$P\Sigma_1$ and Combinatorial Principles

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$P\Sigma_n$

$P\Sigma_n$ is the following arithmetic sentence:

For each a Σ_n partial function f and each k there exists a sequence p (called an approximation) of length $k+1$, s.t.

$$\forall i < k \forall x, y (x \leq p(i) \wedge y = f(x) \downarrow \rightarrow y \leq p(i+1)).$$

Proposition

$(\mathrm{PA}^- + P\Sigma_n)$ *For every Σ_n partial function f and every pair a and b, there exists a finite sequence p of length $b+1$, s.t., $p(0) = a$, for each $i < b$,*

$$p(i+1) = \begin{cases} f(p(i)) & f(p(i)) \text{ is defined} \\ p(i) & \text{otherwise.} \end{cases}$$

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$P\Sigma_n, I\Sigma_n, B\Sigma_n$

Comparing $P\Sigma_n, I\Sigma_n, B\Sigma_n$:

- $I\Sigma_n$: finite length iterations of Σ_n total functions are finite.
- $P\Sigma_n$: finite length iterations of Σ_n partial functions are finite.
- $B\Sigma_n$: Σ_n functions with finite domains have finite graphs.

For $n > 0$, over $\mathrm{PA}^- + I\Sigma_1$,

see the monograph of Hájek and Pudlák (1993).

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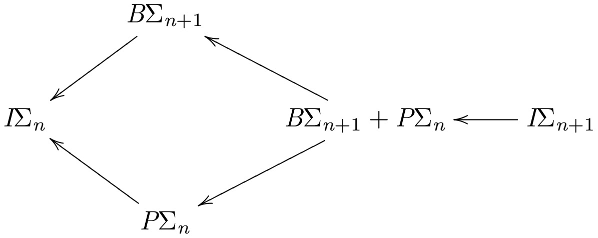

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Variants of $P\Sigma_1$

Theorem (Kreuzer and Yokoyama, 2016)

Over $I\Sigma_1$, the following statements are equivalent:

- $P\Sigma_1$.
- BME_1.
- ω^ω *is well-ordered.*
- *Ackermann-Péter function is total.*

Theorem (Simpson, 1988)

Over RCA_0, the following statements are equivalent:

- ω^ω *is well-ordered.*
- *Hilbert's Basis Theorem (if K is a field and $X_1, \ldots, X_n$ are variable symbols then every ideal in the polynomial ring $K[X_1, \ldots, X_n]$ is finitely generated).*

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- Chong, Slaman and Yang (2017) exploit BME again, and prove that $\mathsf{RCA}_0 + \mathsf{RT}^2_2 \not\vdash I\Sigma_2$.
- Patey and Yokoyama (2018) prove RT^2_2 is Π^0_3-conservative over RCA_0, thus $\mathsf{RCA}_0 + \mathsf{RT}^2_2 \not\vdash P\Sigma_1$.
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- Chong, Li, Wang and Yang (2020) prove that TT^1 is Π^1_1-conservative over $\mathsf{RCA}_0 + B\Sigma^0_2 + P\Sigma^0_1$.
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A Conservation Theorem

We illustrate how to (and why) work with $P\Sigma_1$, by sketching a (new) proof of the following conservation theorem.

Theorem (CLWY, 2020)

TT^1 *is* Π^1_1*-conservative over* $\mathsf{RCA}_0 + B\Sigma^0_2 + P\Sigma^0_1$.

Recall: TT^1 says that every finite coloring of $2^{<\mathbb{N}}$ admits a homogeneous tree isomorphic to $2^{<\mathbb{N}}$.

- ($I\Sigma_2$) Every recursive instance of TT^1 admits a recursive solution.
- (Corduan, Groszek and Mileti, 2010) If $M \models B\Sigma_2 + \neg I\Sigma_2$, then there is an M-recursive instance of TT^1 without M-recursive solution.
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One Step Expansion

A model $\mathcal{M} = (M, \mathcal{S})$ is principal iff $\mathcal{S} = \{Y : Y \leq_T X\}$ for some X.

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Let $\mathcal{M}$ be a countable principal model of $B\Sigma_2^0 + P\Sigma_1^0$ and $C \in \mathcal{M}$ be $C : 2^{<M} \to a$. Then there is a C-homogeneous tree T, s.t. $T \cong 2^{<M}$ and $\mathcal{M}[T] \models B\Sigma_2^0 + P\Sigma_1^0$.

W.l.o.g., assume $\mathcal{M} \not\models I\Sigma_2^0$.

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Big Picture

finite (infinite) means M-finite (M-infinite).

1. Fix $C : 2^{<M} \to a$ in $\mathcal{M}$.
2. Build a Σ^0_2-bounded Σ^0_2 tree $\mathcal{U}$, s.t.
   - each $u \in \mathcal{U}$ is a finite (binary) tree,
   - the successor relation on $\mathcal{U}$ is the end-extension relation on finite trees,
   - if X is an infinite path on $\mathcal{U}$ then $T = \bigcup_{u \in X} u$ is generalized low, $\mathcal{M}[T] \models P\Sigma^0_1$, and for some $c < a$ the set $C^{-1}(c)$ is dense in T.
3. Apply a Basis Theorem by Chong, Slaman and Yang (2012) to obtain an infinite path X on $\mathcal{T}$ s.t. $\mathcal{M}[T] \models B\Sigma^0_2 + P\Sigma^0_1$ for the corresponding T.
4. The density of $C^{-1}(c)$ implies that T can compute a C-homogeneous subtree isomorphic to $2^{<M}$.

The Tree of Trees

The tree $\mathcal{U}$ of finite trees takes its nodes from a Σ^0_2-sequence of finite trees $\vec{T} = (T_\ell : \ell \in M)$ s.t.

1. $T_{\ell+1}$ end-extends T_ℓ.
2. If L is an finite set of leaves of T_ℓ and $|L| > \ell$, then
   - $T_{\ell+1}[L]$ contains a copy of $2^{<\ell}$,
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3. If σ is a leaf of T_ℓ and σ^- its parent then $\mathrm{PL}[\sigma^-, \sigma] = \mathrm{PL}[\sigma, \infty]$, where

$$\mathrm{PL}[\rho, \sigma] = \{ C(\xi) : \rho \preceq \xi \preceq \sigma \}, \quad \mathrm{PL}[\sigma, \infty] = \bigcup_{\tau \succ \sigma} \mathrm{PL}[\sigma, \tau].$$

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Deciding Many Σ^0_1-sentences

For U a tree and $X \subset U$, let $U[X]$ be the set of nodes of U comparable with some nodes in X. So $U[X]$ is a subtree of U.

Fix an finite tree S and $w, b \in M$. Define a Σ^0_1 partial function f as follows.

> *For each finite tree E end-extending S, let $f(E)$ be the first end-extension F of E, s.t., there exists a set of $> w$ many leaves $L \subset S$ and a Σ^0_1-sentence φ below b s.t., $\varphi(F[L])$ holds but $\varphi(E[L])$ does not hold.*

A sufficiently long finite iteration of f starting at S produces U, s.t., U end-extends S, if L is a set of $\geq w$ many leaves of S then $U[L]$ decides every Σ^0_1-sentence below b.

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Many Layers of Trees

With more involved applications of $P\Sigma^0_1$, we can build the desired Σ^0_2-sequence $\vec{T} = (T_\ell : \ell \in M)$.

Recall the expected properties of $\vec{T}$:

1. $T_{\ell+1}$ end-extends T_ℓ.
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Some Combinatorics

Fix a Σ^0_2-cut $I \neq M$ and a cofinal Σ^0_2-sequence $(\ell_i : i \in I)$ growing sufficiently fast.

Then for each $k \in I$, there exists a finite tree S satisfying the following Σ^0_2-conditions,

(U1) each leaf of S is a leaf of T_{ℓ_k};

(U2) if $i < k$ then $S \cap T_{\ell_{i+1}}$ has an initial segment isomorphic to $2^{<\ell_i}$;

(U3) $\bigcap\{PL[\sigma^-, \sigma] : \sigma \text{ is a leaf of } S \cap T_{\ell_i}\} \neq \emptyset$ for each $i \leq k$ (in fact, $\mathrm{PL}[\sigma^-, \sigma]$'s are constant for σ's leaves of $S \cap T_{\ell_k}$);

(U4) if $i < k$ then $S \cap T_{\ell_{i+1}}$ decides each Σ_1-formula with index below ℓ_i and forces $P\Sigma^0_1$-instances below ℓ_i (in fact, $T_{\ell_{i+1}}[S \cap T_{\ell_i}] \subseteq S$).

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An Almost Homogeneous Tree

Let $\mathcal{U}$ be the finite trees satisfying the Σ^0_2-conditions (U1-4). So $\mathcal{U}$ is a Σ^0_2 tree of trees, and it is Σ^0_2-bounded.

Theorem (Chong, Slaman and Yang, 2012)

Let $\mathcal{M}$ be a countable principal model of $\mathsf{RCA}_0 + B\Sigma^0_2$. If $\mathcal{T}$ is a $\Sigma^0_2(\mathcal{M})$-bounded $\Sigma^0_2(\mathcal{M})$ tree, s.t. every infinite path of $\mathcal{T}$ is generalized low, then $\mathcal{T}$ has an infinite path X s.t. $\mathcal{M}[X] \models \mathsf{RCA}_0 + B\Sigma^0_2$.

By (U4) and the above theorem, $\mathcal{U}$ has an infinite path T (a binary tree), s.t. $\mathcal{M}[T] \models \mathsf{RCA}_0 + B\Sigma^0_2$.

By (U4) again, $\mathcal{M}[T] \models P\Sigma^0_1$; and by (U2), $T \cong 2^{<M}$.

A Homogeneous Tree

Now we have $\mathcal{M}[T] \models B\Sigma^0_2$; and by (U3), for each n,

$$\mathrm{PL}_n = \bigcap \{PL[\sigma, \infty] : \sigma \in T \cap 2^n\} \neq \emptyset.$$

Note that the map $n \mapsto \mathrm{PL}_n$ is Δ^0_2 in $C \oplus T$, and non-increasing in n.

Since $\mathrm{PL}_n \subseteq a$, by $\mathcal{M}[T] \models B\Sigma^0_2$,

$$\bigcap_n \mathrm{PL}_n \neq \emptyset.$$

Pick $c \in \bigcap_n \mathrm{PL}_n$. Then T computes a C-homogeneous tree (in color c) isomorphic to $2^{<M}$.

Thank you for your attention.