CHARACTERIZING THE JOIN-IRREDUCIBLE MEDVEDEV DEGREES

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ABSTRACT. We characterize the join-irreducible Medvedev degrees as the degrees of complements of Turing ideals, thereby solving a problem posed by Sorbi. We use this characterization to prove that there are Medvedev degrees above the second-least degree that do not bound any join-irreducible degrees above this second-least degree. This solves a problem posed by Sorbi and Terwijn. Finally, we prove that the filter generated by the degrees of closed sets is not prime. This solves a problem posed by Bianchini and Sorbi.

1. Introduction

We present solutions to three problems concerning the Medvedev degrees. A mass problem is a set \( \mathcal{A} \subseteq \omega^\omega \). For mass problems \( \mathcal{A} \) and \( \mathcal{B} \), we say that \( \mathcal{A} \) Medvedev reduces to \( \mathcal{B} \) (\( \mathcal{A} \leq_M \mathcal{B} \)) if there is a Turing functional \( \Phi \) such that \( \Phi(\mathcal{B}) \subseteq \mathcal{A} \). That is, \( \Phi(f) \in \mathcal{A} \) for all \( f \in \mathcal{B} \). We say that \( \mathcal{A} \) and \( \mathcal{B} \) are Medvedev equivalent (\( \mathcal{A} \equiv_M \mathcal{B} \)) if \( \mathcal{A} \leq_M \mathcal{B} \) and \( \mathcal{B} \leq_M \mathcal{A} \). The equivalence class \( [\mathcal{A}] \) is called the Medvedev degree of \( \mathcal{A} \), and the structure \( \mathfrak{M} = (2^\omega/\equiv_M, \leq_M) \) is called the Medvedev degrees. See Sorbi [15] for a good introduction to the theory of the Medvedev degrees.

For \( f, g \in \omega^\omega \), let \( f \oplus g \) be the function \((f \oplus g)(2n) = f(n) \) and \((f \oplus g)(2n+1) = g(n)\). For \( m \in \omega \) and \( f \in \omega^\omega \), let \( m \cdot f \) be the function \((m \cdot f)(0) = m \) and \((m \cdot f)(n+1) = f(n)\). In general, \( \cdot \) denotes string concatenation. Functions \( f \in \omega^\omega \) are interpreted as \( \omega \)-length strings when appropriate. For a mass problem \( \mathcal{A} \), let \( \cdot \mathcal{A} = \{m \cdot f \mid f \in \mathcal{A} \} \). Given mass problems \( \mathcal{A} \) and \( \mathcal{B} \), let \( \mathcal{A} + \mathcal{B} = \{ f \oplus g \mid f \in \mathcal{A} \land g \in \mathcal{B} \} \) and let \( \mathcal{A} \times \mathcal{B} = 0^\mathcal{A} \cup 1^\mathcal{B} \). Then \( [\mathcal{A}] + [\mathcal{B}] = [\mathcal{A} + \mathcal{B}] \) is the join (i.e., \( \leq_M \)-least upper bound) of \([\mathcal{A}] \) and \([\mathcal{B}] \), while \([\mathcal{A}] \times [\mathcal{B}] = [\mathcal{A} \times \mathcal{B}] \) is the meet (i.e., \( \leq_M \)-greatest lower bound) of \([\mathcal{A}] \) and \([\mathcal{B}] \). Hence \( \mathfrak{M} \) is a lattice. In fact, \( \mathfrak{M} \) is a distributive lattice, meaning that join and meet distribute over each other: \( a + (b \times c) = (a + b) \times (a + c) \) and \( a \times (b + c) = (a \times b) + (a \times c) \). Notation for join and meet appears in the literature variously as +, ×, as \( \lor, \land \), and confusingly as \( \lor, \land \). We choose the +, × notation to avoid conflict with the logical notation and to match Sorbi and Terwijn [16].

\( \mathfrak{M} \) has a least element \( 0 = [\omega^\omega] \) (and any \( \mathcal{A} \) containing a recursive function has this degree), a second-least element \( 0' = [\{ f \mid f >_T 0 \}] \), and a greatest element \( 1 = [\emptyset] \). (The Medvedev degree \( 0' \) has little to do with \( 0' \), the Turing jump of the 0 function. Here \( 0' \) always refers to the second-least Medvedev degree.)

In any lattice, an element \( a \) is called join-reducible if there are \( x, y < a \) such that \( a = x + y \). Otherwise \( a \) is called join-irreducible. Dually, \( a \) is called meet-reducible if there are \( x, y > a \) such that \( a = x \times y \). Otherwise \( a \) is called meet-irreducible. Dyment [3] characterized the meet-reducible Medvedev degrees in the following theorem. Its corollary helps identify meet-irreducible Medvedev degrees.

**Theorem 1.1** ([3]). A Medvedev degree \( a \) is meet-reducible if and only if \( a = [\mathcal{A}] \) for a mass problem \( \mathcal{A} \) for which there are r.e. sets \( V_0, V_1 \subseteq \omega^\omega \) such that

- \( (\forall f \in \mathcal{A})(\exists \sigma \in V_0 \cup V_1)(\sigma \subseteq f) \),
- The following mass problems are \( \leq_M \)-incomparable:

\[ \{ f \in \mathcal{A} \mid (\exists \sigma \in V_0)(\sigma \subseteq f) \} \text{ and } \{ f \in \mathcal{A} \mid (\exists \sigma \in V_1)(\sigma \subseteq f) \} \]
Corollary 1.2 ([3]). If $\mathcal{A}$ is a mass problem such that $\sigma \preceq \mathcal{A} \subseteq \mathcal{A}$ for all $\sigma \in \omega^\omega$, then $[\mathcal{A}]$ is meet-irreducible.

In particular, $0'$ is meet-irreducible because $\sigma \preceq f >_T 0$ whenever $\sigma \in \omega^\omega$ and $f >_T 0$.

The problem of characterizing the join-irreducible Medvedev degrees was posed in [15]. In Section 2, we prove that $a \in \mathfrak{M}$ is join-irreducible if and only if $a = [\omega^\omega - \mathcal{I}]$ for some Turing ideal $\mathcal{I}$.

We have seen that $\mathfrak{M}$ is a distributive lattice with $0$ and $1$. In fact, $\mathfrak{M}$ is a Brouwer algebra. A Brouwer algebra is a distributive lattice with $0$ and $1$ such that for every $a$ and $b$ there is a least $c$ such that $a + c \geq b$. This least $c$ is denoted by $a \rightarrow b$. For mass problems $\mathcal{A}$ and $\mathcal{B}$, define $\mathcal{A} \rightarrow \mathcal{B} = \{ e^c g \mid (\forall f \in \mathcal{A})((\Phi_e(f \oplus g) \in \mathcal{B})\}$. Then $[\mathcal{A}] \rightarrow [\mathcal{B}] = [\mathcal{A} \rightarrow \mathcal{B}]$. A Brouwer algebra is dual to a Heyting algebra, but $\mathfrak{M}$ is proved not to be a Heyting algebra in Sorbi [12].

Brouwer algebras give semantics for propositional logic. For any Brouwer algebra $\mathfrak{B}$, a valuation is a function $\nu$: propositional variables $\rightarrow \mathfrak{B}$. A valuation $\nu$ extends to all propositional formulas $\varphi$ by defining

$$\nu(\varphi \land \psi) = \nu(\varphi) + \nu(\psi),$$
$$\nu(\varphi \lor \psi) = \nu(\varphi) \times \nu(\psi),$$
$$\nu(\varphi \rightarrow \psi) = \nu(\varphi) \rightarrow \nu(\psi),$$
and
$$\nu(\neg \varphi) = \nu(\varphi) \rightarrow 1.$$  

A propositional formula $\varphi$ is called valid in $\mathfrak{B}$ if $\nu(\varphi) = 0$ for every valuation $\nu$. Let $\text{Th}(\mathfrak{B})$ denote the set of propositional formulas valid in $\mathfrak{B}$. The axioms of intuitionistic logic are valid in every Brouwer algebra $\mathfrak{B}$, so $\text{IPC} \subseteq \text{Th}(\mathfrak{B}) \subseteq \text{CPC}$ for every Brouwer algebra $\mathfrak{B}$. Here $\text{IPC}$ denotes intuitionistic logic and $\text{CPC}$ denotes classical logic. Logics $L$ for which $\text{IPC} \subseteq L \subseteq \text{CPC}$ are called intermediate logics.

Providing semantics for propositional logic was one of Medvedev’s main motivations behind introducing $\mathfrak{M}$, and he proved $\text{Th}(\mathfrak{M}) = \text{JAN}$ in Medvedev [8]. JAN denotes the logic $\text{IPC} + \neg p \lor \neg \neg p$ named after Jankov who studied it in Jankov [5]. In any Brouwer algebra $\mathfrak{B}$, the quotient of $\mathfrak{B}$ by the principal filter generated by $a \in \mathfrak{B}$ is denoted by $\mathfrak{B} / a$. The quotient $\mathfrak{B} / a$ is isomorphic to the interval $[0, a]$ which is a Brouwer algebra under the operations inherited from $\mathfrak{B}$. Logics of the form $\text{Th}(\mathfrak{M} / a)$ have been studied in Skvortsova [10], Sorbi [14], and Sorbi and Terwijn [16]. (Skvortsova and Dyment are the same person. Dyment got married and became Skvortsova.) The results in Section 3 and Section 4 are motivated by the following question which remains open:

**Question 1.3 ([16]).** Is $\text{Th}(\mathfrak{M} / a) \subseteq \text{JAN}$ for all $a >_M 0'$?

Sorbi and Terwijn’s study of Question 1.3 in [16] lead them to ask whether every degree $>_M 0'$ bounds a join-irreducible degree $>_M 0'$ because a “yes” answer to this question implies a “yes” answer to Question 1.3. However, Sorbi and Terwijn conjectured that there is a degree $>_M 0'$ that bounds no join-irreducible degree $>_M 0'$, and we prove that this is correct in Section 3. In Section 4 we provide slight extensions to some of the results in [14], thereby widening the class of degrees $a$ for which $\text{Th}(\mathfrak{M} / a) \subseteq \text{JAN}$ is known.

Lastly, in Section 5 we use techniques similar to those used to characterize the join-irreducible degrees to prove that the filter generated by the degrees of mass problems closed in $\omega^\omega$ is not prime. This problem was posed in Bianchini and Sorbi [2] and in Sorbi [15].

2. Characterizing the Join-Irreducible Medvedev Degrees

A Turing ideal is a set $\mathcal{I} \subseteq \omega^\omega$ that is closed downward under $\leq_T$ (i.e., $f \in \mathcal{I} \land g \leq_T f \rightarrow g \in \mathcal{I}$) and closed under $\oplus$ (i.e., $f, g \in \mathcal{I} \rightarrow f \oplus g \in \mathcal{I}$). We prove that $a \in \mathfrak{M}$ is join-irreducible if and only if $a = [\omega^\omega - \mathcal{I}]$ for some Turing ideal $\mathcal{I}$. We frequently use the following well-known lemma without mention:

\[ \text{Lemma:} \quad \forall \varphi \in \mathfrak{B}, \quad \nu(\varphi) = 0 \implies \varphi \text{ is valid in } \mathfrak{B}. \]
Lemma 2.1 (see [1] Section III.2). In a distributive lattice, \( a \) is join-irreducible if and only if for all \( x \) and \( y \), \( a \leq x + y \) implies \( a \leq x \) or \( a \leq y \). Dually, \( a \) is meet-irreducible if and only if for all \( x \) and \( y \), \( a \geq x \times y \) implies \( a \geq x \) or \( a \geq y \).

Proof. Suppose \( a \) is join-irreducible and \( a \leq x + y \). Then
\[
a = a \times (x + y) = (a \times x) + (a \times y).
\]
Thus \( a = a \times x \) or \( a = a \times y \) which means \( a \leq x \) or \( a \leq y \). Conversely, if \( a \) is join-reducible, then by definition there are \( x, y < a \) with \( a = x + y \). The proof for the meet-irreducible case is obtained by dualizing the proof for the join-irreducible case.

For a mass problem \( A \), let \( C(A) \) denote the Turing upward-closure of \( A \): \( C(A) = \{ f \mid (\exists g \in A)(f \geq_T g) \} \). A mass problem \( A \) is called Turing upward-closed if \( A = C(A) \). The identity functional witnesses \( C(A) \subseteq M A \) for any mass problem \( A \), and if \( A \) and \( B \) are mass problems such that \( A \) is Turing upward-closed, then \( A \leq_M B \) if and only if \( B \subseteq A \). Our starting point is the following observation:

Lemma 2.2 ([15]). If \( A \) is a mass problem such that \( [A] \) is join-irreducible, then \( \omega^\omega - C(A) \) is a Turing ideal.

Proof. We prove the contrapositive. If \( \omega^\omega - C(A) \) is not a Turing ideal, then there are \( f, g \notin C(A) \) with \( f \oplus g \in C(A) \). This means that \( \{ f \}, \{ g \} \nsubseteq M A \) but \( \{ f \} + \{ g \} \geq_M A \). Thus \( [A] \) is join-reducible.

The next lemma is the main step in our characterization.

Lemma 2.3. If \( A \) is a mass problem such that \( [A] \) is join-irreducible, then \( A \cong M C(A) \)

Proof. We prove the contrapositive. Suppose \( A \nsubseteq M C(A) \). Then it must be that \( A \nsubseteq C(A) \). We find mass problems \( X \) and \( Y \) such that \( X \), \( Y \nsubseteq M A \) but \( X + Y \geq_M A \). Thus \( [A] \) is join-reducible.

To find \( X \) and \( Y \), first find a sequence \( \langle h_n \mid n \in \omega \rangle \) of functions and a sequence \( \langle e_n \mid n \in \omega \rangle \) of indices such that
(i) \( \Phi_{e_n}(h_n) \in A \) for all \( n \in \omega \),
(ii) \( \Phi_{e_n}(h_{2n}) \notin A \) and \( \Phi_{e_n}(h_{2n+1}) \notin A \) for all \( n \in \omega \), and
(iii) \( h_n(0) = \langle n, e_0, e_1, \ldots, e_{n-1} \rangle \) for all \( n \in \omega \).

We find the desired sequences by iterating the following claim:

Claim. If \( A \nsubseteq M C(A) \), then for every \( e, m \in \omega \) there is an \( h \in C(A) \) such that \( h(0) = m \) and \( \Phi_e(h) \notin A \).

Proof of claim. Suppose not. Then there are \( e, m \in \omega \) such that \( h(0) = m \) implies \( \Phi_e(h) \notin A \) for all \( h \in C(A) \). Thus \( h \mapsto \Phi_e(h) \) is a reduction witnessing \( A \nsubseteq M C(A) \), a contradiction.

Suppose we have \( h_i \) and \( e_i \) for all \( i < n \). To find \( h_n \) and \( e_n \), let \( e = \lfloor n/2 \rfloor \) and let \( m = \langle n, e_0, e_1, \ldots, e_{n-1} \rangle \). By the claim, there is an \( h_n \in C(A) \) such that \( h_n(0) = m \) and \( \Phi_e(h_n) \notin A \). The fact that \( h_n \in C(A) \) means that there is an \( e_n \) such that \( \Phi_{e_n}(h_n) \in A \).

Now set \( X = \{ h_{2n} \mid n \in \omega \} \) and \( Y = \{ h_{2n+1} \mid n \in \omega \} \). Then \( \Phi_{e}(X) \nsubseteq A \) and \( \Phi_{e}(Y) \nsubseteq A \) for each \( e \) by item (ii). Hence \( X, Y \nsubseteq M A \). The following reduction witnesses \( X + Y \geq_M A \).

Given \( h \), decompose \( h \) as \( h = f \oplus g \) and decode \( f(0) \) and \( g(0) \) as \( f(0) = \langle 2n, x_0, x_1, \ldots, x_{2n-1} \rangle \) and \( g(0) = \langle 2m+1, y_0, y_1, \ldots, y_{2m} \rangle \). If either \( f(0) \) or \( g(0) \) is not of the required form, then output the 0 function (as such an \( h \) cannot be in \( X + Y \)). Otherwise output \( \Phi_{x_{2n+1}}(g) \) if \( 2n > 2m + 1 \) and output \( \Phi_{y_{2m}}(f) \) if \( 2m + 1 > 2n \).

Suppose this reduction is applied to some \( h = h_{2n} \oplus h_{2m+1} \in X + Y \). In this case \( f = h_{2n} \), \( g = h_{2m+1} \), and \( f(0) \) and \( g(0) \) are of the required form by item (iii). So if \( 2n > 2m + 1 \) we output \( \Phi_{e_{2m+1}}(h_{2m+1}) \) and if \( 2m + 1 > 2n \) we output \( \Phi_{e_{2n}}(h_{2n}) \). Both alternatives are in \( A \) by item (i). Thus \( X + Y \geq_M A \).
Theorem 2.4. A Medvedev degree $a$ is join-irreducible if and only if $a = [\omega^\omega - I]$ for some Turing ideal $I$.

Proof. Suppose $a$ is join-irreducible, and let $A$ be a mass problem such that $a = [A]$. Then $I = \omega^\omega - C(A)$ is a Turing ideal by Lemma 2.2, $A \equiv_M C(A)$ by Lemma 2.3, and therefore $A \equiv_M C(A) = \omega^\omega - I$. Hence $a = [\omega^\omega - I]$ for the Turing ideal $I$.

Conversely, suppose $I$ is a Turing ideal and let $X$ and $Y$ be mass problems such that $X, Y \not\leq_M \omega^\omega - I$. We show that $X + Y \not\leq_M \omega^\omega - I$. Observe $X, Y \not\leq \omega^\omega - I$ for otherwise the identity functional would witness $X, Y \geq_M \omega^\omega - I$. Let $f \in X \cap I$ and let $g \in Y \cap I$, thereby making $f \oplus g \in (X + Y) \cap I$. The function $f \oplus g$ is in $X + Y$, but it does not compute any member of $\omega^\omega - I$. Therefore $X + Y \not\leq_M \omega^\omega - I$. Hence $[\omega^\omega - I]$ is join-irreducible. \hfill \square

Theorem 2.4 is also valid for the Muchnik degrees $M_w$ in place of $M$, a fact first noticed by Terwijn [17]. $M_w$ is defined just as $M$, but with Muchnik reducibility (also called weak reducibility) $\leq_w$ in place of $\leq_M$: $A \leq_w B$ if for every $f \in B$ there is a $g \in A$ such that $f \geq_T g$. $M_w$ is a Brouwer algebra with $+, \times,$ and $\to$ defined by $[A]_w + [B]_w = [A + B]_w$, $[A]_w \times [B]_w = [A \times B]_w$, and $[A]_w \to [B]_w = \{ (g | (\forall f \in A)(\exists h \in B)(h \leq_T f \oplus g)) \}_w$. The proof of Lemma 2.2 also works for $M_w$, and it is easy to check that $A \equiv_w C(A)$ for any mass problem $A$ (i.e., the $M_w$ analogue of Lemma 2.3 is trivial). This gives the forward direction of Theorem 2.4 for $M_w$. The proof of the reverse direction of Theorem 2.4 also works for $M_w$.

3. Degrees that bound no join-irreducible degrees $>_M 0'$

Recall that JAN is the intermediate logic $IPC + \neg p \lor \neg \neg p$. The results of this section and the next are motivated by Question 1.3: is $Th(\mathcal{M}/a) \subseteq JAN$ for every $a >_M 0'$?

$Th(\mathcal{M}/0') = CPC$ because $\mathcal{M}/0' \cong [0, 0'] = \{0, 0'\}$. In fact, $0'$ is the only degree for which $Th(\mathcal{M}/a) = CPC$. This is because if $a >_M 0'$, then $0' \to a = a$, hence $0' \times (0' \to a) = 0'$. Thus let $p = 0'$ to see that the formula $p \lor \neg p$ is not valid in $Th(\mathcal{M}/a)$.

Furthermore, if $a >_M 0'$, then we cannot have $Th(\mathcal{M}/a) \supsetneq JAN$. It is an easy check that in any Brouwer algebra $\mathcal{B}$ with meet-irreducible $0$ (such as the algebras $\mathcal{M}/a$), $\neg p \lor \neg \neg p \in Th(\mathcal{B})$ if and only if $1$ is join-irreducible. However, if $a >_M 0'$ is join-irreducible, then $Th(\mathcal{M}/a) = JAN$ [14]. Thus if $a >_M 0'$ and $Th(M/a) \supsetneq JAN$, then $\neg p \lor \neg \neg p \in Th(\mathcal{M}/a)$ which implies that $a$ is join-irreducible which implies that $Th(M/a) = JAN$. Thus a “no” answer to Question 1.3 must yield a degree $a$ such that $Th(M/a) \supsetneq JAN$ and $JAN \nsubseteq Th(M/a)$.

The following theorem shows that to give a “yes” answer to Question 1.3 it suffices to show that every $a >_M 0'$ bounds a finite meet of degrees $>_M 0'$.

Theorem 3.1 ([14]). If $a$ is a degree such that $a >_M \prod_{i=0}^n d_i$ for join-irreducible degrees $d_i > M 0'$, $i \leq n$, then $Th(\mathcal{M}/a) \subseteq JAN$.

(The above theorem is stated more generally in [14]. Each degree $d_i$ for $i \leq n$ is allowed to be either join-irreducible or $\mathcal{D}e$-irreducible. See the parenthetical discussion following Theorem 4.1 for the definition of $\mathcal{D}e$-irreducible and an explanation of why we do not consider such degrees here. Theorem 4.1 is a restatement of [14] Theorem 2.11 which is the main tool used to prove Theorem 3.1.)

The degrees of the mass problems $\mathcal{B}_f = \{ g | g \not\leq_T f \}$ play an important role in the study of Question 1.3. The following lemma from Sorbi [13] encapsulates the properties of the $[\mathcal{B}_f]$'s that we need in this section and the next.

Lemma 3.2 ([13]).

(i) Every $[\mathcal{B}_f]$ is join-irreducible.

(ii) Every $\sum_{i=1}^n [\mathcal{B}_{f_i}]$ is meet-irreducible.
(iii) Let $V$ and $J$ be finite sets and let $U_v$ and $I_j$ be finite sets for each $v \in V$ and $j \in J$. Let $x_u^v$ and $y_i^j$ be degrees of the form $[B_f]$ for every $v \in V$, $u \in U_v$, $j \in J$, and $i \in I_j$. Let $a = \sum_{v \in V} \prod_{u \in U_v} x_u^v$ and $b = \sum_{j \in J} \prod_{i \in I_j} y_i^j$. Then $a \leq_M b$ if and only if

$$(\forall v \in V)(\exists j \in J)(\forall i \in I_j)(\exists u \in U_v)(x_u^v \leq_M y_i^j).$$

(iv) In the notation of item (iii),

$$a \rightarrow b = \sum \left\{ \prod_{i \in I_j} y_i^j \mid (\forall v \in V) \left( \prod_{i \in I_j} y_i^j \not\leq_M \prod_{u \in U_v} x_u^v \right) \right\}$$

(where the empty join is 0).

Proof. Item (i) is by Theorem 2.4 and item (ii) is by Corollary 1.2. Item (iv) is proved in [13]. Item (iii) follows from item (iv) because $a \leq_M b$ if and only if $b \rightarrow a = 0$. □

In [16] it is asked if every degree $a >_M 0'$ bounds a join-irreducible degree $>_M 0'$, and it is conjectured that this is not the case based on the evidence provided by the following theorem.

Theorem 3.3 ([16]). There is a degree $a >_M 0'$ such that $a \not\leq_M [B_f]$ for every $f >_T 0$.

Our characterization of the join-irreducible degrees implies that every join-irreducible degree $>_M 0'$ bounds some degree $[B_f]$ with $f >_T 0$. Thus the conjecture is correct.

Corollary 3.4 (to Theorem 2.4). If $a >_M 0'$ is join-irreducible, then $a \geq_M [B_f]$ for some $f >_T 0$.

Proof. If $a$ is join-irreducible, then, by Theorem 2.4, $a = [\omega^\omega - I]$ for some Turing ideal $I$. If $[\omega^\omega - I] >_M 0'$, then $I$ contains some function $f >_T 0$. Thus $\omega^\omega - I \subseteq B_f$. Hence $a = [\omega^\omega - I] \geq_M [B_f]$. □

Theorem 3.5. There is a degree $a >_M 0'$ such that every degree $x$ with $0' <_M x \leq_M a$ is join-irreducible.

Proof. By Theorem 3.3, let $a >_M 0'$ be such that $a \not\geq_M [B_f]$ for every $f >_T 0$. This $a$ is the desired degree because, by Corollary 3.4, if $a \geq_M x$ for some join-irreducible $x >_M 0'$, then $a \geq_M [B_f]$ for some $f >_T 0$. □

The degree $a$ satisfying Theorem 3.3 was constructed by diagonalization in [16]. We can give somewhat more concrete examples of degrees satisfying Theorem 3.3 and Theorem 3.5. Recall the following definitions. Functions $f, g >_T 0$ are a Turing minimal pair if, for all $h$, $h \leq_T f, g$ implies $h \leq_T 0$. A function $f$ has minimal Turing degree if, for all $h$, $h <_T f$ implies $h \leq_T 0$. Minimal pairs and minimal degrees exist. In fact, there are continuum many distinct minimal Turing degrees. See Lerman [6] Section II.4 and Section V.2.

Theorem 3.6. If $f$ and $g$ are a minimal pair, then the degree $a = [B_f] \times [B_g]$ witnesses Theorem 3.5.

Proof. Let $f$ and $g$ be a minimal pair. Then $[B_f], [B_g] >_M 0'$ because $f, g >_T 0$. Thus $[B_f] \times [B_g] >_M 0'$ because $0'$ is meet-irreducible by Corollary 1.2. To show that $[B_f] \times [B_g]$ bounds no join-irreducible degree $>_M 0'$, it suffices by Corollary 3.4 to show that $[B_f] \times [B_g]$ bounds no $[B_h]$ for $h >_T 0$. This is true because $f, g$ is a minimal pair, so for any $h >_T 0$, either $h \not<_T f$ or $h \not<_T g$. Thus either $h \in B_f$ or $h \in B_g$ which means $B_f \times B_g$ contains a function $\equiv_T h$. $B_h$ contains no function $\leq_T h$, therefore $B_f \times B_g \geq_M B_h$. □

We can extend the idea behind Theorem 3.6 to find a degree $a >_M 0'$ that does not bound any finite meet of join-irreducible degrees $>_M 0'$. Several of our examples in this section and the next are of the form $[\bigcup_{i \in \omega} \iota^{-1}D_i]$ for mass problems $D_i$, $i \in \omega$.

Lemma 3.7. Let $d = [\bigcup_{i \in \omega} \iota^{-1}D_i]$ where $[D_i] >_M 0'$ for each $i \in \omega$. Then $d >_M 0'$.
Proof. Suppose for a contradiction that \( \Phi \) is a reduction witnessing \( d \leq_M 0' \) (i.e., \( \Phi(f) \in \bigcup_{i \in \omega} i^\omega D_i \) for all \( f > T 0 \)). Let \( \sigma \in \omega^\omega \) be such that \( \Phi(\sigma)(0) \downarrow \) and let \( i = \Phi(\sigma)(0) \). Then \( f \mapsto \Phi(\sigma \uparrow f) \) is a reduction witnessing \( 0' \geq_M [D_i] \), contradicting \( [D_i] > M 0' \).

**Theorem 3.8.** There is a degree \( a > M 0' \) such that no degree \( x \) with \( 0' <_M x \leq_M a \) is of the form \( \prod_{i\leq n} d_i \) for join-irreducible degrees \( d_i > M 0' \), \( i \leq n \).

Proof. By Corollary 3.4, it suffices to find a degree \( a > M 0' \) which is not above any degree of the form \( \prod_{i\leq n} [B_{f_i}] \) where \( f_i > T 0 \) for each \( i \leq n \). Let \( \{g_i \mid i \in \omega \} \) be a countable collection of functions all of distinct minimal Turing degree. Let \( A = \bigcup_{i \in \omega} i^\omega B_{g_i} \) and put \( a = [A] \). Lemma 3.7 proves that \( a > M 0' \).

Now consider any degree \( \prod_{i=0}^n [B_{f_i}] \), where \( f_i > T 0 \) for each \( i \leq n \). There is a \( j \in \omega \) such that \( g_j \not\equiv_T f_i \) for each \( i \leq n \). Thus for this \( j \), \( [B_{g_j}] \not\geq_M \prod_{i=0}^n [B_{f_i}] \) because \( [B_{g_j}] \) is meet-irreducible. Clearly \( [B_{g_j}] \geq_M a \), so \( a \not\geq_M \prod_{i=0}^n [B_{f_i}] \) as well.

For mass problems \( A_i, i \in \omega \), the Medvedev degree \( \bigcup_{i \in \omega} i^\omega A_i \) is not in general the greatest lower bound of the degrees \( [A_i], i \in \omega \). Such greatest lower bounds need not even exist. For example, the degrees \( [B_{g_i}], i \in \omega \) from Theorem 3.8 do not have a greatest lower bound. This follows from results in Dyment [4] which studies when countable collections of degrees have least upper bounds and greatest lower bounds.

If \( a \) is a degree such that \( a \not\geq_M d \) for all join-irreducible \( d > M 0' \), then \( a \rightarrow d = d \) for all join-irreducible \( d > M 0' \). The degree \( a \) constructed in Theorem 3.8 enjoys a similar property.

**Theorem 3.9.** There is a degree \( a > M 0' \) such that \( a \rightarrow \prod_{i=0}^n d_i = \prod_{i=0}^n d_i \) whenever \( d_i > M 0' \) and is join-irreducible for each \( i \leq n \).

Proof. As in Theorem 3.8, let \( \{g_i \mid i \in \omega \} \) be a countable collection of functions all of distinct minimal Turing degree, let \( A = \bigcup_{i \in \omega} i^\omega B_{g_i} \), and put \( a = [A] \). Suppose \( d_i > M 0' \) and is join-irreducible for each \( i \leq n \). By Theorem 2.4, for each \( i \leq n \) let \( I_i \subseteq \omega^\omega \) be a Turing ideal such that \( d_i = [\omega^\omega - I_i] \). Thus \( \prod_{i=0}^n d_i = [\omega^\omega - \bigcup_{i=0}^n I_i] \) and

\[
a \rightarrow \prod_{i=0}^n d_i = \left\{ e \uparrow g \mid \left( \forall f \in A \right) \left( \Phi_e(f \oplus g) \in \bigcup_{i=0}^n i^\omega (\omega^\omega - I_i) \right) \right\}.
\]

We now describe a reduction witnessing \( a \rightarrow \prod_{i=0}^n d_i \geq_M \prod_{i=0}^n d_i \).

Given \( e \uparrow g \), for each \( i \leq n + 1 \) search for a string \( i \uparrow \sigma_i \) such that \( \Phi_e((i \uparrow \sigma_i) \oplus g)(0) \downarrow \). If there is a \( k \leq n \) such that

\[
\Phi_e((i \uparrow \sigma_i) \oplus g)(0) = \Phi_e((j \uparrow \sigma_j) \oplus g)(0) = k
\]

for two distinct \( i, j \leq n + 1 \), choose the least such \( k \) and output \( k \uparrow g \). Otherwise output 0.

Suppose we apply this reduction to \( e \uparrow g \in A \rightarrow \bigcup_{i=0}^n i^\omega (\omega^\omega - I_i) \). \( \Phi_e(f \oplus g) \) must be total for each \( f \in A \) and for each \( i \in \omega \) there is an \( f \in A \) with \( f(0) = i \). Thus for each \( i \leq n + 1 \) the search finds a string \( i \uparrow \sigma_i \) such that \( \Phi_e((i \uparrow \sigma_i) \oplus g)(0) \downarrow \). Moreover, each \( i \uparrow \sigma_i \) can be extended to a function in \( A \), so \( \Phi_e((i \uparrow \sigma_i) \oplus g)(0) \leq n \) for each \( i \leq n + 1 \). Therefore there is a least \( k \leq n \) for which there are distinct \( i, j \leq n + 1 \) with \( \Phi_e((i \uparrow \sigma_i) \oplus g)(0) = \Phi_e((j \uparrow \sigma_j) \oplus g)(0) = k \). The reduction outputs \( k \uparrow g \), so we must show that \( k \uparrow g \in \bigcup_{i=0}^n i^\omega (\omega^\omega - I_i) \) which means we must show that \( g \not\in I_k \). Suppose for a contradiction that \( g \in I_k \). The functions \( g_i \) and \( g_j \) have distinct minimal degree, so either \( g \not\equiv_T g_i \) or \( g \not\equiv_T g_j \) (\( g > T 0 \) because \( a \not\geq_M \prod_{i=0}^n d_i \) by Theorem 3.8). For the sake of argument, suppose \( g \not\equiv_T g_i \). Then \( \sigma_i \uparrow g \not\equiv_T g_j \) as well, so \( \sigma_\uparrow g \in B_{g_j} \) and \( i \uparrow \sigma_i \uparrow g \in A \). However, \( \Phi_e((i \uparrow \sigma_i \uparrow g) \oplus g) \in k^\omega (\omega^\omega - I_k) \) by the choice of \( i \uparrow \sigma_i \). This cannot be because \( (i \uparrow \sigma_i \uparrow g) \oplus g \in I_k \), thus anything it computes is also in \( I_k \).
By Corollary 4.6 below, the degree \( a = \bigcup_{i \in \omega} i \cdot \mathcal{B}_{g_i} \) used to witness Theorem 3.8 and Theorem 3.9 satisfies \( \text{Th}(\mathcal{M}/a) \subseteq \text{JAN} \) and so does any degree that bounds it. There are, however, degrees \( >_M 0' \) that do not bound any degree of the form \( \bigcup_{i \in \omega} i \cdot D_i \) where \( |D_i| >_M 0' \) and is join-irreducible for each \( i \in \omega \).

**Theorem 3.10.** There is a degree \( a >_M 0' \) such that \( a \not\leq_M \bigcup_{i \in \omega} i \cdot D_i \) whenever \( |D_i| >_M 0' \) and is join-irreducible for each \( i \in \omega \).

**Proof.** Let \( D_i \) be such that \( |D_i| >_M 0' \) and is join-irreducible for each \( i \in \omega \). By Corollary 3.4, for every \( i \in \omega \) there is an \( f_i >_T 0 \) such that \( D_i \geq_M B_{f_i} \). The mass problem \( B_{f_i} \) is Turing upward-closed for each \( i \in \omega \), so \( D_i \subseteq B_{f_i} \) for each \( i \in \omega \). Thus \( \bigcup_{i \in \omega} i \cdot D_i \subseteq \bigcup_{i \in \omega} i \cdot B_{f_i} \). Hence it suffices to find a degree \( a >_M 0' \) that does not bound any degree of the form \( \bigcup_{i \in \omega} i \cdot B_{f_i} \), where \( f_i >_T 0 \) for each \( i \in \omega \).

We use the same construction used in [16] to prove Theorem 3.3. Build mass problems \( A_s \subseteq \{ g \mid g >_T 0 \} \) such that \( \{ g \mid g >_T 0 \} - A_s \) is finite for each \( s \in \omega \). Set \( A_0 = \{ g \mid g >_T 0 \} \). At stage \( s + 1 \), choose \( h_s >_T 0 \) such that \( h_s \) does not compute any of the (finitely many) functions in \( \{ g \mid g >_T 0 \} - A_s \). If \( \Phi_s(h_s) \) is total and \( >_T 0 \), let \( g_s = \Phi_s(h_s) \) and set \( A_{s+1} = A_s - \{ g_s \} \). Otherwise set \( A_{s+1} = A_s \). Put \( A = \bigcap_{i \in \omega} A_i \) and put \( a = [A] \).

To see \( a >_M 0' \), observe that by construction \( \Phi_e(h_s) \notin A \) for each \( s \in \omega \). Now let \( f_i >_T 0 \) for each \( i \in \omega \). We need to show that \( \Phi_e(A) \not\subseteq \bigcup_{i \in \omega} i \cdot B_{f_i} \) for every index \( e \). To do this, we first show that the functions in \( \{ g \mid g >_T 0 \} - A \) have distinct Turing degree. Suppose that \( g_i \) leaves \( A \) at stage \( i+1 \) and \( g_j \) leaves \( A \) at stage \( j+1 \). Suppose \( \Phi_e(g_i) \) is total and \( >_T 0 \) for \( i+1 < j+1 \) (i.e., at stage \( i+1 \) we had \( \Phi_e(h_i) = g_i >_T 0 \), and at stage \( j+1 \) we had \( \Phi_e(h_j) = g_j >_T 0 \). Then \( g_i \not<_T g_j \) because otherwise \( g_i \not<_T g_j \leq_T h_j \), contradicting that \( h_j \) was chosen \( \not<_T g_i \) at stage \( j+1 \). Now suppose \( \Phi_e(g) \) is total for all \( g \in A \). Fix any \( \sigma \in \omega^{<\omega} \) such that \( \Phi_e(\sigma)(0)_\downarrow \), and let \( n \) be such that \( \Phi_e(\sigma)(0)_\downarrow = n \). \( A \) is missing at most one function \( \equiv_T f_n \), so let \( g \in A \) be such that \( \sigma < g \) and \( g \equiv_T f_n \). Then \( \Phi_e(g)(0) = n \), but \( \Phi_e(g) \notn\cdot B_{f_n} \). Hence \( \Phi_e(A) \not\subseteq \bigcup_{i \in \omega} i \cdot B_{f_i} \).

**Question 3.11.** Let \( a \) be the degree constructed in Theorem 3.10. Does \( a \rightarrow \bigcup_{i \in \omega} i \cdot D_i \) whenever \( |D_i| >_M 0' \) and is join-irreducible for each \( i \in \omega \)? Is \( \text{Th}(\mathcal{M}/a) \subseteq \text{JAN} \)?

Finally, we note that the answer to Question 1.3 is “no” for \( \mathcal{M}_w \) in place of \( \mathcal{M} \). Let \( f >_T 0 \) have minimal Turing degree, and let \( a = [B_f]_w \). Then, in \( \mathcal{M}_w \), \( [0,a] = \{0,0',a\} \) and \( \text{JAN} \subseteq \text{Th}(\mathcal{M}_w/a) \not\subseteq \text{CPC} \).

4. NEW DEGREES WHOSE CORRESPONDING LOGIC IS CONTAINED IN JAN

We extend Theorem 3.1 by proving \( \text{Th}(\mathcal{M}/a) \subseteq \text{JAN} \) for degrees \( a \) such that \( a >_M \bigcup_{i \in \omega} i \cdot D_i \) for some collection of join-irreducible degrees \( \{D_i\} \).

A propositional formula is called *positive* if the connective ‘\(-\)’ does not appear in it. For a logic \( L \) let \( L^+ \) denote the positive formulas in \( L \), and for a Brouwer algebra \( \mathcal{B} \) let \( \text{Th}^+(\mathcal{B}) \) denote the set of positive formulas valid in \( \mathcal{B} \). JAN is the maximum intermediate logic \( L \) for which \( L^+ = \text{IPC}^+ \). This means that \( L^+ = \text{IPC}^+ \) implies \( L \subseteq \text{JAN} \) for any intermediate logic \( L \). Thus \( \text{Th}^+(\mathcal{B}) = \text{IPC}^+ \) implies \( \text{Th}(\mathcal{B}) \subseteq \text{JAN} \) for any Brouwer algebra \( \mathcal{B} \).

Let \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) be Brouwer algebras. An injection \( f : \mathcal{B}_1 \rightarrow \mathcal{B}_2 \) is called a *B-embedding* if it preserves \( 0, 1, +, x, \) and \( \rightarrow \) (and therefore also \( \neg \)). An injection \( f : \mathcal{B}_1 \rightarrow \mathcal{B}_2 \) is called a *B+ embedding* if it preserves \( 0, +, x, \) and \( \rightarrow \) (but not necessarily \( 1 \) or \( \neg \)). If \( \mathcal{B}_1 \) -B embeds into \( \mathcal{B}_2 \), then \( \text{Th}(\mathcal{B}_2) \subseteq \text{Th}(\mathcal{B}_1) \), and if \( \mathcal{B}_1 \) -B+ embeds into \( \mathcal{B}_2 \), then \( \text{Th}(\mathcal{B}_2) \subseteq \text{Th}(\mathcal{B}_1) \). Both of these facts are easily checked in light of [9] Theorem VI.2.4. If \( a \leq b \) are in a Brouwer algebra \( \mathcal{B} \), then \( \mathcal{B}/a \) -B+ embeds into \( \mathcal{B}/b \) by the identity. This implies that \( \text{Th}^+(\mathcal{B}/b) \subseteq \text{Th}^+(\mathcal{B}/a) \), and it follows that the \( a \) for which \( \text{Th}(\mathcal{B}/a) \subseteq \text{JAN} \) is upward-closed in any Brouwer algebra \( \mathcal{B} \).

Our goal is to \( B^+ \)-embed a certain class of Brouwer algebras into \( \mathcal{M}/a \). For any set \( X \), let \( \text{Fr}(X) \) denote the free distributive lattice generated by \( X \) and let \( 0 \oplus \text{Fr}(X) \) denote \( \text{Fr}(X) \) with a new
bottom element 0. The elements of Fr(X) are all of the form \( \sum_{v \in V} \prod_{u \in U_v} x^v_u \) where \( V \) and the \( U_v \) are finite sets of indices and the \( x^v_u \) are all in \( X \) (see for example Balbes and Dwyer [1] Section V.3). For such representations, \( \sum_{v \in V} \prod_{u \in U_v} x^v_u \leq \sum_{j \in J} \prod_{i \in I_j} y^j_i \) if and only if

\[
(\forall v \in V)(\exists j \in J)(\forall i \in I_j)(\exists u \in U_v) \left( x^v_u \leq y^j_i \right).
\]

If \( a, b \in \text{Fr}(X) \) are such that \( a \nleq b \), then \( a \rightarrow b \) exists. To see this, let \( a = \sum_{v \in V} \prod_{u \in U_v} x^v_u \) and \( b = \sum_{j \in J} \prod_{i \in I_j} y^j_i \) be representations for \( a \) and \( b \). Then check

\[
a \rightarrow b = \sum \left\{ \prod_{i \in I_j} y^j_i \mid (\forall v \in V) \left( \prod_{i \in I_j} y^j_i \nleq \prod_{u \in U_v} x^v_u \right) \right\}.
\]

If \( a \geq b \) are in \( \text{Fr}(X) \) for an infinite \( X \), then \( a \rightarrow b \) fails to exist because in this case \( \text{Fr}(X) \) has no least element. We see then that \( a \rightarrow b \) exists for every \( a, b \in 0 \oplus \text{Fr}(X) \). If \( X \) is finite, then so are \( \text{Fr}(X) \) and \( 0 \oplus \text{Fr}(X) \). Hence both are Brouwer algebras. Let \( \text{Fr}(n) \) denote the free distributive lattice with \( n \) generators. The logic \( \text{LM} = \bigcap_{n \in \omega} \text{Th}(0 \oplus \text{Fr}(n)) \) is called the Medvedev logic of finite problems. (LM is usually defined in terms of Brouwer algebras isomorphic to the \( 0 \oplus \text{Fr}(n) \). See [16] for details.) We take advantage of the fact that \( \text{LM}^+ = \text{IPC}^+ [8] \).

If \( X \) is infinite, then \( 0 \oplus \text{Fr}(X) \) fails to be a Brouwer algebra only because it lacks a top element. Therefore the notion of a \( B^+ \)-embedding makes sense when we allow \( \mathcal{B}_1 \) or \( \mathcal{B}_2 \) to be \( 0 \oplus \text{Fr}(X) \). If we let \( 0 \oplus \text{Fr}(X) \oplus 1 \) denote \( \text{Fr}(X) \) with a new bottom element 0 and a new top element 1, then \( 0 \oplus \text{Fr}(X) \oplus 1 \) is always a Brouwer algebra.

For any partial order \( (P, \leq_P) \), let \( \text{Fr}(P, \leq_P) \) denote the free distributive lattice generated by \( (P, \leq_P) \). \( \text{Fr}(P, \leq_P) \) is the quotient \( \text{Fr}(P)/\equiv \) where, for \( a = \sum_{v \in V} \prod_{u \in U_v} x^v_u \) and \( b = \sum_{j \in J} \prod_{i \in I_j} y^j_i \) in \( \text{Fr}(P) \), \( a \equiv b \) if and only if \( (a \leq b) \wedge (b \leq a) \) and \( a \preceq b \) if and only if

\[
(\forall v \in V)(\exists j \in J)(\forall i \in I_j)(\exists u \in U_v) \left( x^v_u \leq_P y^j_i \right).
\]

\( \text{Fr}(P, \leq_P) \) is always a distributive lattice, and \( 0 \oplus \text{Fr}(P, \leq_P) \oplus 1 \) is always a Brouwer algebra; also see [13].

The following lemmas facilitate our embeddings. Lemma 4.3 is a slight generalization of the claim in the proof of [13] Lemma 2.3 and of [10] Lemma 6. The embedding is done in Theorem 4.4 which is nearly identical to [14] Theorem 2.11. Part of the reason for reproducing the proof here is to (hopefully) correct the notational inconsistencies in the proof in [14]. We restate [14] Theorem 2.11 for reference.

**Theorem 4.1** ([14] Theorem 2.11). Let \( d = \prod_{i=0}^n d_i \) where \( d_i > M 0' \) and \( d_i \) is join-irreducible for each \( i \leq n \). Then \( 0 \oplus \text{Fr}(P, \leq_P) \oplus 1 \) \( B \)-embeds into \( \mathcal{M}/d \) for every countable partial order \( (P, \leq_P) \).

(The above theorem is stated more generally in [14]. Each degree \( d_i \) for \( i \leq n \) is allowed to be either join-irreducible or \( \mathcal{D} \)-irreducible. A degree \( a \) is dense if it is of the form \( [4] \) where \( \mathcal{A} \) is dense in \( \omega^\omega \), and a degree \( d \) is \( \mathcal{D} \)-irreducible if \( a \rightarrow d = d \) for all dense degrees \( a \). We do not consider \( \mathcal{D} \)-irreducible degrees in our version of [14] Theorem 2.11, which is Theorem 4.4 below, because in Theorem 4.4 we require that the mass problems \( D_i \) (which play the role of the degrees \( d_i \) in [14] Theorem 2.11) are Turing upward-closed. Mass problems that are Turing upward-closed are dense and hence their degrees are not \( \mathcal{D} \)-irreducible.)

**Lemma 4.2** ([3]). If \( \mathcal{X} \not\preceq_M \mathcal{Y} \) are mass problems, then there is a \( \mathcal{W} \subseteq \mathcal{X} \) with \( |\mathcal{W}| \leq \omega \) such that \( \mathcal{W} \not\preceq_M \mathcal{Y} \).

**Proof.** \( \mathcal{X} \not\preceq_M \mathcal{Y} \) means that there is no Turing functional \( \Phi \) such that \( \Phi(\mathcal{X}) \subseteq \mathcal{Y} \). Thus for each functional \( \Phi_e \) there must be some function \( f_e \in \mathcal{X} \) such that \( \Phi_e(f_e) \notin \mathcal{Y} \). Let \( \mathcal{W} \) consist of a choice of one such \( f_e \in \mathcal{X} \) for each functional \( \Phi_e \). \( \square \)
Lemma 4.3. Let $U$, $V$, and $F_i$ for $i \in \omega$ be mass problems such that $\bigcup_{i \in \omega} i^{-1} F_i \leq_M U + V$ and $\sigma^{-1} U \subseteq U$ for all $\sigma \in \omega^{<\omega}$. Then there are mass problems $V_i$ for $i \in \omega$ such that $\bigcup_{i \in \omega} i^{-1} V_i \equiv_M V$ and $F_i \leq_M U + V_i$ for each $i \in \omega$.

Proof. Let $U$, $V$, and $F_i$ for $i \in \omega$ be as in the statement of the lemma. Let $\Phi$ be such that $\Phi(U + V) \subseteq \bigcup_{i \in \omega} i^{-1} F_i$. For each $i \in \omega$, define $V_i = \{g \in V \mid (\exists \sigma \in \omega^{<\omega})(\Phi(\sigma \oplus g)(0) = i)\}$. Then $V \leq_M \bigcup_{i \in \omega} i^{-1} V_i$ is clear, $\bigcup_{i \in \omega} i^{-1} V_i \leq_M V$ by the reduction which, given $g$, searches for a $\sigma \in \omega^{<\omega}$ such that $\Phi(\sigma \oplus g)(0) \downarrow$ and outputs $\Phi(\sigma \oplus g)(0)^\frown g$. To see $i^{-1} F_i \leq_M U + V_i$, consider the reduction which, given $f \oplus g$, searches for a $\sigma \in \omega^{<\omega}$ such that $\Phi(\sigma \oplus g)(0) = i$ and outputs $\Phi((\sigma \frown f) \oplus g)$. If $f \oplus g \in U + V_i$, then such a $\sigma$ is found, $\sigma \frown f$ is in $U$, and $\Phi((\sigma \frown f) \oplus g)$ is in $i^{-1} F_i$. □

Theorem 4.4. Let $d = [\bigcup_{i \in \omega} i^{-1} D_i]$ where $[D_i] >_M 0'$, $[D_i]$ is join-irreducible, and $D_i$ is Turing upward-closed for each $i \in \omega$. Then $0 \oplus \Fr(2^{\omega})$ $B^+$-embeds into $\MM / d$.

Proof. Let $D_i$ for $i \in \omega$ be as in the statement of the theorem, let $D = \bigcup_{i \in \omega} i^{-1} D_i$, and let $d = [D]$. Lemma 3.7 proves that $d >_M 0'$. By Lemma 4.2, let $A \subseteq \{f \mid f >_T 0\}$ be a countable mass problem such that $A \not\subseteq \MM$. Let $\{f_x \mid x \in 2^{\omega}\}$ be a collection of functions such that $f_x \equiv_T f_y$ for all $x, y \in 2^{\omega}$ with $x \neq y$ and that $f \not\leq_T f_x$ for all $f \in A$ and $x \in 2^{\omega}$. Such a sequence can be constructed via standard recursion-theoretic techniques: build a perfect tree whose paths are Turing incomparable and do not compute any functions in $A$. See for example [6] Section II.4. Notice that $B_{f_x} \leq_M A$ (because $A \subseteq \BB_{f_x}$) for each $x \in 2^{\omega}$.

Define $G: 0 \oplus \Fr(2^{\omega}) \to \MM$ as follows. Let $G(0) = 0$ and let $G(x) = [B_{f_x}]$ on the generators $x \in 2^{\omega}$. Then extend $G$ to all of $0 \oplus \Fr(2^{\omega})$ so that $G(\sum_{v \in V} \prod_{u \in U_v} x_u^v) = \sum_{v \in V} \prod_{u \in U_v} G(x_u^v)$. $G$ preserves $0$, $+$, and $\times$ by definition, and $G$ is injective and preserves $\to$ by Lemma 3.2 items (iii) and (iv). Hence $G$ is a $B^+$-embedding (this is essentially [13] Corollary 2.5). Now define $H: 0 \oplus \Fr(2^{\omega}) \to \MM / d$ by $H(a) = G(a) \times d$ for all $a \in 0 \oplus \Fr(2^{\omega})$. This $H$ is the desired $B^+$-embedding. By definition, $H$ preserves $0$, $+$, and $\times$. We must show that $H$ is injective and that $H$ preserves $\to$.

Clearly $H(a) = 0$ if and only if $a = 0$, so to show that $H$ is injective we let $a, b \in \Fr(2^{\omega})$ be such that $H(a) \leq_M H(b)$ and show that $a \leq b$. Let $a = \sum_{v \in V} \prod_{u \in U_v} x_u^v$ be a representation for $a$ and let $b = \sum_{j \in J} \prod_{i \in I_j} y_i^j$ be a representation for $b$. $H(a) \leq_M H(b)$ means that

$$\sum_{v \in V} \prod_{u \in U_v} G(x_u^v) \times d \leq_M \sum_{j \in J} \prod_{i \in I_j} G(y_i^j) \times d.$$

Therefore

$$\sum_{v \in V} \prod_{u \in U_v} G(x_u^v) \times d \leq_M \sum_{j \in J} \prod_{i \in I_j} G(y_i^j) = \prod \left\{ \sum_{j \in J} G(y_i^j) \mid \alpha \in \prod_{j \in J} I_j \right\}.$$

where the equality is by distributivity ($\prod_{j \in J} I_j$ denotes the Cartesian product of the $I_j$’s). Hence

$$\sum_{v \in V} \prod_{u \in U_v} G(x_u^v) \times d \leq_M \sum_{j \in J} G(y_i^j) \text{ for each } \alpha \in \prod_{j \in J} I_j.$$

Thus, $\sum_{j \in J} G(y_i^j)$ is meet-irreducible by Lemma 3.2 item (ii). Also, $d \not\leq_M \sum_{j \in J} G(y_i^j)$ for each $\alpha \in \prod_{j \in J} I_j$ because $\sum_{j \in J} G(y_i^j) \leq_M [A]$ but $d \not\leq_M [A]$. Thus

$$\sum_{v \in V} \prod_{u \in U_v} G(x_u^v) \leq_M \sum_{j \in J} G(y_i^j) \text{ for each } \alpha \in \prod_{j \in J} I_j.$$
and this implies that

\[ \sum_{v \in V} \prod_{u \in U_v} G(x_u^v) \leq M \prod J \sum_{j \in J} \left( \sum_{j \in J} G(y_{\alpha(j)}^j) \right) \]

The left-hand side of the above inequality is \( G(a) \) and the right-hand side is \( G(b) \). \( G \) is a \( B^+ \)-embedding, so we conclude \( a \leq b \).

If either of \( a, b \in \emptyset \oplus \text{Fr}(2^\omega) \) is \( \emptyset \), then clearly \( H(a \rightarrow b) = H(a) \rightarrow H(b) \). So as before, let \( a = \sum_{v \in V} \prod_{u \in U_v} x_u^v \) and let \( b = \sum_{j \in J} \prod_{i \in I_j} y_i^j \) be in Fr\( (2^\omega) \). We see \( H(a \rightarrow b) \geq M H(a) \rightarrow H(b) \) because

\[ H(a \rightarrow b) = H((a \rightarrow b) + a) \geq M H(b). \]

To show that \( H(a \rightarrow b) \leq_M H(a) \rightarrow H(b) \), we show that if \( z \in \mathfrak{M} \) is such that \( H(b) \leq_M H(a) + z \), then \( H(a \rightarrow b) \leq_M z \). Suppose \( H(b) \leq_M H(a) + z \). That is,

\[ \sum_{j \in J} \prod_{i \in I_j} G(y_i^j) \times d \leq_M \left( \sum_{v \in V} \prod_{u \in U_v} G(x_u^v) \times d \right) + z. \]

Since \( a \rightarrow b = \sum \{ \prod_{i \in I_j} y_i^j \mid (\forall v \in V) (\prod_{i \in I_j} y_i^j \notin \prod_{u \in U_v} x_u^v) \} \), we have

\[ H(a \rightarrow b) = G(a \rightarrow b) \times d \]

\[ = \sum \left\{ \prod_{i \in I_j} G(y_i^j) \mid (\forall v \in V) \left( \prod_{i \in I_j} G(y_i^j) \notin_M \prod u \in U_v G(x_u^v) \right) \right\} \times d. \]

It suffices to show that, given \( j \in J \), if \( \prod_{i \in I_j} G(y_i^j) \) satisfies

\[ (\forall v \in V) \left( \prod_{i \in I_j} G(y_i^j) \notin_M \prod u \in U_v G(x_u^v) \right), \]

then \( \prod_{i \in I_j} G(y_i^j) \times d \leq_M z \). Suppose \( \prod_{i \in I_j} G(y_i^j) \) is such a meet. Then we know

\[ (\forall v \in V) \left( \exists u \in U_v \left( \prod_{i \in I_j} G(y_i^j) \notin M G(x_u^v) \right) \right). \]

By choosing such an \( u \in U_v \) for every \( v \in V \) and by appealing to Lemma 3.2 items (i) and (ii), we see that there is an \( \alpha \in \prod_{v \in V} U_v \) such that

\[ \prod_{i \in I_j} G(y_i^j) \notin_M \sum_{v \in V} G(x_{\alpha(v)}^v). \]

Distributing \( \sum_{v \in V} \prod_{u \in U_v} G(x_u^v) \) in the right-hand side of (1) yields

\[ \prod_{i \in I_j} G(y_i^j) \times d \leq_M \sum_{v \in V} G(x_{\alpha(v)}^v) + z. \]

The degree \( \sum_{v \in V} G(x_{\alpha(v)}^v) \) is a finite join of degrees of the form \( [\mathcal{B}_f] \) and thus has a representative \( \mathcal{U} \) such that \( \sigma^{-1} \mathcal{U} \subseteq \mathcal{U} \) for all \( \sigma \in \omega^{<\omega} \). So by Lemma 4.3 there are mass problems \( \mathcal{Z}_i \) for \( i \in I_j \) and
Consider the filters $\mathcal{F}$ and $\mathcal{G}$ for each $i \in \omega$ such that

$$z = \left( \prod_{i \in I_j} [Z_i] \right) \times \left( \bigcup_{i \in I_j} i^{-1}  \hat{Z}_i \right),$$

$$G(y_i^j) \leq_M \sum_{v \in V} G(x_{\alpha(v)}^v) + [Z_i] \text{ for each } i \in I_j, \text{ and}$$

$$[D_i] \leq_M \sum_{v \in V} G(x_{\alpha(v)}^v) + [\hat{Z}_i] \text{ for each } i \in \omega.$$  

Each $G(y_i^j)$ is join-irreducible, and $G(y_i^j) \not\leq_M \sum_{v \in V} G(x_{\alpha(v)}^v)$ by (2). Thus $G(y_i^j) \leq_M [Z_i]$ for each $i \in \omega$, so $\prod_{i \in I_j} G(y_i^j) \leq_M \prod_{i \in I_j} [Z_i]$. Each $[D_i]$ is join-irreducible by assumption, and also $[D_i] \not\leq_M \sum_{v \in V} G(x_{\alpha(v)}^v)$ because the right-hand side is $\leq_M [A]$ but the left-hand side is not. It follows that $[D_i] \leq_M [\hat{Z}_i]$ for each $i \in \omega$, and so $\hat{Z}_i \subseteq D_i$ for each $i \in \omega$ because each $D_i$ is Turing upward-closed. Thus $\bigcup_{i \in \omega} i^{-1}  \hat{Z}_i \subseteq D$, so $d \leq_M \left( \bigcup_{i \in \omega} i^{-1}  \hat{Z}_i \right)$. Therefore

$$\prod_{i \in I_j} G(y_i^j) \times d \leq_M \left( \prod_{i \in I_j} [Z_i] \right) \times \left( \bigcup_{i \in \omega} i^{-1}  \hat{Z}_i \right) = z$$

as desired. \hfill \Box

**Corollary 4.5.** If $a \geq_M d$ are degrees such that $\exists \in \bigcup_{i \in \omega} i^{-1}  \hat{D}_i$ where $[D_i] >_M 0'$ and is join-irreducible for each $i \in \omega$, then $0 \oplus \text{Fr}(2^\omega)$ $B^+$-embeds into $\mathcal{M} / a$.

**Proof.** Let $a$, $d$, and $D_i$ for $i \in \omega$ be as in the statement of the corollary. Let $d_0 = \left( \bigcup_{i \in \omega} i^{-1} C(D_i) \right)$ and notice that $d \geq_M d_0$. $D_i \equiv C(D_i)$ for each $i \in \omega$ by Lemma 2.3, so $d_0$ satisfies the hypotheses of Theorem 4.4. Thus $0 \oplus \text{Fr}(2^\omega)$ $B^+$-embeds into $\mathcal{M} / d_0$ which $B^+$-embeds into $\mathcal{M} / a$. \hfill \Box

**Corollary 4.6.** If $a \geq_M d$ are degrees such that $d = \left( \bigcup_{i \in \omega} i^{-1}  \hat{D}_i \right)$ where $[D_i] >_M 0'$ and is join-irreducible for each $i \in \omega$, then $\text{Th}(\mathcal{M} / a) \subseteq \text{JAN}$. 

**Proof.** The Brouwer algebra $0 \oplus \text{Fr}(n)$ $B^+$-embeds into $0 \oplus \text{Fr}(2^\omega)$ for each $n$, and $0 \oplus \text{Fr}(2^\omega)$ $B^+$-embeds into $\mathcal{M} / a$ by Corollary 4.5. Thus $\text{Th}^+(\mathcal{M} / a) \subseteq \bigcap_{n \in \omega} \text{Th}^+(0 \oplus \text{Fr}(n)) = LM^+ = \text{IPC}^+$. So $\text{Th}(2\mathcal{M} / a) \subseteq \text{JAN}$. \hfill \Box

Theorem 4.4 can be modified to $B$-embed $0 \oplus \text{Fr}(2^\omega) \oplus 1$ into $\mathcal{M} / d$ for degrees $d$ as in the statement of Theorem 4.4. However, if $a \leq b$ in a Brouwer algebra $\mathcal{B}$, it is not in general the case that $\mathcal{B} / a$ $B$-embeds into $\mathcal{B} / b$. So the proof of Corollary 4.5 fails for $B$-embedding $0 \oplus \text{Fr}(2^\omega) \oplus 1$. Theorem 4.4 can also be modified to prove a more precise analogue of [14] Theorem 2.11 (restated as Theorem 4.1 above). Let $d = \left( \bigcup_{i \in \omega} i^{-1} \hat{D}_i \right)$ where $[D_i] >_M 0'$, $[D_i]$ is join-irreducible, and $D_i$ is Turing upward-closed for each $i \in \omega$. Then $0 \oplus \text{Fr}(P, \leq_P) \oplus 1$ $B$-embeds into $\mathcal{M} / d$ for every countable partial order $(P, \leq_P)$.

5. $\mathfrak{F}_c\mathfrak{I}$ is not prime

Recall that a filter $\mathfrak{F}$ in a lattice is called prime if $a + b \in \mathfrak{F} \rightarrow a \in \mathfrak{F} \vee b \in \mathfrak{F}$ for all $a$ and $b$ in the lattice. Theorem 2.4 can be phrased as a characterization of the prime principal filters in $\mathcal{M}$: a degree $a$ generates a prime filter if and only if $a = [\omega^\omega - I]$ for some Turing ideal $I$. There is little general theory of the non-principal filters in $\mathcal{M}$, but several specific cases have been studied in Dyment [3], Sorbi [11], Bianchini and Sorbi [2], and Lewis, Shore, and Sorbi [7]. See also [15] for a summary of many of the results appearing in these works. We consider the filters $\mathfrak{F}$ and $\mathfrak{F}_c\mathfrak{I}$:

**Definition 5.1.**

- A degree $a$ is called dense (closed) if $a = [A]$ for an $A$ that is dense (closed) in $\omega^\omega$.
- $\mathfrak{I}$ denotes the ideal generated by $\{a \mid a$ is dense$\}$. 
contains the recursive function \( \Phi \). 

[Case 1] must not have occurred because otherwise

\[ A \]
for each

\[ B \]
the least string such that \( \Phi \left( e, s \right) \). 

The basic properties of 

\[ I \]
proof that 

\[ F \]
is not principal \[ 3 \],

[2]. Thus \( I \), \( \mathfrak{F} \), and \( \mathfrak{F}_{cl} \) are as follows: \( I \) is a prime ideal \[ 11 \], \( \mathfrak{F} \) is a prime filter \[ 2 \], \( \mathfrak{J} \) is not principal \[ 3 \], \( \mathfrak{F} \) and \( \mathfrak{F}_{cl} \) are not principal \[ 2 \], and \( \mathfrak{F}_{cl} \subseteq \mathfrak{F} \) \[ 2 \]. Both \[ 2 \] and \[ 15 \] ask for a proof that \( \mathfrak{F}_{cl} \) is not prime. We provide a proof of this fact now.

**Lemma 5.2.** For any \( f \in \omega^\omega \) there are \( A, B \subseteq \omega^\omega \) such that \( A + B \geq_M \{ f \} \) and, for any closed \( C \subseteq \omega^\omega \), if \( A \geq_M C \) or \( B \geq_M C \), then \( C \) contains a recursive function.

**Proof.** Fix a recursive bijection \( \omega \leftrightarrow \omega^{<\omega} \). For \( e, n \in \omega \), if

\[
\forall m \forall \sigma \exists \tau \geq \sigma (\Phi_e(n^\sigma)(m)^\uparrow),
\]
then define \( \eta(e,n,i) \in \omega^{<\omega} \) by induction on \( i \in \omega \) as follows. Let \( \eta(e,n,0) = n^\sigma \), where \( \sigma \) is the least string such that \( \Phi_e(n^\sigma)(0)^\downarrow \). Given \( \eta(e,n,i) \), let \( \eta(e,n,i+1) = \eta(e,n,i)^{0^\sigma} \), where \( \sigma \) is the least string such that \( \Phi_e(\eta(e,n,i)^{0^\sigma})(i+1)^\downarrow \).

Let \( f \in \omega^\omega \). We construct \( A \) and \( B \) such that:

- If \( g \in A \), then \( g(0) \) has the form
  \[
g(0) = \langle \ell, \langle n_0,x_0,y_0 \rangle, \ldots, \langle n_{\ell-1},x_{\ell-1},y_{\ell-1} \rangle \rangle,
\]
  where \( \ell \in \omega \) and \( n_i \in \omega \), \( x_i \in \{ 0,1 \} \), and \( y_i \in \omega \) for each \( i < \ell \).
- If \( g \in A \) and \( \langle n_e,0,y_e \rangle \) is in the \( e \)th position of \( g(0) \), then
  \[
  \exists m \exists \sigma (\forall \tau \geq \sigma (\Phi_e(n_e^\sigma)(m)^\uparrow))
  \]
  - Any \( h \in B \) with \( h(0) = n_e \) is of the form \( h = n_e^\sigma \) \( f \), where \( |\sigma| = y_e \).
- If \( g \in A \) and \( \langle n_e,1,y_e \rangle \) is the \( e \)th position of \( g(0) \), then
  \[
  \forall m \forall \sigma \exists \tau \geq \sigma (\Phi_e(n_e^\sigma)(m)^\downarrow)
  \]
  - Any \( h \in B \) with \( h(0) = n_e \) is of the form \( h = \eta(e,n,e)^{1^\sigma} \) \( f \) for some \( i \in \omega \).
- The above properties hold with the roles of \( A \) and \( B \) reversed.

We construct \( A \) and \( B \) in stages. The construction is similar to the construction in Lemma 2.3 in that if \( g \) goes into \( A \) before \( h \) goes into \( B \), then \( h(0) \) codes how to recover \( f \) from \( g \), and similarly with the roles of \( A \) and \( B \) reversed. Start at stage 0 with \( A = \emptyset \), \( B = \emptyset \), \( s = \langle \rangle \), and \( t = \langle \rangle \).

Stage \( e+1 \): Set \( n_e = e^\ell \).

Case 1: \( \exists m \exists \sigma (\forall \tau \geq \sigma (\Phi_e(n_e^\sigma)(m)^\uparrow)) \). Choose such a \( \sigma \) and put \( n_e^\sigma \) \( f \) in \( A \). Update \( s = s^\langle n_e,0,|\sigma| \rangle \).

Case 2: \( \forall m \forall \sigma (\exists \tau \geq \sigma (\Phi_e(n_e^\sigma)(m)^\downarrow)) \). Put the functions \( \eta(e,n,e)^{1^\sigma} f \) in \( A \) for each \( i \in \omega \). Update \( s = s^\langle n_e,1,0 \rangle \).

Repeat the above procedure with the roles of \( A \) and \( B \) reversed and the roles of \( s \) and \( t \) reversed. This completes stage \( e+1 \). Then go on to stage \( e+2 \). This completes the construction.

Suppose \( A \geq_M C \) where \( C \) is closed. We show that \( C \) contains a recursive function. The proof with \( B \) in place of \( A \) is the same. Let \( \Phi_e(A) \subseteq C \). Consider stage \( e+1 \) of the above construction. Case 1 must not have occurred because otherwise \( A \) would contain a function \( n_e^\sigma \) \( f \) such that \( \Phi_e(n_e^\sigma f) \) is not total. Thus case 2 occurred, and so \( A \) contains the function \( \eta(e,n,e)^{1^\sigma} f \) for each \( i \in \omega \). Let \( k \) be the recursive function \( k = n_e^\sigma \sigma_0^\sigma \cdots 0^\sigma \sigma_2^\sigma \cdots 0^\sigma \) \( f_i \) for each \( i \in \omega \) (think of \( k \) as the “limit” of the strings \( \eta(e,n,e) \) as \( i \to \infty \)). Then \( \Phi_e(\eta(e,n,e)^{1^\sigma} f) \in C \) and \( \Phi_e(\eta(e,n,e)^{1^\sigma} f) \uparrow i = \Phi_e(k) \uparrow i \) for each \( i \in \omega \). Thus \( C \) contains the recursive function \( \Phi_e(k) \) because \( C \) is closed.

We now describe a uniform procedure for producing \( f \) from \( g \oplus h \in A + B \). First decode \( h(0) \) as \( h(0) = \langle \ell, \langle n_0,x_0,y_0 \rangle, \ldots, \langle n_{\ell-1},x_{\ell-1},y_{\ell-1} \rangle \rangle \) and look for \( g(0) \) among the \( n_e \). If \( \langle g(0),0,y_e \rangle \) appears in \( h(0) \) at position \( e \), then output \( g \) from position \( y_e+1 \) onward as in this case \( g = \sigma^\ell \) for
some string $\sigma$ of length $y_e+1$. If $(g(0), 1, 0)$ appears in $h(0)$ at position $e$, then $g = \eta(e, g(0), i)^{\sim 1}f$ for some $i \in \omega$. Compute which $i$ by successively computing the $\eta(e, g(0), j)$, matching them against $g$, and checking if the next bit of $g$ is 0 (in which case compute $\eta(e, g(0), j+1)$) or 1 (in which case $j = i$). Output $f$ once $i$ is found.

The number $g(0)$ appears among the $n_e$ coded into $h(0)$ if $g$ went into $A$ before $h$ went into $B$. Otherwise $h$ went into $B$ before $g$ went into $A$, so $h(0)$ appears among the $n_e$ coded in $g(0)$. In this case, switch the roles of $g$ and $h$ and apply the above procedure to compute $f$.

**Theorem 5.3.** $\mathfrak{F}_{cl}$ is not prime. In fact, if $G \subseteq \mathfrak{F}_{cl}$, $G \neq \{1\}$ is a filter, then $G$ is not prime.

**Proof.** Suppose $G \subseteq \mathfrak{F}_{cl}$ is a filter such that $G \neq \{1\}$. Let $f >_T 0$ be such that $\{\{f\}\} \in G$. Let $A, B \subseteq \omega^\omega$ be as in Lemma 5.2 for this $f$. Let $a = [A]$ and $b = [B]$. Then $a, b \notin G$ because $a, b \notin \mathfrak{F}_{cl}$, but $a + b \in G$ because $a + b \geq_M \{\{f\}\}$. □

If $x$ and $y$ are degrees such that $y$ is closed and $y \not\leq_M x$, then there is no dense degree $z$ such that $y \leq_M x + z$ [7]. Therefore, if $G \subseteq \mathfrak{F}_{cl}$, $G \neq \{1\}$ is a filter, then any degrees $a$ and $b$ witnessing that $G$ is not prime must both be in $\mathfrak{G} - G$.

The results of Section 3 suggest two new filters to study:

**Definition 5.4.**

- $G$ denotes the filter generated by
  $$\{d \mid d >_M 0' \text{ and is join-irreducible}\}.$$
- $\mathfrak{H}$ denotes the filter generated by
  $$\left\{\left[\bigcup_{i \in \omega} D_i\right] \mid (\forall i \in \omega)([D_i] >_M 0' \text{ and is join-irreducible})\right\}.$$

$G$ is exactly the set of all degrees $b$ for which $b \geq_M \prod_{i=0}^{n-1} d_i$ for some join-irreducible degrees $d_i >_M 0'$, $i \leq n$, and $\mathfrak{H}$ is exactly the set of all degrees $b$ for which $b \geq_M \left[\bigcup_{i \in \omega} i^{-D_i}\right]$ for some join-irreducible degrees $[D_i] >_M 0'$, $i \in \omega$.

**Theorem 5.5.** $\mathfrak{F}_{cl} \subseteq G \subseteq \mathfrak{H} \subseteq \{a \mid a >_M 0'\}$. $G \not\subseteq \mathfrak{F}$ (hence also $\mathfrak{H} \not\subseteq \mathfrak{F}$). Neither $G$ nor $\mathfrak{H}$ is principal.

**Proof.** Every closed degree $>_M 0'$ bounds a join-irreducible degree $>_M 0'$ [16]. Hence $\mathfrak{F}_{cl} \subseteq G$. $G \subseteq \mathfrak{H}$ is clear. To see $G \not\subseteq \mathfrak{F}$, observe that every $B_f$ is dense, so if $f >_T 0$, then $[B_f] \in G$. $\not\in \mathfrak{F}_{cl}$. This also shows $G \not\subseteq \mathfrak{F}_{cl}$. The degree constructed in Theorem 3.8 witnesses $\mathfrak{H} \not\subseteq G$. The degree constructed in Theorem 3.10 witnesses $\{a \mid a >_M 0'\} \not\subseteq \mathfrak{H}$. We show that $G$ is not principal. The proof for $\mathfrak{H}$ is the same. First, if $A$ is countable and contains no recursive functions, then there is a function $f >_T 0$ such that $g \not\leq_T f$ for all $g \in A$. Thus $B_f \leq_M A$ (as $A \subseteq B_f$) for this $f$. Every $B_f$ for $f >_T 0$ is in $G$, so every $[A]$ where $A$ is countable and contains no recursive function is in $G$. If $G$ were principal, it would be generated by a degree $x$ such that $x \leq_M [A]$ for all countable $A$ not containing a recursive function. By Lemma 4.2, the only such $x$ are $0$ and $0'$. We know $0$ and $0'$ are not in $G$, so $G$ cannot be principal. □

We end with a question.

**Question 5.6.**

- Is $\mathfrak{F} \subseteq G$? Is $\mathfrak{F} \subseteq \mathfrak{H}$?
- Is $G$ prime? Is $\mathfrak{H}$ prime?
- Is $\{a \mid \text{Th}(\mathcal{M}/a) \subseteq \text{JAN}\}$ a filter?

To prove that $\{a \mid \text{Th}(\mathcal{M}/a) \subseteq \text{JAN}\}$ is a filter, it suffices to prove that $\text{Th}(\mathcal{M}/(a \times b)) \subseteq \text{JAN}$ whenever both $\text{Th}(\mathcal{M}/a)$ and $\text{Th}(\mathcal{M}/b)$ are $\subseteq \text{JAN}$ because $\{a \mid \text{Th}(\mathcal{M}/a) \subseteq \text{JAN}\}$ is upward-closed in $\mathcal{M}$. 
Acknowledgements

Thanks to my advisor, Richard Shore, for introducing me to the Medvedev degrees and for many helpful discussions about them. Thanks also to Andrea Sorbi and Sebastiaan Terwijn for suggesting several of the problems considered here. This research was partially supported by NSF grants DMS-0554855 and DMS-0852811.

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