New operators on Weihrauch degrees and their applications to separation results

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$$a \quad \not\leq \quad b$$

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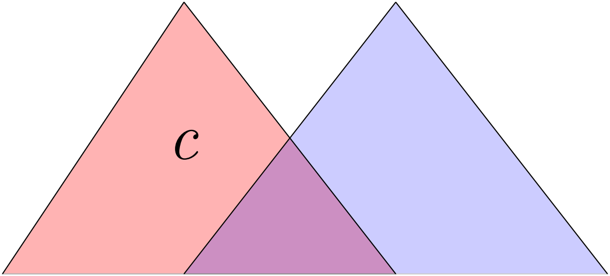

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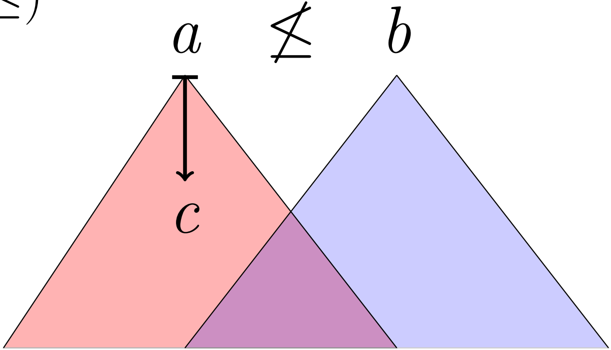

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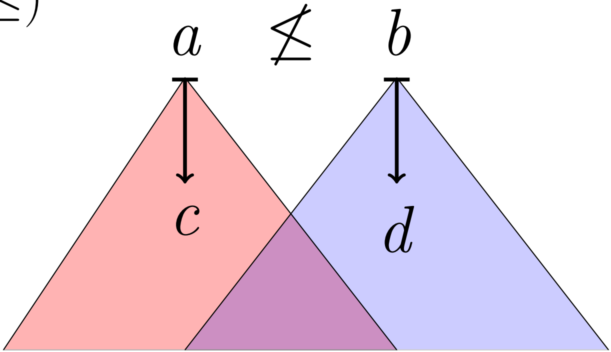

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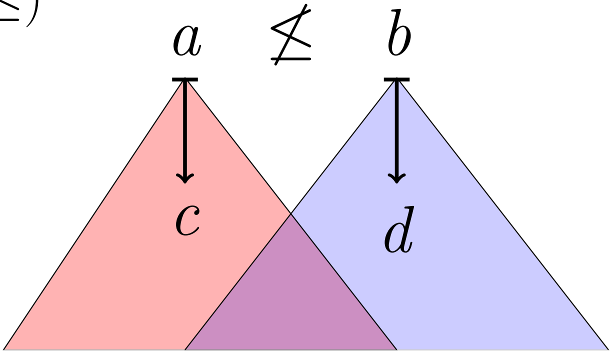

If c and d are maxima (in the resp. lower cones) satisfying some property φ then

$$c \not\leq d \Rightarrow a \not\leq b$$

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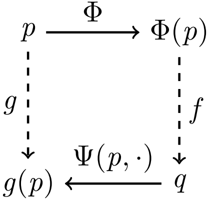

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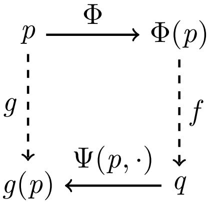

$g \leq_{\mathrm{sW}} f :\iff g \leq_{\mathrm{W}} f$ and Ψ does not depend on p.

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If g has codomain Y and there is a computable injection $Y \to \mathbb{N}$ with computable inverse we say that it is *first-order*.

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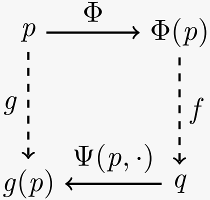

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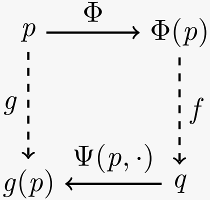

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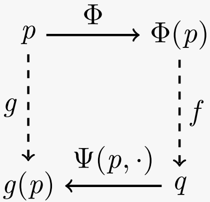
$g(p) \in \mathbb{N}$, hence for every $q \in f(\Phi(p))$,

$$\Phi_w(q)(0) = \Psi(p, q)(0) \downarrow\in g(p)$$



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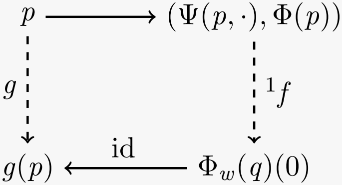
It follows that $g \leq_{\mathrm{W}} {}^1f \leq_{\mathrm{W}} f$.



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${}^1(\cdot)$ is an interior operator:

- ${}^1({}^1 f) \equiv_{\mathrm{W}} {}^1 f \leq_{\mathrm{W}} f$
- $f \leq_{\mathrm{W}} g \Rightarrow {}^1 f \leq_{\mathrm{W}} {}^1 g$

In particular, ${}^1 f \not\equiv_{\mathrm{W}} {}^1 g \Rightarrow f \not\equiv_{\mathrm{W}} g$.

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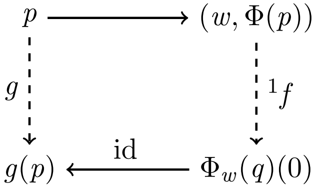

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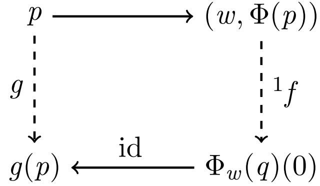
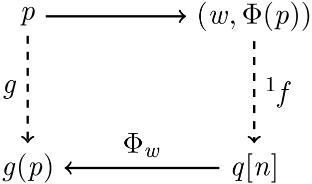


By the continuity of Ψ, only a prefix of q is needed to solve g.

$q[n]$ is sufficiently long so that Φ_w converges on 0.



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This is an interior operator, in particular

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- $\boldsymbol{\Sigma}^1_1$-$\mathsf{C}_{\mathbb{N}}$: given a list of subtrees of $\mathbb{N}^{<\mathbb{N}}$, find the index of an ill-founded one.

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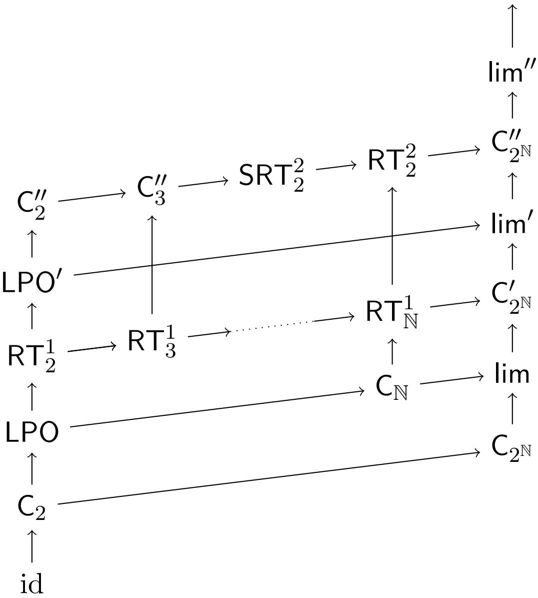
f'	:	*jump in the Weihrauch lattice*
$\vdots$		

Overview of the Weihrauch lattice



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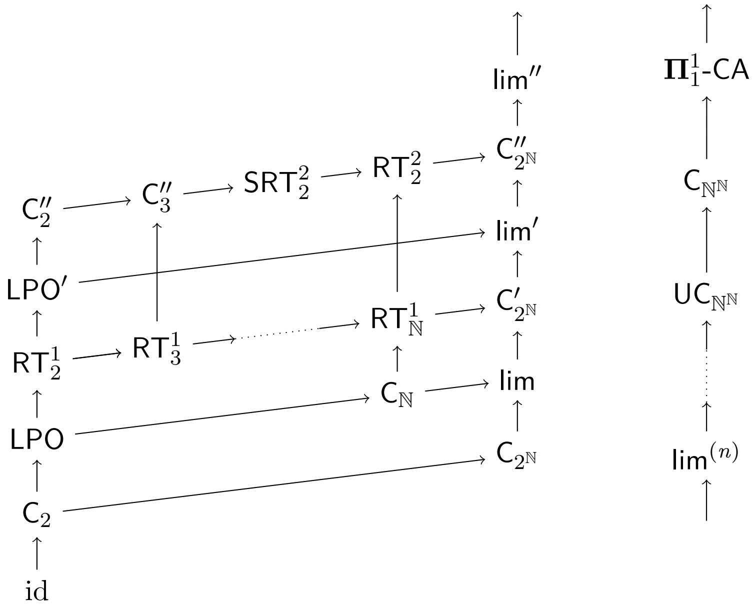

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(Kihara, Marcone, Pauly): $\mathbf{\Pi}^1_1{-}\mathsf{Bound} \not\leq_{\mathrm{W}} {}^1\mathbf{\Sigma}^1_1\text{-}\mathsf{WKL}$

Finding descending sequences

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DS is the problem of finding a descending sequence in a given linear order.

$$\mathsf{DS}(\leq_L) := \{x \in \mathbb{N}^\mathbb{N} : (\forall i)(x(i+1) \leq_L x(i))\}$$

How complicated is DS?

Theorem (Goh, Pauly, V.)

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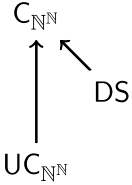

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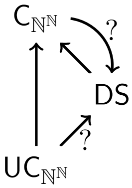
$\mathsf{DS} \le_W \mathsf{C}_{\mathbb{N}^\mathbb{N}}$ *and* $\mathsf{DS} \not\le_W \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$



Answer: NO to both the questions.

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$\mathbf{\Pi}^1_1$−Bound: given a finite $\mathbf{\Pi}^1_1$ subset of $\mathbb{N}$, produce an upper bound

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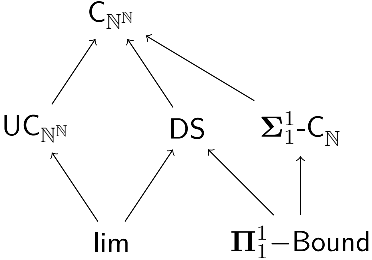
- $^1\mathsf{DS} \equiv_W \mathbf{\Pi}^1_1\text{−}\mathsf{Bound} <_W {}^1\mathsf{C}_{\mathbb{N}^\mathbb{N}} \equiv_W \mathbf{\Sigma}^1_1\text{-}\mathsf{C}_\mathbb{N}$
- $\mathrm{Det}(\mathsf{DS}) \equiv_W \mathsf{lim} <_W \mathrm{Det}(\mathsf{UC}_{\mathbb{N}^\mathbb{N}}) \equiv_W \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$



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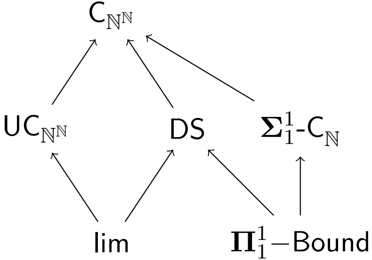

Characterizing the FO/Det part is hard in general.

Some algebraic properties

Proposition (Soldà, V.)

The first-order part distributes over joins and meets.

$$^1(f \sqcup g) \equiv_W {}^1f \sqcup {}^1g \qquad {}^1(f \sqcap g) \equiv_W {}^1f \sqcap {}^1g$$

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Proposition (Soldà, V.)

$$\begin{array}{ccc} {}^1f * {}^1g & \longrightarrow & {}^1(f * g) \\ \uparrow & & \uparrow \\ {}^1f \times {}^1g & \longrightarrow & {}^1(f \times g) \end{array}$$

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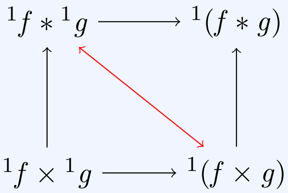

The unbounded-* operator

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input : $(w, (x_n)_{n\in\mathbb{N}})$ s.t. $(x_n)_n \in \operatorname{dom}(\widehat{f})$ and for every $(y_n)_n \in \widehat{f}((x_n)_n)$, there is $k \in \mathbb{N}$ s.t

$$\Phi_w(\langle y_i \rangle_{i<k})(0)\downarrow \text{ in } k \text{ steps}$$

output : every finite sequence $(y_n)_{n<k}$ s.t. for every n, $y_n \in f(x_n)$ and $\Phi_w(\langle y_i \rangle_{i<k})(0)\downarrow$ in k steps.

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$(\cdot)^{u*}$ *is a closure operator:*

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If f is first-order then so is f^{u*}.

FOP and parallelization

Theorem (Soldà, V.)

For every f, ${}^{1}(\widehat{f}) \equiv_{\mathrm{W}} {}^{1}(f^{u*})$.

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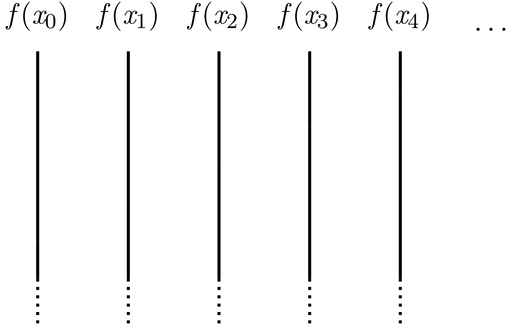

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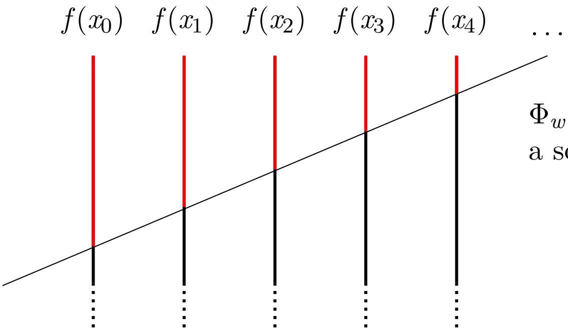

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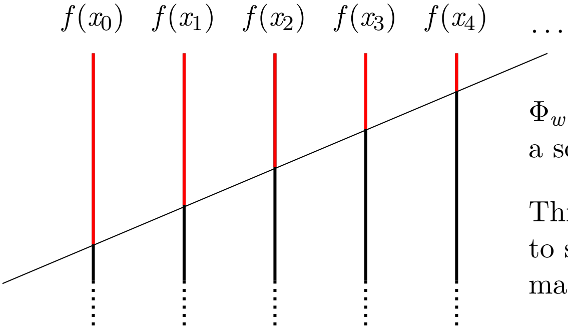

Φ_w selects a prefix of a solution.

This corresponds to selecting finitely many columns.

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For every first-order f,

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If $\mathrm{id}_2 \leq_{\mathrm{sW}} f$ *then this lifts to jumps: for every* n

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Is this peculiar of first-order problems?

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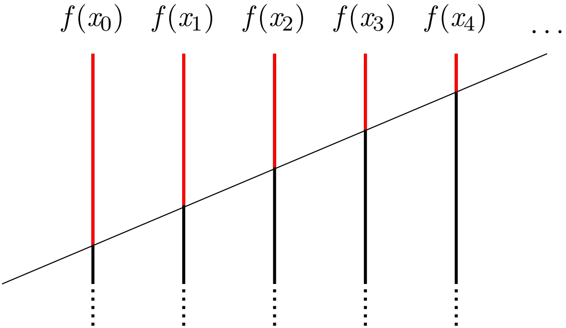

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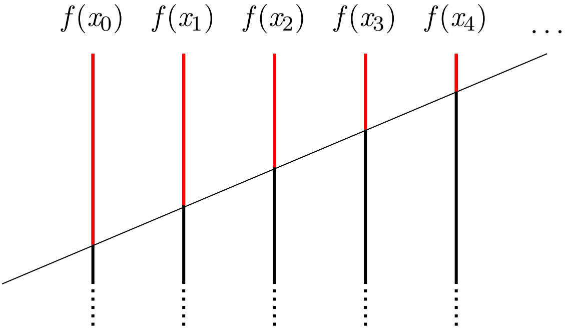
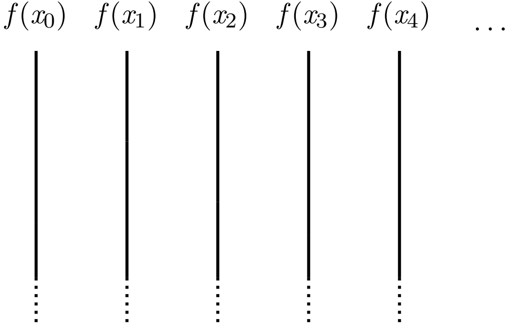


Φ_w selects a prefix of a solution.

The prefix of $f(x_i)$ may depend on the solution to x_j.

FOP and unbounded-*

On the other hand, an input for $({}^1f)^{u*}$ is $(w, (w_n, x_n)_n)$ s.t.



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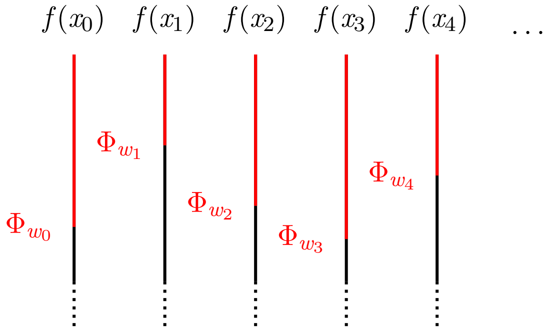

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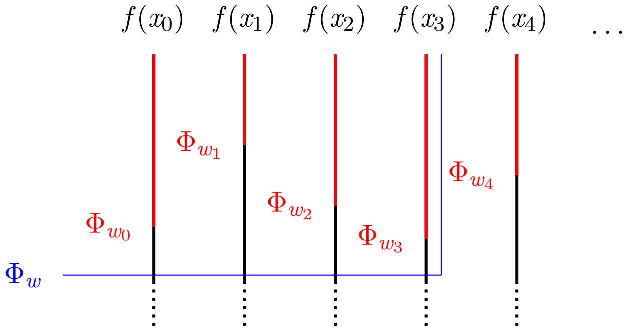

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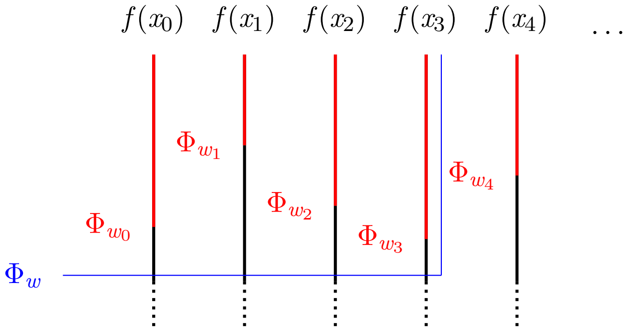

The prefix of $f(x_i)$ is independent of the solution of x_j.

FOP and unbounded-*

In some cases, we have a work around. E.g. if $f :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ is finitely valued (for every $p \in \operatorname{dom}(f)$, $|f(p)| < \infty$) then

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Lemma

There are two sequences $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ of subsets of $\mathbb{N}$ s.t.

- *for every n, $\emptyset' \not\le_T A_n$, $\emptyset' \not\le_T B_n$, but $\emptyset' \le_T A_n \oplus B_n$;*
- *for every n and every computable functional Ψ s.t. $\emptyset' = \Psi(\langle A_i \rangle, B_n)$, the map sending x to the prefix of B_n used in the computation of $\emptyset'(x)$ is not B_n-computable.*

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$$^1(\mathsf{WKL}^{(n)}) \equiv_W (\mathsf{C}_2^{(n)})^{u*} \equiv_W (\mathsf{C}_2^{*})^{(n)} \equiv_W (\mathsf{C}_2^{u*})^{(n)}$$

Ramsey's theorem

Theorem (Brattka, Rakotoniaina)

For every $n > 1$ *and* $k \geq 2$, $\mathsf{C}_k^{(n)} \leq_{\mathrm{W}} \widehat{\mathsf{SRT}_k^n} \leq_{\mathrm{W}} \widehat{\mathsf{RT}_k^n} \equiv_{\mathrm{W}} \mathsf{WKL}^{(n)}$

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Are the last two reductions strict?

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Open question (Brattka, Rakotoniaina): $\mathsf{C}'_{\mathbb{N}} \le_{\mathrm{W}} \mathsf{RT}^2_2$?

Theorem (Pauly, Soldà, V.)

For every n and $k > 1$, $\mathsf{C}^{(n)}_{\mathbb{N}} \not\le_{\mathrm{W}} \mathsf{RT}^{n+1}_k$.

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In particular, for $n = 2$, ${}^1\mathsf{SRT}^2_2 <_{\mathrm{W}} {}^1\mathsf{RT}^2_2$ (Brattka, Rakotoniaina), hence

Theorem (Soldà, V.)

$\mathsf{C}_2'' <_{\mathrm{W}} {}^1\mathsf{SRT}^2_2 <_{\mathrm{W}} {}^1\mathsf{RT}^2_2 <_{\mathrm{W}} (\mathsf{C}_2^*)'' \equiv_{\mathrm{W}} {}^1(\mathsf{WKL}'')$

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Open question (Brattka, Rakotoniaina): $\mathsf{C}'_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{RT}^2_2$?

Theorem (Pauly, Soldà, V.)

For every n and $k > 1$, $\mathsf{C}^{(n)}_{\mathbb{N}} \not\leq_{\mathrm{W}} \mathsf{RT}^{n+1}_k$.

In particular, for $n = 2$, ${}^1\mathsf{SRT}^2_2 <_{\mathrm{W}} {}^1\mathsf{RT}^2_2$ (Brattka, Rakotoniaina), hence

Theorem (Soldà, V.)

$\mathsf{C}''_2 <_{\mathrm{W}} {}^1\mathsf{SRT}^2_2 <_{\mathrm{W}} {}^1\mathsf{RT}^2_2 <_{\mathrm{W}} (\mathsf{C}^*_2)'' \equiv_{\mathrm{W}} {}^1(\mathsf{WKL}'')$

Can we fully characterize ${}^1\mathsf{RT}^n_k$?

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$g \leq_{\mathrm{W}} f^\diamond$ iff Player 2 has a computable winning strategy for $G(f \to g)$

unbounded-* and diamond

The diamond is essentially an "unbounded compositional product".

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Theorem (Soldà, V.)

If $f \colon \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}$ *is s.t.* $\{(x,n) \,:\, n \in f(x)\} \in \Pi^0_1$ *then* $f^{u*} \equiv_{\mathrm{W}} f^{\diamond}$

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Idea: we guess the possible answers to the oracle calls and use the effective closedness of Graph(f) to discard wrong guesses.

Examples: C_k for every $k \in \mathbb{N}$.

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Question: can we do better?

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