma^0_1 \), \( \psi \) is \( \Pi^0_1 \), and \( X \) does not occur freely in \( \varphi \). \( \text{RCA}_0 \) is the standard weak system for the purpose of reversals. \( \text{RCA}_0 \) proves that the function \( \langle i, j \rangle \mapsto (i + j)^2 + i \) is injective (see [9] Section II.2). For \( X \subseteq \mathbb{N} \) and \( n \in \mathbb{N} \), we define

\[
(X)_n = \{i \mid \langle i, n \rangle \in X\} \quad \text{and} \quad (X)^n = \{\langle i, m \rangle \mid \langle i, m \rangle \in X \wedge m < n\}.
\]

\( \text{RCA}_0 \) proves that if \( X \) exists, then so do \((X)_n\) and \((X)^n\). We interpret \((X)_n\) as the \((n+1)\)th column of \( X \) and \((X)^n\) as the set of the first \( n \) columns of \( X \).

\( \text{ACA}_0 \) (for arithmetical comprehension axiom) consists of the basic axioms, the induction axiom, and the arithmetical comprehension scheme. The arithmetical comprehension scheme is the restriction of the comprehension scheme to formulas \( \varphi \) that are arithmetical.

\( \text{ATR}_0 \) (for arithmetical transfinite recursion) consists of \( \text{ACA}_0 \) plus an axiom scheme that says if a set can be constructed by iterating arithmetical comprehension along an existing well-order, then that set exists. Let \( \text{LO}(X, <_X) \) be a formula that says “\( X \) is a linear order on the set \( X \)” and let \( \text{WO}(X, <_X) \) be a formula that says “\( X \) is a well-order on the set \( X \).” Given a formula \( \theta(n,Y) \), let \( H_\theta(X,<_X,Y) \) be a formula that says “\( \text{LO}(X,<_X) \) and \( Y = \{\langle n, j \rangle \mid j \in X \land \theta(n,\{\langle m,i \rangle \mid i <_X j\})\} \).” The axioms of \( \text{ATR}_0 \) consist of those of \( \text{ACA}_0 \) plus all universal closures of formulas of the form

\[
\forall X \forall <_X (\text{WO}(X,<_X) \rightarrow \exists Y \text{H}_\theta(X,<_X,Y))
\]

where \( \theta \) is arithmetical. An easier-to-understand equivalent of \( \text{ATR}_0 \) is the system \( \Sigma^1_1 \) \textit{separation}, which consists of the axioms of \( \text{RCA}_0 \) plus the all universal closures of formulas of the form

\[
\neg \exists n(\varphi_0(n) \land \varphi_1(n)) \rightarrow \exists Z \forall n((\varphi_0(n) \rightarrow n \in Z) \land (\varphi_1(n) \rightarrow n \notin Z)),
\]

where \( \varphi_0 \) and \( \varphi_1 \) are \( \Sigma^1_1 \) and \( Z \) does not occur freely in either \( \varphi_0 \) or \( \varphi_1 \) (see [9] Theorem V.5.1).

\( \Sigma^1_1 \text{-DC}_0 \) (for \( \Sigma^1_1 \) dependent choice) consists of \( \text{ACA}_0 \) and the scheme of \( \Sigma^1_1 \) dependent choice. The scheme of \( \Sigma^1_1 \) dependent choice consists of all universal closures of formulas of the form

\[
\forall \forall n \forall X \forall Y \eta(n,X,Y) \rightarrow \exists Z \forall n \forall \eta(n,\langle Z\rangle^n,\langle Z\rangle^n)
\]

where \( \eta \) is \( \Sigma^1_1 \) and \( Z \) does not occur freely in \( \eta \).

\( \Pi^1_1 \text{-CA}_0 \) (for \( \Pi^1_1 \) comprehension axiom) consists of the basic axioms, the induction axiom, and the \( \Pi^1_1 \) comprehension scheme. The \( \Pi^1_1 \) comprehension scheme is the restriction of the comprehension scheme to formulas \( \varphi \) that are \( \Pi^1_1 \).

\( \text{RCA}_0 \) is strictly weaker than \( \text{ACA}_0 \), which is strictly weaker than both \( \text{ATR}_0 \) and \( \Sigma^1_1 \text{-DC}_0 \). \( \text{ATR}_0 \) and \( \Sigma^1_1 \text{-DC}_0 \) are independent over \( \text{RCA}_0 \). However, \( \text{ATR}_0 \) proves the consistency of \( \Sigma^1_1 \text{-DC}_0 \). Both \( \text{ATR}_0 \) and \( \Sigma^1_1 \text{-DC}_0 \) are strictly weaker than \( \Pi^1_1 \text{-CA}_0 \).

Our proof of Menger’s theorem in \( \Pi^1_1 \text{-CA}_0 \) relies on two key meta-mathematical facts. The first key fact concerns the existence of \( \beta \)-models. The second key fact concerns the existence of models of \( \Sigma^1_1 \text{-DC}_0 \).

**Definition 2.1.** A countable coded \( \omega \)-model is a set \( X \subseteq \mathbb{N} \) viewed as coding the structure \( \mathcal{M} = (\mathbb{N}, \{(X)_n \mid n \in \mathbb{N}\}, +, \times, 0, 1, <) \).

We usually identify a countable coded \( \omega \)-model \( X \) with the structure \( \mathcal{M} \) that it codes.

**Definition 2.2.** A countable coded \( \beta \)-model is a countable coded \( \omega \)-model \( \mathcal{M} \) that is absolute for \( \Sigma^1_1 \) formulas with parameters from \( \mathcal{M} \). That is, if \( \varphi \) is a \( \Sigma^1_1 \) formula with parameters from \( \mathcal{M} \), then \( \mathcal{M} \models \varphi \) if and only if \( \varphi \) is true.
Theorem 2.3 (see [9] Theorem VII.2.10). The statement “for every $X$ there is a countable coded $\beta$-model $M$ with $X \in M$” is equivalent to $\Pi^1_1-\text{ACA}_0$ over $\text{ACA}_0$.

It is helpful to keep in mind that $\text{ACA}_0$ proves that every countable coded $\beta$-model is a model of $\text{ATR}_0$ (see [9] Theorem VII.2.7).

Theorem 2.4 (see [9] Theorem VIII.4.20). $\text{ATR}_0$ proves that for every $X$ there is a countable coded $\omega$-model $M$ of $\Sigma^1_1-\text{DC}_0$ with $X \in M$.

The statement “for every $X$ there is a countable coded $\omega$-model $M$ of $\Sigma^1_1-\text{DC}_0$ with $X \in M$” is in fact equivalent to $\text{ATR}_0$ over $\text{RCA}_0$. See [9] Lemma VIII.4.15 for the reversal.

3. Warps, Waves, and Alternating Walks

In this section we use $\text{ACA}_0$ to develop the basic tools we need to prove Menger’s theorem in $\Pi^1_1-\text{CA}_0$. Our notation and terminology mostly follows [2] with some ideas borrowed from [5] Section 8.4.

Definition 3.1. A warp in a web $G = (G, A, B)$ is a subgraph $W$ of $G$ such that

- $A \subseteq V(W)$,
- every $x \in V(W)$ has in-deg$_W(x) \leq 1$ and out-deg$_W(x) \leq 1$, and
- every $x \in V(W)$ is reachable from some $a \in A$ by a path in $W$.

A warp is thus a collection of disjoint paths in $G$ with each path starting at a distinct vertex in $A$ and such that for every $a \in A$ there is a path in the warp starting at $a$. Such paths may be one-way infinite. It is often convenient to think of a warp $W$ as the collection of its component paths $\{P_a \mid a \in A \land \text{in}(P_a) = a\}$ with the understanding that this collection is coded by the set $\{(a, \langle n, x \rangle) \mid x = \text{n}^{th}\text{vertex of } P_a\}$. “$P$ is a path in $W$” always means that $P$ is one of these component paths.

If $W$ is a warp, then let $\text{ter}(W) = \{x \in V(W) \mid \text{out-deg}_W(x) = 0\}$. That is, $\text{ter}(W)$ is the set of terminal vertices of the paths in $W$. The statement “if $W$ is a warp then $\text{ter}(W)$ exists” is equivalent to $\text{ACA}_0$ over $\text{RCA}_0$, hence our assumption of $\text{ACA}_0$ throughout this section.

Definition 3.2. A wave in a web $G = (G, A, B)$ is a warp $W$ such that $\text{ter}(W)$ is an $A$-$B$ separator.

It is important to note that “$X$ is an $A$-$B$ separator in $(G, A, B)$” is an arithmetical property. $A$-$B$ paths are finite, and quantification over them can be coded by quantification over $\mathbb{N}$. Thus “$W$ is a wave in $G$” is also an arithmetical property.

The warp $\{P_a \mid a \in A\}$ in which each path $P_a$ is the trivial path $(a)$ is always a wave, and we call it the trivial wave.

Definition 3.3. For warps $W$ and $Y$ in a web $G = (G, A, B)$, $Y$ is an extension of $W$ (written $W \leq Y$) if and only if $W$ is a subgraph of $Y$.

Definition 3.4. If $\{W_i \mid i \in \mathbb{N}\}$ is a sequence of warps such that $W_i \leq W_{i+1}$ for each $i \in \mathbb{N}$, then $\bigcup_{i \in \mathbb{N}} W_i$ denotes the limit warp defined by $V(\bigcup_{i \in \mathbb{N}} W_i) = \bigcup_{i \in \mathbb{N}} V(W_i)$ and $E(\bigcup_{i \in \mathbb{N}} W_i) = \bigcup_{i \in \mathbb{N}} E(W_i)$.

It is easy to check in $\text{RCA}_0$ that a limit warp, if it exists, is indeed a warp. However, the statement “if $\{W_i \mid i \in \mathbb{N}\}$ is a sequence of warps such that $W_i \leq W_{i+1}$ for each $i \in \mathbb{N}$, then $\bigcup_{i \in \mathbb{N}} W_i$ exists” is equivalent to $\text{ACA}_0$ over $\text{RCA}_0$.

Definition 3.5. Let $W = \{P_a \mid a \in A\}$ be a wave in a web $G = (G, A, B)$. Then

- $P_a$ is $W$-essential in $G$ if and only if $P_a$ is finite and there is a $\text{ter}(P_a)$-$B$ path in $G$ disjoint from $V(W) - \{\text{ter}(P_a)\}$,
- $a \in A$ is $W$-essential in $G$ if and only if $P_a$ is $W$-essential in $G$, and
• \( \text{ess}_G(W) = \{ a \in A \mid a \text{ is } W\text{-essential in } G \} \).

The motivation behind the definition of \( W\)-essential in \( G \) is that if \( P \) is a path in a wave \( W \) that is \( W\)-essential in \( G \), then \( \text{ter}(W) \) needs \( \text{ter}(P) \) to separate \( A \) from \( B \). If \( Q \) is a \( \text{ter}(P)\)-\( B \) path disjoint from \( V(W) - \{ \text{ter}(P) \} \), then \( PQ \) is an \( A\)-\( B \) path disjoint from \( \text{ter}(W) - \{ \text{ter}(P) \} \). One readily checks that \( \{ \text{ter}(P) \mid P \text{ is } W\text{-essential in } G \} \) is an \( A\)-\( B \) separator.

**Definition 3.6.** If \( W \) and \( Y \) are waves in a web \( G = (G, A, B) \) with \( W \leq Y \), then \( Y \) is a **good** extension of \( W \) if and only if \( \text{ess}_G(W) = \text{ess}_G(Y) \) and \( Y \) is a **bad** extension of \( W \) if and only if \( \text{ess}_G(W) \neq \text{ess}_G(Y) \).

If \( W \) and \( Y \) are waves in a web \( G = (G, A, B) \) with \( W \leq Y \), then it is always the case that \( \text{ess}_G(Y) \subseteq \text{ess}_G(W) \). Thus \( Y \) is a good extension of \( W \) if and only if \( \text{ess}_G(W) \subseteq \text{ess}_G(Y) \).

**Lemma 3.7** (in \( \mathsf{ACA}_0 \); see [2] Lemma 2.5). If \( \{W_i \mid i \in \mathbb{N}\} \) is a sequence of waves in a web \( G = (G, A, B) \) such that \( W_i \leq W_{i+1} \) for each \( i \in \mathbb{N} \), then \( \bigcup_{i \in \mathbb{N}} W_i \) is a wave in \( G \).

**Proof.** Let \( W = \bigcup_{i \in \mathbb{N}} W_i \). As mentioned above, it is easy to check that \( W \) is a warp. We need to show that \( \text{ter}(W) \) is an \( A\)-\( B \) separator. Let \( P \) be an \( A\)-\( B \) path, and let \( X = \{ \langle x, i \rangle \mid x \in V(P) \cap \text{ter}(W_i) \} \) which exists by arithmetical comprehension. Each \( W_i \) is a wave, hence \( X \) is infinite. As \( V(P) \) is finite, there must be an \( x \in V(P) \) such that \( \{ i \mid x \in \text{ter}(W_i) \} \) is infinite. Then \( x = \text{ter}(Q) \) for the path \( Q \) in \( W \) containing \( x \). If not, then there is a vertex following \( x \) on \( Q \), the corresponding edge must appear in \( W_n \) for some \( n \), and so \( x \notin \text{ter}(W_i) \) for all \( i \geq n \).

**Definition 3.8.** Let \( W \) and \( Y \) be warps in a web \( G = (G, A, B) \) with \( W \leq Y \). Let \( Q \) be a finite path in both \( Y \) and \( W \) (i.e., the path \( Q \) is in \( W \) and is not properly extended in \( Y \)). A **\( (Y-W)\)-alternating walk** from \( \text{ter}(Q) \) is a walk \( R = x_0 e_0 x_1 \cdots e_{n-1} x_n \) such that

(i) \( x_0 = \text{ter}(Q) \),
(ii) for all \( i \leq n \), \( x_i \in (V(G) - V(W)) \cup \text{ter}(W) \),
(iii) for all \( i < n \), if \( e_i \notin E(Y) \), then \( e_i = (x_i, x_{i+1}) \),
(iv) for all \( i < n \), if \( e_i \in E(Y) \), then \( e_i = (x_{i+1}, x_i) \) (i.e., \( R \) traverses \( e_i \) backwards),
(v) for all \( i, j \leq n \) with \( i \neq j \), if \( x_i = x_j \), then \( x_i \notin V(Y) \), and
(vi) for all \( i, j \leq n \) with \( i \neq j \), \( e_i \neq e_j \).
(vii) for all \( 0 < i \leq n \), if \( x_0 \notin V(Y) \), then either \( e_{i-1} \) or \( e_i \) is in \( E(Y) \).

Note that if \( x_n \) is the last vertex on a **\( (Y-W)\)-alternating walk** from \( \text{ter}(Q) \) and \( x_n \in V(Y) \), then item (vii) implies that \( e_{n-1} \in E(Y) \), and by item (iv) it must also be that \( e_{n-1} = (x_n, x_{n-1}) \).

A **\( (Y-W)\)-alternating walk** from \( \text{ter}(Q) \) is similar to a **\( Y\)-walk** as defined in [2] and to a walk which alternates with respect to \( Y \) as defined in [5] Section 3.3. The difference is that a **\( (Y-W)\)-alternating walk** from \( \text{ter}(Q) \) is not allowed to use the vertices in \( V(W) - \text{ter}(W) \), hence the notation “\( Y-W \).”

**Definition 3.9.** Let \( W = \{ P_a \mid a \in A \} \) and \( Y = \{ Q_a \mid a \in A \} \) be warps in a web \( G = (G, A, B) \) with \( W \leq Y \). Let \( Q_{a_0} \) be a finite path in both \( Y \) and \( W \). Then \( \text{alt}_G(Y-W, \text{ter}(Q_{a_0})) \) denotes the warp \( \{ Q'_a \mid a \in A \} \) where \( Q'_a = Q_{a_0} x \) if \( x \) is the last vertex on \( Q_{a_0} \) which lies on a **\( (Y-W)\)-alternating walk** from \( \text{ter}(Q_{a_0}) \) and \( Q'_a = P_a \) if no such \( x \) exists.

Our definition of \( \text{alt}_G(Y-W, \text{ter}(Q_{a_0})) \) is analogous to the definition of \( M(a_0, W) \) in [2]. Also, note that \( W \leq \text{alt}_G(Y-W, \text{ter}(Q_{a_0})) \leq Y \). The first inequality is by Definition 3.8 item (ii) and the second inequality is clear.

The crucial lemma from this section is **Lemma 3.12** below. **Lemma 3.10** and **Lemma 3.11** are used to prove **Lemma 3.12**.

**Lemma 3.10** (in \( \mathsf{ACA}_0 \)). Let \( W \) and \( Y \) be warps in a web \( G = (G, A, B) \) with \( W \leq Y \). Let \( Q \) be a finite path in both \( Y \) and \( W \). Let \( R \) be a **\( (Y-W)\)-alternating walk** from \( \text{ter}(Q) \). Then there is a \( warp \ Z \geq W \) in \( G \) with \( \text{ter}(Z) = \{ \text{ter}(Y) - \{ \text{ter}(Q) \} \} \cup \{ \text{ter}(R) \} \).
Proof. Let \( R = x_0e_0x_1e_1 \cdots e_{n-1}x_n \) where \( x_0 = \text{ter}(Q) \) and \( x_n = \text{ter}(R) \). Assume \( n > 0 \), for otherwise we may take \( Z = Y \). Let \( Z' \) be the subgraph of \( G \) with \( E(Z') = E(Y) \triangle E(R) \) and \( V(Z') = A \cup \{ x \mid (3e \in E(Z'))(x \text{ is a vertex of } e) \} \). One readily checks the following equalities:

- if \( x \in V(Y) - V(R) \), then in-deg\(_Z'(x) = \text{in-deg}_Y(x) \) and out-deg\(_Z'(x) = \text{out-deg}_Y(x) \),
- for \( 0 < i < n \), if \( x_i \in V(R) - V(Y) \), then in-deg\(_Z'(x_i) = 1 \) and out-deg\(_Z'(x_i) = 1 \),
- for \( 0 < i < n \), if \( x_i \in V(R) \cap V(Y) \) is in \( V(Z') \), then in-deg\(_Z'(x_i) = \text{in-deg}_Y(x_i) \) and out-deg\(_Z'(x_i) = \text{out-deg}_Y(x_i) \),
- in-deg\(_Z'(x_0) = \text{in-deg}_Y(x_0) \) and out-deg\(_Z'(x_0) = 1 \), and
- in-deg\(_Z'(x_n) = 1 \) and out-deg\(_Z'(x_n) = 0 \).

It follows that in-deg\(_Z'(x) \leq 1 \) and out-deg\(_Z'(x) \leq 1 \) for all \( x \in V(Z') \), which means that every component of \( Z' \) is either a path or a cycle. Let \( Z \) be the subgraph of \( Z' \) consisting of the component paths of \( Z' \) (i.e., \( Z \) is the subgraph of \( Z' \) induced by \( \{ x \in V(Z') \mid x \text{ is not on a cycle in } Z' \} \)). \( Z \) contains every vertex \( x \in V(Z') \) with in-deg\(_Z'(x) = 0 \) or out-deg\(_Z'(x) = 0 \). In particular, \( A \subseteq V(Z) \) and \( \text{ter}(Z) = (\text{ter}(Y) - \{ \text{ter}(Q) \}) \cup \{ \text{ter}(R) \} \). To show that \( Z \) is a warp, we need only show that in\((P) \) exists and is in \( A \) for every path \( P \) in \( Z \). The above equations imply that if \( x \in V(P) - A \), then in-deg\(_Z(x) \neq 0 \) and hence that \( x \) has an immediate predecessor on \( P \). This fact together with the fact that \( R \) is finite implies that there is an \( x \in V(P) \) such that \( (V(Px) \cap V(R)) - A = \emptyset \). Thus the edges of \( Px \) must all be edges of \( Y \), which means that \( Px \) must be an initial segment of some path in \( Y \). Hence in\((P) \) exists and is in \( A \). Finally, \( Z \geq W \) by Definition 3.8 item (ii). \( \square \)

Lemma 3.11 (in \( \text{ACA}_0 \); see [2] Lemma 2.7). Let \( W \) and \( Y \) be waves in a web \( G = (G, A, B) \) with \( W \leq Y \). Let \( Q \) be a finite path in both \( Y \) and \( W \). Then \( \text{alt}_G(Y - W, \text{ter}(Q)) \) is a wave.

Proof. Let \( U = \text{alt}_G(Y - W, \text{ter}(Q)) \). Suppose for a contradiction that \( P \) is an \( A-B \) path disjoint from ter\((U) \). Let \( w \) be the last vertex on \( P \) that is in \( V(W) \), and let \( S \) be the path in \( W \) containing \( w \). It must be that \( w = \text{ter}(S) \), for otherwise \( SwP \) is an \( A-B \) path disjoint from ter\((W) \), contradicting that \( W \) is a wave. The path \( SwP \) is, however, an \( A-B \) path disjoint from ter\((U) \). \( Y \) is a wave, so \( wP \) intersects ter\((Y) \), which must happen at a vertex in \( V(Y) - V(U) \). Let \( y \) be the first vertex on \( wP \) in \( V(Y) - V(U) \). Let \( z \) be the last vertex on \( wPy \) in \( V(U) \), which exists because \( w \in V(U) \).

Claim. There is a \((Y - W)\)-alternating walk from ter\((Q) \) ending at \( z \).

Proof of claim. Let \( Q' \) be the path in \( Y \) containing \( z \). We show that there is a \((Y - W)\)-alternating walk \( R \) from ter\((Q) \) that meets \( Q' \) at a vertex \( r \) which is past \( z \) on \( Q' \). If \( r \) is the first such vertex on \( R \), then \( RrQ'z \) (following the edges of \( Q' \) backwards) is the desired walk. If \( z = w \), then \( Q' \) extends \( S \), so if there is no such walk \( R \) then by Definition 3.9 \( S \) is a path in \( U \) which contradicts that \( P \) is disjoint from ter\((U) \). On the other hand, if \( z \neq w \), then \( z \notin V(W) \) by choice of \( w \). As \( z \in V(U) - V(W) \) and \( z \notin \text{ter}(U) \), again by Definition 3.9 it must be the case that some \((Y - W)\)-alternating walk \( R \) from ter\((Q) \) meets \( Q' \) at a vertex past \( z \). \( \square \)

Now let \( R \) be the walk provided by the claim, let \( r \) be the last vertex of \( zPy \) on \( R \), and let \( y' \) be the vertex immediately preceding \( y \) on the path in \( Y \) containing \( y \). Then \( RrPy(y', y) \) is a \((Y - W)\)-alternating walk from ter\((Q) \) on which \( y \) lies which contradicts \( y \notin V(U) \). \( \square \)

Lemma 3.12 (in \( \text{ACA}_0 \); see [2] Lemma 2.8). Let \( W \) be a wave in a web \( G = (G, A, B) \) that has no bad extensions in \( G \). Let \( x \in V(G) - V(W) \) be such that there is a wave \( Y \geq W \) in \( G - \{ x \} = (G - \{ x \}, A - \{ x \}, B - \{ x \}) \) with ess\(_G-\{x\}(Y) \subseteq \text{ess}_G(W) \). Then there is a wave \( Z \geq W \) in \( G \) with \( x \in \text{ter}(Z) \).

Proof. Let \( G = (G, A, B) \), \( W = \{ P_a \mid a \in A \} \), \( x \), and \( Y = \{ Q_a \mid a \in A \} \) be as in the statement of the lemma. Let \( a_0 \in \text{ess}_G(W) - \text{ess}_G-\{x\}(Y) \). If we replace \( Q_{a_0} \) with \( P_{a_0} \) in \( Y \), then we retain that this path is not \( Y \)-essential in \( G - \{ x \} \). Thus we may assume \( Q_{a_0} = P_{a_0} \). In particular, \( Q_{a_0} \) is a finite path that is not \( Y \)-essential in \( G - \{ x \} \).
Claim. In $G$, there is an alternating $(Y - W)$-walk from $\text{ter}(Q_{a_0})$ ending at $x$.

Proof of claim. If there is a $\text{ter}(Q_{a_0})$-$x$ path disjoint from $V(Y) - \{\text{ter}(Q_{a_0})\}$, then this path is the desired walk. So suppose instead there is no such path. Let $U = \text{alt}_{G -\{x\}}(Y - W, \text{ter}(Q_{a_0}))$. $U$ is a wave in $G - \{x\}$ by Lemma 3.11. Furthermore, $a_0 \notin \text{ess}_{G -\{x\}}(Y)$ implies that $a_0 \notin \text{ess}_{G -\{x\}}(U)$ because if $P$ is a $\text{ter}(Q_{a_0})$-$B$ path in $G - \{x\}$, then the first vertex on $P$ in $V(Y) - \{\text{ter}(Q_{a_0})\}$ is also in $V(U)$. We prove that $U$ is not a wave in $G$. To do this, it suffices to show that every $\text{ter}(Q_{a_0})$-$B$ path in $G$ intersects $V(U) - \{\text{ter}(Q_{a_0})\}$. Therefore if $U$ were a wave in $G$, it would be a bad extension of $W$ in $G$ because $a_0$ would be in $\text{ess}_{G}(W) - \text{ess}_{G}(U)$. This is a contradiction.

Consider a $\text{ter}(Q_{a_0})$-$B$ path $P$. If $x \notin V(P)$, then $P$ is a path in $G - \{x\}$ and hence $P$ intersects $V(U) - \{\text{ter}(Q_{a_0})\}$ because $a_0 \notin \text{ess}_{G -\{x\}}(U)$. If $x \in V(P)$, then by assumption $Px$ intersects $V(Y) - \{\text{ter}(Q_{a_0})\}$. Again, the first vertex on $P$ in $V(Y) - \{\text{ter}(Q_{a_0})\}$ is also in $V(U)$.

We now know that $U$ is a wave in $G - x$ but not in $G$. Thus there is an $A$-$B$ path $S$ in $G$ avoiding $\text{ter}(U)$, and $x$ must lie on $S$. Let $z$ be the last vertex of $Sx$ that is in $V(U)$. It must be that $z \in ((V(U) - W(V)) \cup \text{ter}(W)) - \text{ter}(U)$. Hence there must be an alternating $(Y - W)$-walk $R$ from $\text{ter}(Q_{a_0})$ to $z$. Let $y$ be the last vertex of $zRx$ which lies on $R$. Then $Ryx$ is the desired alternating $(Y - W)$-walk from $\text{ter}(Q_{a_0})$ to $x$. \hfill $\square$

By the claim, let $R$ be an alternating $(Y - W)$-walk from $\text{ter}(Q_{a_0})$ ending at $x$. Apply Lemma 3.10 to get a warp $Z \geq W$ in $G$ with $\text{ter}(Z) = (\text{ter}(Y) - \{\text{ter}(Q_{a_0})\}) \cup \{x\}$. $Z$ is a wave because $\text{ter}(Y) - \{\text{ter}(Q_{a_0})\}$ is an $(A - \{x\})-(B - \{x\})$ separator in $G - \{x\}$, thus $(\text{ter}(Y) - \{\text{ter}(Q_{a_0})\}) \cup \{x\}$ is an $A$-$B$ separator in $G$. \hfill $\square$

4. Menger’s Theorem in $\Pi_1^1$-$\mathsf{CA}_0$

We plan to prove Menger’s theorem as follows. Given a web $G = (G, A, B)$, start with $W$ a $\leq$-maximal wave in $G$. Let $C$ be the terminal vertices of the paths in $W$ that are $W$-essential. Then extend these $W$-essential paths to be the collection of disjoint $A$-$B$ paths $M$. Lemma 4.1 below provides the $\leq$-maximal wave $W$, and Lemma 4.2 below is the tool we use to extend the $W$-essential paths to a collection of disjoint $A$-$B$ paths. The proof of Lemma 4.1 is the only argument in which we employ the full strength of $\Pi_1^1$-$\mathsf{CA}_0$.

Lemma 4.1 (in $\Pi_1^1$-$\mathsf{CA}_0$; see [2] Corollary 2.5a). In every web there is a $\leq$-maximal wave.

Proof. Let $G = (G, A, B)$ be a web, let $(g_n | n \in \mathbb{N})$ be an enumeration of $V(G)$, and by Theorem 2.3 let $\mathcal{M}$ be a countable coded $\beta$-model with $(G, A, B) \in \mathcal{M}$. Using $\mathsf{ACA}_0$ outside $\mathcal{M}$, we construct a sequence of integers $(i_n | n \in \mathbb{N})$ such that $(\mathcal{M})_{i_n}$ is a wave for each $n \in \mathbb{N}$ and $(\mathcal{M})_{i_n} \leq (\mathcal{M})_{i_{n+1}}$ for each $n \in \mathbb{N}$. Let $i_0$ be an index such that $(\mathcal{M})_{i_0}$ is the trivial wave $\{(a) | a \in A\}$. Suppose we have $i_0, \ldots, i_n$. If there is an $i \in \mathbb{N}$ such that $(\mathcal{M})_i$ is a wave with $(\mathcal{M})_i \geq (\mathcal{M})_{i_n}$ and $g_n \in V((\mathcal{M})_i)$, then let $i_{n+1}$ be such an $i$. Otherwise let $i_{n+1} = i_n$. With the desired sequence $(i_n | n \in \mathbb{N})$ in hand, let $W$ be the limit $W = \bigcup_{n \in \mathbb{N}}(\mathcal{M})_{i_n}$, which is a wave by Lemma 3.7. This $W$ is $\leq$-maximal in $(G, A, B)$. If not, there is a wave $Y \geq W$ with some $g_n \in V(Y) - V(W)$. As $(\mathcal{M})_{i_n} \leq W \leq Y$, at stage $n + 1$ in the construction the $\Sigma_1^1$ formula $(\exists Y)(Y \text{ is a wave } \land Y \geq (\mathcal{M})_{i_n} \land g_n \in V(Y))$ is true and hence is true in $\mathcal{M}$ because $\mathcal{M}$ is a $\beta$-model. Therefore we chose $i_{n+1}$ so that $g_n \in V((\mathcal{M})_{i_{n+1}})$, contradicting $g_n \notin V(W)$. \hfill $\square$

In [2] and [5], Lemma 4.1 is obtained by a simple application of Zorn’s lemma. Our proof above is the most effective proof possible, in the sense that Lemma 4.1 is equivalent to $\Pi_1^1$-$\mathsf{CA}_0$ over $\mathsf{RCA}_0$ (see Corollary 5.3 below).

The following Lemma 4.2 is the key tool used to complete the proof of Menger’s theorem. We first give a proof of Lemma 4.2 in the style of ordinary mathematics for the sake of clarity. We then explain how to formalize Lemma 4.2 in a way that will allow us to complete the proof of Menger’s theorem in $\Pi_1^1$-$\mathsf{CA}_0$. 
Lemma 4.2 (see [2] Theorem 3.4). Let $W$ be a wave in $G = (G, A, B)$ that has no bad extensions in $G$, let $a \in \text{ess}_{G}(W)$, and let $P_a$ be the component path of $W$ starting at $a$. Then there is a finite $a$-$B$ tree $T$ with trunk $P_a$ such that $V(T) \cap (V(W) - V(P_a)) = \emptyset$, and there is a wave $Y$ in $G - T$ such that $Y \geq W - P_a$, $\text{ess}_{G- T}(Y) = \text{ess}_{G}(W) - \{a\}$, and $Y$ has no bad extensions in $G - T$.

Proof. We assume that the conclusion of the lemma is false and construct a bad extension of $W$ in $G$, which is a contradiction.

Let $(Q_n \mid n \in \mathbb{N})$ list all the $\text{ter}(P_a)$-$B$ paths with each path occurring on the list infinitely often. We construct sequences $(T_n \mid n \in \mathbb{N})$ and $(Y_n \mid n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$

(i) $T_n$ is a finite tree in $G$ with trunk $P_a$,
(ii) $Y_n$ is a wave in $G - T_n$,
(iii) $T_{n-1} \subseteq T_n$ (if $n > 0$),
(iv) $Y_{n-1} \subseteq Y_n$ (if $n > 0$),
(v) $\text{ess}_{G- T_n}(Y_n) = \text{ess}_{G}(W) - \{a\}$, and
(vi) $Y_n$ has no bad extensions in $G - T_n$.

Start with $T_0 = P_a$ and $Y_0 = W - P_a$. Items (i), (ii), (iii), and (v) are easily checked for $n = 0$. Furthermore, if $Y$ were a bad extension of $Y_0$ in $G - T_0$, then $Y \cup \{P_a\}$ would be a bad extension of $W$ in $G$. Hence we have item (vi) for $n = 0$ as well.

Suppose we have constructed $T_n$ and $Y_n$. If $V(Q_n) \cap V(Y_n) \neq \emptyset$ or $T = T_n \cup Q_n$ is not a tree with trunk $P_a$, set $T_{n+1} = T_n$ and $Y_{n+1} = Y_n$. Otherwise $V(Q_n) \cap V(Y_n) = \emptyset$ and $T = T_n \cup Q_n$ is a tree with trunk $P_a$. The situation now is that

$\bullet$ $T$ is a finite $a$-$B$ tree with trunk $P_a$ such that $V(T) \cap (V(W) - V(P_a)) = \emptyset$ and
$\bullet$ $Y_n$ is a wave in $G - T$. But both cases imply the existence of a wave $Y \geq Y_n$ in $G - T$ with $\text{ess}_{G- T}(Y) \subseteq \text{ess}_{G}(W) - \{a\}$ (where the equality is by item (v)). Let $x$ be the first vertex on $Q_n$ such that there exists a wave $Y \geq Y_n$ in $G - (T_n \cup Q_n x)$ with $\text{ess}_{G- (T_n \cup Q_n x)}(Y) \subseteq \text{ess}_{G- T}(Y_n)$ (note $x \neq \text{ter}(P_a)$ by item (vi)). Let $T_{n+1} = T_n \cup Q_n x$. By the choice of $x$, $Y_n$ has no bad extensions in $G - T_{n+1}$, but there is an extension $Y \geq Y_n$ in $(G - T_{n-1}) - x$ with $\text{ess}(G - T_{n+1}) - x(Y) \subseteq \text{ess}_{G- T_{n+1}}(Y_n)$. Thus by Lemma 3.12 there is a wave $Y_{n+1} \geq Y_n$ in $G - T_{n+1}$ with $x \in \text{ter}(Y_{n+1})$. With this $T_{n+1}$ and $Y_{n+1}$, items (i)-(iv) are clear for $n + 1$. Item (v) is by the choice of $x$, which implies that $\text{ess}_{G- T_{n+1}}(Y_{n+1}) = \text{ess}_{G- T_n}(Y_n) = \text{ess}_{G}(W) - \{a\}$. Item (vi) is again by the choice of $x$ because a bad extension $Y \geq Y_{n+1}$ in $G - T_{n+1}$ would be a $Y \geq Y_n$ in $G - T_{n+1}$ with $\text{ess}_{G- T_{n+1}}(Y) \subseteq \text{ess}_{G- T_{n+1}}(Y_n)$.

Let $T = \bigcup_{n \in \mathbb{N}} T_n$, and let $Y = \bigcup_{n \in \mathbb{N}} Y_n$. By construction, $V(Y_n) \cap V(T_m) = \emptyset$ for all $n, m \in \mathbb{N}$, which means that each $Y_n$ is a wave in $G - T$. Therefore $Y$ is a wave in $G - T$ by Lemma 3.7. It remains to show that $\text{ter}(Y)$ is an $A$-$B$ separator in $G$. Our desired contradiction follows because then $Y \cup \{P_a\}$ would be a bad extension of $W$ in $G$ because $a \in \text{ess}_{G}(W) - \text{ess}_{G}(Y \cup \{P_a\})$.

Let $P$ be an $A$-$B$ path in $G$. To show that $P$ intersects $\text{ter}(Y)$, we show that there is a final segment $S$ of $P$ that lies in $G - T$ and intersects $V(Y)$. This suffices to finish the proof because if $x$ is the last vertex of $S$ in $V(Y)$ and $Q$ is the component path of $Y$ containing $x$, then $QxS$ is an $A$-$B$ path in $G - T$ which means that $xS$ (and hence $P$) must intersect $\text{ter}(Y)$. Thus let $x$ be the last vertex of $P$ on $T$ (if there is no such $x$, then $P$ is a path in $G - T$ and thus intersects $\text{ter}(Y)$), let $n$ be such that $x \in T_n$, and let $m > n$ be such that $Q_n = \text{ter}(P_n)T_n x P$. Consider stage $m + 1$ of the construction. If $V(Q_m) \cap V(Y_m) \neq \emptyset$, then it must be that $xP$ intersects $Y_m$ and hence intersects $Y$ as desired. Otherwise $V(Q_m) \cap V(Y_m) = \emptyset$ and $T_m \cup Q_m$ is a tree with trunk $P_a$. Thus we choose $Y_{m+1}$ to contain a vertex of $xP$, so $Y$ intersects $xP$ as desired.

\begin{proof}
Consider the formula $\varphi(G, Y, P, x)$ which says there exists a number $z$ such that

\end{proof}

Lemma 4.3 (in $\mathbb{A} \mathbb{C}_{0}$). If $\mathcal{M}$ is a countable coded $\omega$-model of $\Sigma^1_1$-$\mathcal{D} \mathcal{C}_{0}$, then Lemma 4.2 holds in $\mathcal{M}$.

Proof. Consider the formula $\varphi(G, Y, P, x)$ which says there exists a number $z$ such that
\( (i) \) \( z \) codes a finite subset of \( V(P) \),

\( (ii) \)

\[ \forall y (\forall Y' (Y' \text{ is a wave } \geq Y \text{ in } G - \overline{P}y \Rightarrow \text{ess}_{G - \overline{P}y}(Y') = \text{ess}_G(Y)) \Rightarrow y \in z \),

\( (iii) \)

\[ \forall s (s \text{ codes a finite set } \wedge \forall y (\exists Y' (Y' \text{ is a wave } \geq Y \text{ in } G - \overline{P}y \wedge \text{ess}_{G - \overline{P}y}(Y') \subseteq \text{ess}_G(Y)) \Rightarrow y \in s) \]

\[ \rightarrow V(P) - z \subseteq s \), \text{ and} \]

\( (iv) \) \( x \) is the first vertex on \( P \) not in \( z \).

The reason for the somewhat convoluted definition of \( \varphi \) is that prefixing this \( \varphi \) yields a \( \Sigma_1^1 \) formula.

**Claim.** In \( \mathcal{M} \), suppose that \( G = (G, A, B) \) is a web, \( Y \) is a wave in \( G \), and \( P \) is a finite path in \( G \) disjoint from \( V(Y) \) such that \( Y \) has no bad extensions in \( G \) but there exists a wave \( Y' \geq Y \) in \( G - P \) with \( \text{ess}_{G - P}(Y') \subseteq \text{ess}_G(Y) \). Then \( \mathcal{M} \models \varphi(G,Y,P,x) \) if and only if, in \( \mathcal{M} \), \( x \) is the first vertex on \( P \) such that there exists a wave \( Y' \geq Y \) in \( G - Px \) with \( \text{ess}_{G - Px}(Y') \subseteq \text{ess}_G(Y) \).

**Proof of claim.** For the forward direction, using \( \text{ACA}_0 \) outside \( \mathcal{M} \), let

\[ Z = \{ y \in V(P) \mid \text{ess}_{G - \overline{P}y}(Y') = \text{ess}_G(Y) \text{ for all waves } Y' \geq Y \text{ in } G - \overline{P}y \text{ that are in } \mathcal{M} \} \]

Let \( z \) be a number coding \( Z \) and let \( s \) be a number coding \( V(P) - Z \). In \( \mathcal{M} \), \( z \) and \( s \) code the same sets that they do outside of \( \mathcal{M} \), and \( \mathcal{M} \) interprets that

\( z \) codes \( \{ y \in V(P) \mid \text{ess}_{G - \overline{P}y}(Y') = \text{ess}_G(Y) \text{ for all waves } Y' \geq Y \text{ in } G - \overline{P}y \} \)

\( s \) codes \( \{ y \in V(P) \mid \text{ess}_{G - \overline{P}y}(Y') \subseteq \text{ess}_G(Y) \text{ for some wave } Y' \geq Y \text{ in } G - \overline{P}y \} \).

Hence in \( \mathcal{M} \), this \( z \) is the only \( z \) which satisfies items \( (i)-(iii) \). Thus if \( \varphi(G,Y,P,x) \) holds in \( \mathcal{M} \), \( x \) must be the first vertex on \( P \) not in \( z \) for this \( z \). Thus \( x \) must be the first vertex on \( P \) such that, in \( \mathcal{M} \), there exists a wave \( Y' \geq Y \) in \( G - P - x \) with \( \text{ess}_{G - Px}(Y') \subseteq \text{ess}_G(Y) \).

For the converse, by using \( \text{ACA}_0 \) outside of \( \mathcal{M} \), let \( x \) be the first vertex on \( P \) such that, in \( \mathcal{M} \), there exists a wave \( Y' \geq Y \) in \( G - P - x \) with \( \text{ess}_{G - Px}(Y') \subseteq \text{ess}_G(Y) \). Let \( z \) be a number coding \( V(Px) \). In \( \mathcal{M} \), \( z \) also codes \( V(Px) \), and this \( z \) witnesses \( \mathcal{M} \models \varphi(G,Y,P,x) \). \( \square \)

Suppose for a contradiction that Lemma 4.2 is false in \( \mathcal{M} \) and, in \( \mathcal{M} \), let \( W, G = (G, A, B) \), and \( P_0 \) be a counterexample to Lemma 4.2. We use \( \Sigma_1 \text{-DC}_0 \) in \( \mathcal{M} \) to run the construction from Lemma 4.2. This produces in \( \mathcal{M} \) a bad extension of \( W \) in \( G \), which is a contradiction.

We apply \( \Sigma_1 \text{-DC}_0 \) to the formula \( \eta(n,X,Y) \) below. Our \( \eta \) has fixed parameters \( G, W, P_a, \) and \( (Q_n \mid n \in \mathbb{N}) \) (a list of all \( \text{ter}(P_a) \)-\( B \) paths with each occurring infinitely often). We think of a set \( Y \subseteq \mathbb{N} \) as coding a pair \( Y = (tY, wY) \) where \( wY \) is a wave in \( G - tY \). Formally, \( tY = (Y)_0 \) and \( wY = (Y)_1 \). Our formula \( \eta(n,X,Y) \) says that if \( t(X)_{n-1} \) is the tree and \( w(X)_{n-1} \) is the wave constructed at stage \( n - 1 \) in Lemma 4.2, then \( tY \) is the tree and \( wY \) is the wave constructed at stage \( n \) in Lemma 4.2. Formally, \( \eta(n,X,Y) \) says:

- If \( n = 0 \), then \( tY = P_a \) and \( wY = W - P_a \).
- If \( n > 0 \), then \( t(X)_{n-1} \) is a finite tree with trunk \( P_a \) such that \( V(t(X)_{n-1}) \cap (V(W) - V(P_a)) = \emptyset \), \( w(X)_{n-1} \geq W - P_a \) is a wave in \( G - t(X)_{n-1} \), \( \text{ess}_{G - t(X)_{n-1}}(w(X)_{n-1}) = \text{ess}_G(W) - \{ a \} \), and \( w(X)_{n-1} \) has no bad extensions in \( G - t(X)_{n-1} \), then
  - if \( V(Q_{n-1}) \cap V(w(X)_{n-1}) \neq \emptyset \) or \( t(X)_{n-1} \cup Q_{n-1} \) is not a tree, then \( tY = t(X)_{n-1} \) and \( wY = w(X)_{n-1} \), and
  - if \( V(Q_{n-1}) \cap V(w(X)_{n-1}) = \emptyset \) and \( t(X)_{n-1} \cup Q_{n-1} \) is a tree, then there is an \( x \) such that \( tY = t(X)_{n-1} \cup Q_{n-1} x \), \( wY \) is a wave in \( G - tY \), \( x \in \text{ter}(wY) \), \( wY \geq w(X)_{n-1} \), and \( \varphi(G - t(X)_{n-1}, w(X)_{n-1}, Q_{n-1}, x) \).
Prenexing \( \eta \) yields a \( \Sigma^1_1 \) formula. To see this, observe that all the subformulas of \( \eta \) are arithmetical, with the exception of “\( w(X)_{n-1} \) has no bad extensions in \( G - t(X)_{n-1} \)” which is \( \Pi^1_1 \) and appears in the antecedent of \( \eta \), and \( \varphi \), which is \( \Sigma^1_1 \) and appears in the consequent of \( \eta \).

We show that \( M \models \forall n \forall X \exists Y \eta(n, X, Y) \). The interesting case is when \( n > 0 \) and \( X \in M \) is such that, in \( M \),

- \( t(X)_{n-1} \) is a finite tree in \( G \) with trunk \( P_a \) such that \( V(t(X)_{n-1}) \cap (V(W) - V(P_a)) = \emptyset \),
- \( w(X)_{n-1} \geq W - P_a \) is a wave in \( G - t(X)_{n-1} \),
- \( \text{ess}_{G - t(X)_{n-1}}(w(X)_{n-1}) = \text{ess}_{G}(W) - \{a\} \),
- \( w(X)_{n-1} \) has no bad extensions in \( G - t(X)_{n-1} \),
- \( V(Q_n) \cap V(w(X)_{n-1}) = \emptyset \), and
- \( t(X)_{n-1} \cup Q_{n-1} \) is a tree in \( G \) with trunk \( P_a \).

By applying \( \mathsf{ACA}_0 \) outside \( M \), let \( x \) be the first vertex on \( Q_{n-1} \) such that, in \( M \), there is a wave \( Z \geq w(X)_{n-1} \) in \( G - (t(X)_{n-1} \cup Q_{n-1}) \) with \( \text{ess}_{G - (t(X)_{n-1} \cup Q_{n-1})}(Z) \subseteq \text{ess}_{G - t(X)_{n-1}}(w(X)_{n-1}) \).

Such an \( x \) exists by the assumption that \( G, W, \) and \( P_a \) are a counterexample to Lemma 4.2 in \( M \). By the claim, \( \varphi(G - t(X)_{n-1}, w(X)_{n-1}, Q_{n-1}, x) \) holds in \( M \). As \( M \models \mathsf{ACA}_0 \), apply Lemma 3.12 inside \( M \) to get a wave \( Z \geq t(X)_{n-1} \) in \( G - (t(X)_{n-1} \cup Q_{n-1}) \) with \( x \in \text{ter}(Z) \). Set \( tY = t(X)_{n-1} \cup Q_{n-1} - x \) and \( wY = Z \) to get a \( Y \in M \) witnessing \( \exists Y \eta(n, X, Y) \).

Now apply \( \Sigma^1_1 \)-\( \mathsf{DC}_0 \) inside \( M \) to conclude that \( M \models \exists Z \forall n \exists \eta(n, (Z)^n, (Z)_n) \), and let \( Z \in M \) be such a \( Z \). By induction, for all \( n \in \mathbb{N} \),

- \( t(Z)_n \) is a finite tree in \( G \) with trunk \( P_a \) such that \( V(t(Z)_n) \cap (V(W) - V(P_a)) = \emptyset \),
- \( w(Z)_n \geq W - P_a \) is a wave in \( G - t(Z)_n \),
- \( t(Z)_{n-1} \subseteq t(Z)_n \) (if \( n > 0 \)),
- \( w(Z)_{n-1} \leq w(Z)_n \) (if \( n > 0 \)),
- \( \text{ess}_{G - t(Z)}(w(Z)_n) = \text{ess}_{G}(W) - a_0 \),
- \( w(Z)_n \) has no bad extensions in \( G - t(Z)_n \) that are in \( M \),

and additionally if \( V(Q_n) \cap V(w(Z)_n) = \emptyset \) then \( t(Z)_n \cup Q_n \) is a finite tree in \( G \) with trunk \( P_a \), then \( \text{ter}(w(Z)_{n+1}) \) contains a vertex of \( Q_n \). Notice that although the statement “\( w(Z)_n \) has no bad extensions in \( G - t(Z)_n \)” that are in \( M \) is \( \Pi^1_1 \) from \( M \)’s perspective, it is arithmetical outside of \( M \) because quantifying over the sets inside \( M \) from outside \( M \) amounts to quantifying over (the indices of) the columns of the set coding \( M \). Thus the above-listed properties can be proven (in \( \mathsf{ACA}_0 \)) outside of \( M \) by induction using an appropriate arithmetical formula.

Inside \( M \), let \( T = \bigcup_{n \in \mathbb{N}} t(Z)_n \) and \( Y = \bigcup_{n \in \mathbb{N}} w(Z)_n \). Just as in the proof of Lemma 4.2, \( Y \) is a wave in \( G - T \) and \( \text{ter}(Y) \) is an \( A-B \) separator in \( G \). Thus \( Y \cup \{P_a\} \in M \) is the desired bad extension of \( W \) in \( G \), which gives the contradiction.

\textbf{Theorem 4.4.} Menger’s theorem for countable webs is provable in \( \Pi^1_1 \)-\( \mathsf{CA}_0 \).

\textit{Proof.} Let \( G = (G, A, B) \) be a countable web. By Lemma 4.1, let \( W = \{P_a \mid a \in A\} \) be a \( \leq \)-maximal wave in \( G \). Let \( C = \{\text{ter}(P_a) \mid a \in \text{ess}_G(W)\} \). We extend the paths in \( \{P_a \mid a \in \text{ess}_G(W)\} \) to be a collection of disjoint \( A-B \) paths \( M \). \( M \) and \( C \) then witness Menger’s theorem for \( G \).

By Theorem 2.4, let \( M \) be a countable coded \( \omega \)-model of \( \Pi^1_1 \)-\( \mathsf{DC}_0 \) containing \( G \) and \( W \). By Lemma 4.3, Lemma 4.2 holds in \( M \). Also, \( M \models \text{“} W \text{ is a } \leq \text{-maximal wave”} \), therefore \( M \models \text{“} W \text{ has no bad extensions in } G \text{”} \) because \( W \) has no proper extensions in \( G \) whatsoever. Let \( \{a_n \mid n \in \mathbb{N} \} \) enumerate \( \text{ess}_G(W) \). Outside \( M \), we construct sequences \( (X_n \mid n \in \mathbb{N}), (Y_n \mid n \in \mathbb{N}) \), and \( (Q_n \mid n \in \mathbb{N}) \) such that, for all \( n \in \mathbb{N} \),

- \( X_n \in M, Y_n \in M, \) and \( Q_n \in M, \)
- \( X_n \subseteq V(G) \) is a finite set, \( X_n \cap A = \{a_i \mid i \leq n\}, \) and \( X_n \subseteq X_{n+1}, \)
- \( Y_n \) is a wave in \( G - X_n \) such that \( Y_n \geq W \setminus \bigcup_{i \leq n} P_{a_i}, \) \( Y_n \) has no bad extensions in \( G - X_n, \)
- and \( \text{ess}_{G - X_n}(Y_n) = \{a_i \mid i > n\}, \) and
- \( Q_n \) is an \( A-B \) path extending \( P_{a_n}. \)
To get started, by Lemma 4.3 let $T \in \mathcal{M}$ be a finite $a_0$-$B$ tree in $G$ with trunk $P_{a_0}$ such that $V(T) \cap (V(W) - V(P_{a_0})) = \emptyset$, and let $Y_0 \in \mathcal{M}$ be a wave in $G - T$ such that $Y_0 \geq W - P_{a_0}$, $\text{ess}_{G - T}(Y_0) = \{a_i \mid i > 0\}$, and $Y_0$ has no bad extensions in $G - T$. Let $X_0 = T$, and let $Q_0$ be the $a_0$-$B$ path in $T$. Suppose we have $X_n$, $Y_n$ and $Q_n$. Let $P'_{a_{n+1}}$ be the path in $Y_n$ starting at $a_{n+1}$, and note that $P'_{a_{n+1}}$ extends $P_{a_{n+1}}$. By Lemma 4.3, let $T \in \mathcal{M}$ be a finite $a_{n+1}$-$B$ tree in $G - X_n$ with trunk $P'_{a_{n+1}}$ such that $V(T) \cap (V(Y_n) - V(P'_{a_{n+1}})) = \emptyset$, and let $Y_{n+1} \in \mathcal{M}$ be a wave in $G - (X_n \cup T)$ such that $Y_{n+1} \geq Y_n - P'_{a_{n+1}}$, $\text{ess}_{G - (X_n \cup T)}(Y_{n+1}) = \{a_i \mid i > n + 1\}$, and $Y_{n+1}$ has no bad extensions in $G - (X_n \cup T)$. Let $X_{n+1} = X_n \cup T$ and let $Q_{n+1}$ be the $a_{n+1}$-$B$ path in $T$. In the end, the collection $M = \{Q_n \mid n \in \mathbb{N}\}$ consists of disjoint $A$-$B$ paths in $G$, and $C = \{\text{ter}(P_a) \mid a \in \text{ess}_G(W)\}$ is an $A$-$B$ separator containing exactly one vertex from each path in $M$.

\[\square\]

5. Extended Menger’s theorem

Although Menger’s theorem for countable webs cannot be equivalent to $\Pi_1^1$-$\text{CA}_0$ over $\text{RCA}_0$ as discussed in the introduction, the proof given in Theorem 4.4 is equivalent to $\Pi_1^1$-$\text{CA}_0$ in the sense that it proves a stronger statement, called extended Menger’s theorem, that is equivalent to $\Pi_1^1$-$\text{CA}_0$ over $\text{RCA}_0$. This additional strength comes from our application of Lemma 4.1.

Extended Menger’s Theorem. Let $(G, A, B)$ be a countable web. Then there is a set of disjoint $A$-$B$ paths $M$ and an $A$-$B$ separator $C$ such that $C$ consists of exactly one vertex from each path in $M$. Furthermore, $C$ is the set of terminal vertices of the essential paths in a $\leq$-maximal wave.

Let $G$ be a graph. For $x \in V(G)$ and $X \subseteq V(G)$, $N(x) = \{y \in V(G) \mid (x, y) \in E(G)\}$ denotes the set of neighbors of $x$ and $D(X) = \{y \in V(G) \mid N(y) \subseteq X\}$ denotes the demand of $X$. In the proof of König’s duality theorem for countable bipartite graphs in [4], the following lemma plays the role that Lemma 4.1 plays in the proof of Menger’s theorem for countable webs given in Theorem 4.4.

Lemma 5.1 ([4] Lemma 3.2). Let $(X, Y, E)$ be a countable bipartite graph. Then there is a $\leq$-maximum $Y_0 \subseteq Y$ for which there is a matching of $Y_0$ into $D(Y_0)$.

The application of Lemma 5.1 yields a stronger form of König’s duality theorem, called extended König’s duality theorem.

Extended König’s Duality Theorem. Let $(X, Y, E)$ be a countable bipartite graph. Then there is a matching $M$ and a cover $C$ such that $C$ consists of exactly one vertex from each edge in $E$. Furthermore, for every $y \in Y$, $y \in C$ if and only if there is a $Y_0 \subseteq Y$ containing $y$ and a matching of $Y_0$ into $D(Y_0)$.

Extended König’s duality theorem is equivalent to $\Pi_1^1$-$\text{CA}_0$ over $\text{RCA}_0$ by [4] Theorem 4.18. In fact, Lemma 5.1 itself is equivalent to $\Pi_1^1$-$\text{CA}_0$ over $\text{RCA}_0$ by [4] Corollary 4.20. In contrast, recall from the introduction that König’s duality theorem is equivalent to $\text{ATR}_0$ over $\text{RCA}_0$. We show that the existence of a $\leq$-maximal wave, that is, Lemma 4.1, implies Lemma 5.1 over $\text{RCA}_0$. It follows that both Lemma 4.1 and extended Menger’s theorem are equivalent to $\Pi_1^1$-$\text{CA}_0$ over $\text{RCA}_0$.

Lemma 5.2. Lemma 4.1 implies Lemma 5.1 over $\text{RCA}_0$.

Proof. We prove the lemma in two steps. First, we prove that Lemma 4.1 implies $\text{ACA}_0$ over $\text{RCA}_0$. Second, we prove that Lemma 4.1 implies Lemma 5.1 over $\text{ACA}_0$.

First work in $\text{RCA}_0$. We use the fact that $\text{ACA}_0$ is equivalent to the statement “for every injection $f \colon \mathbb{N} \rightarrow \mathbb{N}$ there is a $Z \subseteq \mathbb{N}$ such that $\forall n(m \in Z \leftrightarrow \exists m(f(m) = n))$” (see [9] Lemma III.1.3). So let $f \colon \mathbb{N} \rightarrow \mathbb{N}$ be an injection. Let $(X, Y, E)$ be the bipartite graph with sides $X = \{x_n \mid n \in \mathbb{N}\}$ and $Y = \{y_n \mid n \in \mathbb{N}\}$ and edges $E = \{(x_n, y_n) \mid f(m) = n\}$. Let $G$ be the web $G = ((X, Y, E), X, Y)$, and by Lemma 4.1 let $W$ be a $\leq$-maximal wave in $G$. Let $Z = \{n \mid y_n \in V(W)\}$. We show that $\forall n(n \in Z \leftrightarrow \exists m(f(m) = n))$. If $f(m) = n$, then $(x_m, y_n)$ is the only edge incident to either $x_n$ or
Corollary 5.3. Lemma 4.1 is equivalent to $\Pi^1_1$-$\text{CA}_0$ over RCA$_0$.

Proof. The given proof of Lemma 4.1 is in $\Pi^1_1$-$\text{CA}_0$. By Lemma 5.2, Lemma 4.1 implies Lemma 5.1 over RCA$_0$. By [4] Corollary 4.20, Lemma 5.1 is equivalent to $\Pi^1_1$-$\text{CA}_0$ over RCA$_0$. 

Corollary 5.4. Extended Menger’s theorem is equivalent to $\Pi^1_1$-$\text{CA}_0$ over RCA$_0$.

Proof. Theorem 4.4 proves extended Menger’s theorem in $\Pi^1_1$-$\text{CA}_0$. Extended Menger’s theorem asserts the existence of a $\leq$-maximal wave, which is equivalent to $\Pi^1_1$-$\text{CA}_0$ over RCA$_0$ by Corollary 5.3.

Acknowledgements

Thanks to my advisor Richard A. Shore for introducing me to reverse mathematics and for many helpful discussions.

References


Department of Mathematical Sciences, Appalachian State University, 121 Bodenheimer Drive, Walker Hall, Boone, NC 28608, USA

E-mail address: shaferpe@appstate.edu

URL: http://www.appstate.edu/~shaferpe/