# Applications of single sky to r.e. sets

V. Yu. Shavrukov
v.yu.shavrukov@gmail.com

Leeds Computability Days 2022




- Good evening, Leeds.
- Is the sound OK?
- Thanks to organizers for inviting me.
- We are going to talk about ultrafilters in the algebra of recursive sets, the corresponding analogues of ultrapowers of the natural numbers, and a couple of properties they may or may not possess. After that, we shall discuss whether these considerations can be useful in the study of recursively enumerable sets.
- Without further ado, . . .

$\mathcal{R}^*$ — recursive subsets of $\omega$ mod finite

Ultrafilters $u$ in $\mathcal{R}^*$ are *recursive ultrafilters*

*Recursive ultrapowers* of $\mathbb{N} = (\omega, 0, 1, +, \times)$ (J. Hirschfeld, Nerode):

$$\mathbb{N}[u] = \Big(\{f : \omega \to \omega \mid f \text{ total recursive}\}/\text{agreement mod } u,\ \ 0, 1, +, \times\Big)$$

Fact (J. Hirschfeld). $\mathbb{N}[u] \models \mathrm{Th}_{\forall\exists}\mathbb{N}$. ■

$\mathrm{Th}_{\forall\exists}\mathbb{N} = \mathrm{Th}_{\Pi_2}\mathbb{N} = \mathrm{TA}_2 \vdash \mathrm{I}\Delta_0 + \exp$



$\mathcal{R}^*$ — recursive subsets of $\omega$ mod finite

Ultrafilters $u$ in $\mathcal{R}^*$ are *recursive ultrafilters*

*Recursive ultrapowers* of $\mathbb{N} = (\omega, 0, 1, +, \times)$ (J. Hirschfeld, Nerode):

$\mathbb{N}[u] = \Big(\{f : \omega \to \omega \mid f \text{ total recursive}\}/\text{agreement mod } u,\ \ 0, 1, +, \times\Big)$

Fact (J. Hirschfeld). $\mathbb{N}[u] \models \mathrm{Th}_{\forall\exists}\mathbb{N}$. ■

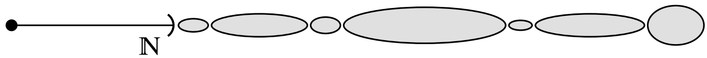
$\mathrm{Th}_{\forall\exists}\mathbb{N} = \mathrm{Th}_{\Pi_2}\mathbb{N} = \mathrm{TA}_2 \vdash \mathrm{I}\Delta_0 + \exp$

- The lattice of recursive sets modulo finite differences is denoted by R-asterisk. We do away with finite differences because we wish to exclude principal ultrafilters.
- The ultrafilters in R-asterisk are called *recursive ultrafilters*. (No, it is not me who is responsible for the term.)
- In some sense, all these utrafilters look the same: R-asterisk is the countable atomless Boolean algebra, so any two ultrafilters are taken one the other by an appropriate automorphism. We shall nevertheless see that they can still exhibit distinct behaviour when one takes into account the ultrafilters' recursive content.
- Joram Hirschfeld studied *recursive ultrapowers* of the natural numbers together with the arithmetical structure:
- Into the recursive ultrapower, one only allows total recursive functions as elements.
- These are then quotiented by agreement on some set in the recursive ultrafilter.
- As usual, the operations are pointwise.
- What you get is a nonstandard model of the true forall-exists arithmetic.
- The true forall exists arithmetic gives you the same theory as the true $\Pi_2$ arithmetic and is denoted TA-2,
- so you get, for example, $\Delta_0$-induction, as well as the totality of all functions that are total recursive in the real world. This suffices for coding sequences, truth predicates, and a healthy portion of finite combinatorics.

Let $a, b \in M \models \mathrm{TA}_2$.

$$b \lll a \iff \exists \text{ total recursive } f \quad M \models f(a) \geq b$$

$\lll$ is a linear preorder; equivalence classes are (total recursive) *skies*




Theorem. *If $M \models \mathrm{TA}_2$ has at least two distinct non-standard skies, then the skies of $M$ are densely ordered (including $\mathbb{N}$)* ∎

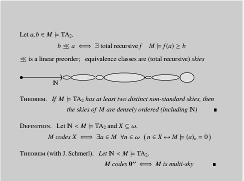
Definition. Let $\mathbb{N} < M \models \mathrm{TA}_2$ and $X \subseteq \omega$.

$$M \textit{ codes } X \iff \exists a \in M \;\; \forall n \in \omega \;\; \big(n \in X \leftrightarrow M \models (a)_n = 0\big)$$

Theorem (with J. Schmerl). *Let $\mathbb{N} < M \models \mathrm{TA}_2$.*

$$M \textit{ codes } \mathbf{0}'' \iff M \textit{ is multi-sky}$$ ∎




Let $a, b \in M \models \mathrm{TA}_2$.
$b \lll a \iff \exists$ total recursive $f$ $M \models f(a) \geq b$
$\lll$ is a linear preorder; equivalence classes are (total recursive) *skies*
Theorem. If $M \models \mathrm{TA}_2$ has at least two distinct non-standard skies, then the skies of $M$ are densely ordered (including ℕ)
Definition. Let $\mathbb{N} < M \models \mathrm{TA}_2$ and $X \subseteq \omega$.
$M$ codes $X \iff \exists a \in M \; \forall n \in \omega \; (n \in X \leftrightarrow M \models (a)_n = 0)$
Theorem (with J. Schmerl). Let $\mathbb{N} < M \models \mathrm{TA}_2$.
$M$ codes $\mathbf{0}'' \iff M$ is multi-sky

- Take an arbitrary model $M$ of forall-exists arithmetic (on particular, a recursive ultrapower). We consider a certain preorder on the elements of $M$.
- Say that $b$ *is at most total recursively larger than* $a$ if you can get from $a$ past $b$ by some total recursive function. When we say ‘recursive function’, we mean an absolutely standard recursive function, that is, one with a standard index — it just operates in a non-standard model.
- The equivalence classes of this preorder are called *(total recursive) skies*. All numbers that are at at most total recursive distance from one another go into the same sky.
- Each sky is convex and the standard numbers form the lowermost sky.
- After that come the non-standard skies.
- Here we have a dichotomy: Either our model $M$ has a single non-standard sky, or all the skies of $M$, incuding the standard one, are densely ordered, so there are infinitely many skies.
- There is a useful criterion to distinguish between the two cases. First, a definition:
- Suppose $M$ is a non-standard model of forall-exists arithmetic, and $X$ is a set of natural numbers.
- We say that $M$ *codes* $X$ if there is an element $a$ of $M$, the *code of* $X$, representing a sequence whose first $\omega$ terms are the exact printout of the content of the set $X$. This is an absolutely classical definition.
- Our theorem established together with James Schmerl says that a nonstandard $\mathrm{TA}_2$-model $M$ has infinitely many skies if and only if $M$ codes $\mathbf{0}''$ — the collection of coded sets is always closed under Turing reducibility, so one can talk about a model coding a given Turing degree.
- If $\mathbf{0}''$ is left uncoded, the model has just a single non-standard sky in addition to the standard one, so all the nonstandard elements are at most total recursively far away from one another — it’s a very compressed situation.

**Definition.** Suppose $f$ is total recursive and $R$ is recursive.

$$f \text{ is } \textit{ef21 on } R \iff \exists_{\text{t.r.}} G \ \forall y \in \omega \ \ G(y) = \left(f|_R\right)^{-1}(y)$$

($G(y)$ is the canonical code of a finite set)

$$\iff \exists_{\text{t.r.}} g \ \forall x \in R \ \ g \circ f(x) \geq x$$

($g \geq \max G$)

**Observation.** (a) *Ef21 images $f[R]$ of recursive sets are recursive.*

(b) *Ef21 images of co-r.e. sets are co-r.e.* ∎

**Proposition.** $\mathbb{N}[u]$ *is single-sky*

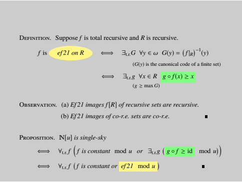
$$\iff \forall_{\text{t.r.}} f \left( f \textit{ is constant} \mod u \ \ \textit{or} \ \ \exists_{\text{t.r.}} g \left( g \circ f \geq \text{id} \mod u \right) \right)$$

$$\iff \forall_{\text{t.r.}} f \left( f \textit{ is constant or } \textit{ef21} \mod u \right)$$ ∎

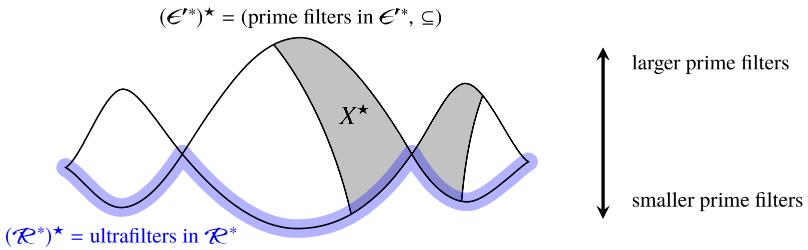
Applications of single sky to r.e. sets

Dichotomy

Effectively finite-to-one functions


Definition. Suppose $f$ is total recursive and $R$ is recursive.
$f$ is ef21 on $R$ $\iff \exists_{\text{t.r.}} G \ \forall y \in \omega \ G(y) = (f|_R)^{-1}(y)$
($G(y)$ is the canonical code of a finite set)
$\iff \exists_{\text{t.r.}} g \ \forall x \in R \ g \circ f(x) \geq x$
($g \geq \max G$)
Observation. (a) Ef21 images $f[R]$ of recursive sets are recursive.
(b) Ef21 images of co-r.e. sets are co-r.e. ∎
Proposition. $\mathbb{N}[u]$ is single-sky
$\iff \forall_{\text{t.r.}} f \left( f \text{ is constant} \mod u \text{ or } \exists_{\text{t.r.}} g \left( g \circ f \geq \text{id} \mod u \right) \right)$
$\iff \forall_{\text{t.r.}} f \left( f \text{ is constant or ef21} \mod u \right)$ ∎

- We now introduce an important definition. Suppose we have a total recursive function $f$ and a recursive set $R$. One says that $f$ is *effectively finite-to-one on* $R$ if one can effectively compute the pre-image of any element for the restriction of $f$ to $R$.
- We require that the set of pre-images is given by a total recursive function which produces the canonical code of the finite pre-image — even when this set is empty.
- 't.r.' in the subscript to the quantifier stands for 'total recursive'.
- An equivalent condition is the existence of a total recursive function $g$ which, looking at a value of $f$, gives an upper bound on those arguments of $f$ that lie in $R$.
- Here is the connection between capital-$G$ and lowercase-$g$: When you can compute the finite pre-image, it suffices to take the maximum of capital-$G$ to get an upper bound. Conversely, when you have an upper bound on the elements in the inverse image, you know when you can stop your exhaustive search for elements of the full pre-image — everything you search through is recursive.
- The ecursive set $R$ is going to be quite important — we are going to vary it much more intensively than the function $f$, probing for regions where a given function is effectively finite-to-one.
- Effectively finite-to-one functions are nice functions — the forward image of a recursive set is recursive once we know that the function is effectively finite-to-one on that recursive set. This happens for the same reason as with monotone functions.
- Similarly, forward images preserve co-recursive enumerability of subsets of those recursive sets where the function is effectively finite-to-one.
- How is this related to ultrapowers? Let us look at a single-sky ultrapower $\mathbb{N}[u]$.
- Single sky says that each element, represented by some total recursive function $f$, is either standard, in which case $f$ will have to equal a particular integer on some recursive set in the ultrafilter $u$, or, $f(x)$ must belong to the same sky as $x$. This means that some total recursive function $g$ maps $f(x)$ to some number at least as large as $x$. For this to happen in the ultrapower, it has to hold on some recursive set in the ultrafilter. ' $\mod u$', of course, means 'on some recursive set in $u$'.
- But we have already discussed how this condition is the same as effective finite-to-oneness modulo $u$.
- Thus a recursive ultrapower $\mathbb{N}[u]$ is single-sky if and only if each total recursive function is either constant or effectively finite-to-one modulo the recursive ultrafilter.




$(\mathcal{R}^*)^\star \approx \min(\mathcal{E}'^*)^\star$

Let $X$ be r.e. $X^\star$ is the *picture* of $X$. $p \ni X \iff p \in X^\star$

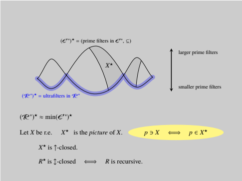
$X^\star$ is $\uparrow$-closed.

$R^\star$ is $\updownarrow$-closed $\iff$ $R$ is recursive.

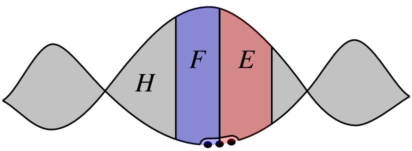
Applications of single sky to r.e. sets

Duality

$(\mathcal{R}^*)^\star$ and $(\mathcal{E}'^*)^\star$



- We are now going to actively look for single-sky recursive ultrafilters.
- The whole totality of recursive ultrafilters on $R$-asterisk is denoted by $R$-asterisk-star. The star symbolizes the passage to the dual space of a Boolean algebra, or, more generally, of a distributive lattice.
- In my mind's eye, that space looks somewhat like this. However, I prefer to see it as a quotient of a larger space,
- namely, the space of prime filters in the lattice of recursively enumerable sets modulo finite differences.
- This space is called $E$-asterisk-star.
- This larger space comes with a natural ordering, which is just the ordinary inclusion among prime filters.
- We think of larger prime filters as situated higher in the picture than the smaller ones.
- The recursive ultrafilters are essentially the same thing as the minimal prime filters on $E$-asterisk-star. This is because the minimal prime filters are exactly those that have a base of recursive elements. The recursive elements of a prime filter always form a recursive ultrafilter.
- One can use this ordered space as a canvas for drawing pictures of, say, recursively enumerable sets $X$. This is similar in spirit to the old Venn diagrams.
- The picture of $X$ is denoted by $X^\star$. It consists of exactly those prime filters that contain $X$. This equivalence is probably the central slogan of duality for distributive lattices.
- The picture of a recursively enumerable set must be upwards-closed. This is because once an r.e. set belong to a prime filter, it must belong to all larger ones.
- The picture of a recursive set must be closed in both directions. That's because the complement of a picture is the picture of the recursive complement, which must also be upwards-closed.




r.e. $H$ hhsimple
$\Leftrightarrow\ \overline{H^\star} \subseteq \min(\in'^*)^\star$

$H$ is *hhsimple* $\iff$ $\{$r.e. $X \mid X \supseteq H\}$ is a Boolean lattice

PROPOSITION. *$H$ is hhsimple* $\Longrightarrow$ *all $h \notin H^\star$ are single-sky.*

PROOF MAP. $\mathbb{N}[h]$ is an existentially closed model of $\mathrm{TA}_2 + \mathbb{N} < x \notin H$;
$\mathbb{N}[h]$ is recursively saturated for $\Sigma_1$ formulas; Not coding $\mathbf{0}''$. ∎

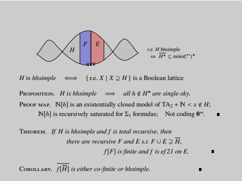
THEOREM. *If $H$ is hhsimple and $f$ is total recursive, then there are recursive $F$ and $E$ s.t. $F \cup E \supseteq \overline{H}$, $f[F]$ is finite and $f$ is ef21 on $E$.* ∎

COROLLARY. *$\overline{f[\overline{H}]}$ is either co-finite or hhsimple.* ∎

Applications of single sky to r.e. sets

Hyperhypersimple sets

Hyperhypersimplicity and ef21

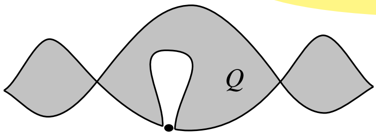


- Now, the picture of a hyperhypersimple set looks like this.
- Recall that a set $H$ is hyperhypersimple if the recursively enumerable supersets of $H$ form a non-trivial Boolean lattice.
- The subspace of $E$-asterisk-star corresponding to the lattice of supersets is exactly the complement of the picture. Since the lattice is Boolean, that complement must be an antichain. Since the picture of $H$ must be upwards closed, the complementary antichain can only lie at the bottom.
- Thus an r.e. set is hyperhypersimple if and only if all points outside the picture of $H$ are minimal points of $(\in'^*)^\star$.
- Well, all these points on the outside are single-sky.
- The proof involves noticing that the corresponding recursive ultrapowers are existentially closed models of $\mathrm{TA}_2 + \mathbb{N} < x \notin H$ where the bold $\boldsymbol{x}$ is a new constant. It can then be shown that existentially closed models enjoy recursive saturation for $\Sigma_1$ formulas (with parameters).

  If such a model coded $\mathbf{0}''$ then it cannot possibly have a topmost total recursive sky, which is not the case — recursive ultrapowers do each of them have a topmost sky. So the ultrapower cannot code $\mathbf{0}''$, so it must be single-sky.
- Using this, the following theorem can be shown: Given a hyperhypersimple set $H$, take an arbitrary total recursive function $f$.
- Then you can find two disjoint recursive sets which together cover the complement of the picture of $H$ such that the image under $f$ of one of these recursive sets is finite, and $f$ is effectively finite-to-one on the other recursive set.
- Here is how this works: Each outsider point, being single-sky, contains a recursive set on which it is either constant or ef21, so you get a covering of the complement of $H^\star$ by recursive sets. Using a compactness principle, finitely many outsider points still ensure a covering, so the recursive sets in the finite covering on which $f$ is constant unite to produce the set on which $f$ only has finitely many values, and the other recursive sets go to the other side — $f$ is ef21 on their union. Notice that if a function is ef21 on two sets, then it is also ef21 on their union.
- As a corollary, we get that any recursive image of the complement of any hyperhypersimple set is either finite or it complements some hyperhypersimple set. In particular, you cannot shake off the co-recursive enumerability by recursive images.
- Notice that the statements of the theorem and the corollary do not mention recursive ultrafilters, ultrapowers, or anything like that.

$$R_e = \{ n \in \omega \mid \forall k \leq n \ \{e\}(k)\downarrow \ \& \ \{e\}(n) = 0 \}$$

$$\text{ix}\, u = \{ e \in \omega \mid R_e \in u \} \qquad (u \in (\mathcal{R}^*)^\star)$$




$h_Q$, the *heel* of $Q$

r.e. $Q$ r-maximal
$\Leftrightarrow \ |\min \overline{Q}^\star| = 1$

**Lemma** (Lerman–Shore–Soare). *$Q$ is r-maximal* $\implies$ $\text{ix}\, h_Q$ *is* $\Delta^0_3$.

**Proof.** $e \in \text{ix}\, u \iff R_e \in h_Q \iff R_e \cup Q =^* \omega$

$e \notin \text{ix}\, u \iff R_e \notin h_Q \iff R_e \subseteq^* Q$ ∎

**Lemma.** *$\mathbb{N}[u]$ does NOT code* $\text{ix}\, u$ *for any $u \in (\mathcal{R}^*)^\star$.* ∎

**Theorem.** *$Q$ is r-maximal* $\implies$ *$\mathbb{N}[h_Q]$ is single-sky.*

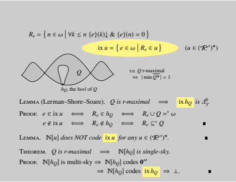
**Proof.** $\mathbb{N}[h_Q]$ is multi-sky $\Rightarrow$ $\mathbb{N}[h_Q]$ codes $\mathbf{0}''$

$\Rightarrow$ $\mathbb{N}[h_Q]$ codes $\text{ix}\, h_Q$ $\Rightarrow \perp$. ∎

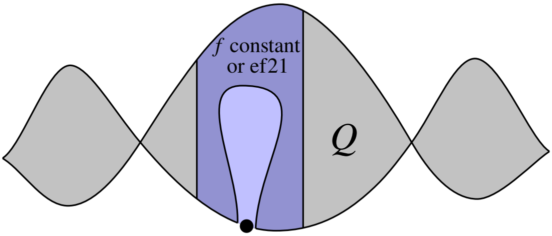
Applications of single sky to r.e. sets

r-Maximal sets

r-maximality and single skies



- Recall that a recursively enumerable set is r-maximal if you cannot nontrivially split its complement by a recursive set.
- In the picture, this corresponds to the complement having exactly one minimal point (which then necessarily must be minimal in $(\mathcal{E}'^*)^\star$). You cannot split such a complement by a set whose picture is both upwards- and downwards-closed (recall that these are the recursive sets). If you had more than one minimal point, then including one point and not including the other would not be a problem.
- The minimal point of the complement is the *heel* of the r-maximal set. Now we are going to show that the heels of r-maximal sets are always single-sky.
- We fix one of the conventional enumerations $R_e$ of all recursive sets — for such an enumeration, the membership of a number in the $e$th set is a recursively enumerable binary relation. It cannot be recursive on pain of diagonalization.
- The *index set* of an ultrafilter $u$ on the recursive set is the collection of those $R$-indices whose recursive sets lie in the ultrafilter.
- A key observation was made by Lerman, Shore and Soare: The $R$-index set of the recursive ultrafilter corresponding to the heel must lie in $\mathbf{0}''$.
- Indeed, $R_e$ is an element of $u$ if and only if the point $u$ lies in the picture of $R_e$ which will then have to cover the whole complement of $Q$ because pictures are upwards closed. This is equivalent to the union of $R_e$ and $Q$ being cofinite.
- If $R_e$ lies outside $u$, that is, the picture of $R_e$ does not cover the point $u$, then the whole picture of $R_e$ must lie within the picture $Q$. That's because no recursive set splits the complement of $Q$ non-trivially. This means that $R_e$ is a subset of $Q$ modulo some finite set.
- We see that both these conditions are $\Sigma^0_3$, so this proves the lemma.
- The next lemma says that a recursive ultrapower never codes the $R$-index set of the ultrafilter. It's one of those situations where an object cannot describe itself: its proof employs 2nd Recursion.
- We are now ready to see why the heel of an r-maximal set has to be single-sky.
- If there were many skies, then the ultrapower would code $\mathbf{0}''$. The index set of $u$ would then also be coded. But this contradicts the previous lemma.
- So here we have an example of complexity influencing structure.




Theorem. *Q is r-maximal*

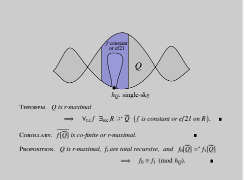
$$\Longrightarrow \quad \forall_{\text{t.r.}} f \ \ \exists_{\text{rec.}} R \supseteq^* \overline{Q} \ \ \big(f \text{ is constant or ef21 on } R\big). \qquad \blacksquare$$

Corollary. $\overline{f[\overline{Q}]}$ *is co-finite or r-maximal.* $\blacksquare$

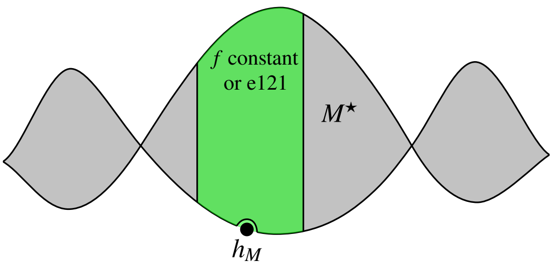
Proposition. *Q is r-maximal,* $f_i$ *are total recursive, and* $f_0[\overline{Q}] =^* f_1[\overline{Q}]$

$$\Longrightarrow \quad f_0 \equiv f_1 \ \ (\text{mod } h_Q). \qquad \blacksquare$$





- So the heels of all r-maximal sets are single-sky.
- The next theorem is just a reformulation of this fact: If $Q$ is r-maximal and $f$ is any total recursive function, then there is a recursive set $R$ including almost all of the complement of $Q$ such that $f$ is constant or effectively finite-to-one on $R$. This is because containing almost all of the complement of $Q$ is the same as belonging to the ultrafilter at the heel.
- This theorem directly answers a question posed by Eberhard Herrmann.
- As a corollary, we obtain that any total recursive image of the complement of each r-maximal set is either finite or itself complements some r-maximal set.
- A further consequence which uses single sky together with some r-maximal specifics says that the total recursive mapping from a first r-maximal complement onto a second one, if such a mapping exists, is always unique up to finite variations — any two such mappings must agree on almost all of the complement of the first r-maximal set.




r.e. $M$ maximal
$\Leftrightarrow$ $|\overline{M^\star}| = 1$

**Proposition.** $\mathbb{N}[h_M]$ *is* single-constellation*:*

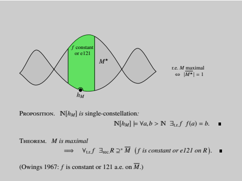
$$\mathbb{N}[h_M] \models \forall a, b > \mathbb{N} \;\; \exists_{\text{t.r.}} f \;\; f(a) = b. \qquad \blacksquare$$

**Theorem.** *$M$ is maximal*

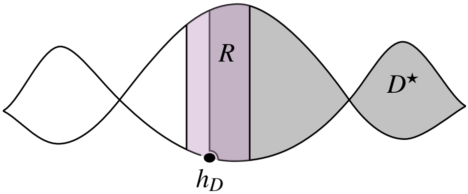
$$\Longrightarrow \quad \forall_{\text{t.r.}} f \;\; \exists_{\text{rec.}} R \supseteq^* \overline{M} \;\; \big(f \textit{ is constant or e121 on } R\big). \qquad \blacksquare$$

(Owings 1967: $f$ is constant or 121 a.e. on $\overline{M}$.)





- Next we take a look at maximal sets. Here is the picture. An r.e. set is *maximal* if and only if there is exactly one point in the complement of the picture of the set.
- Maximal sets are a subclass of both r-maximal and hyperhypersimple sets, so everything in earlier slides still applies. In particular, the recursive ultrafilter at the heel is single-sky.
- For a maximal set however the heel posesses an even stronger property: the ultrafilter is *single-constellation* (a.k.a. *tame*): This means that whenever you have any two nonstandard elements, there is a total recursive function that takes you from one to the other. In other words, there exactly two substructures: the standard numbers and the whole ultrapower.
- It follows that maximal sets possess the following property: Each total recursive $f$ is either constant or effectively one-to-one on some recursive set containing almost all of the complement of $M$. 'Effectively one-to-one' is exactly like 'effectively finite-to-one' except the recursive inverse function gives you the empty set or a singleton rather than just a finite set.
- Incidentally, 'e121' can be replaced by 'strictly monotone' if desired.
- This theorem almost coincides with the property established by Owings in 1967, who showed that each total recursive function is either constant or strictly monotone almost everywhere on the complement of the maximal set. Here we have effective one-to-oneness on a whole recursive set which almost includes the complement.




r.e. $D$ is D-maximal
$\Leftrightarrow\ |{\downarrow}D^\star - D^\star| = 1$

**Proposition.** *$D$ is D-maximal $\implies$ $h_D$ is single-sky.* ∎

**Open question.** Must $h_D$ be single-constellation?

**Proposition.** *$D$ is D-maximal*

$$\implies \forall_{\text{t.r.}} f\ \exists_{\text{rec.}} R \in h_D\ \big(f \text{ is e121 on } R \text{ or } f[R - D] \text{ is recursive}\big).$$ ∎

**Definition.**

$$X \leq_{Q_1} Y \Leftrightarrow \exists_{\text{disjoint r.e. family}} (U_i)_{i\in\omega}\ \forall i\ \ (i \in X \Leftrightarrow U_i \subseteq Y)$$
$$\Leftrightarrow \exists_{\text{p.r.}} g\ \ g[\overline{Y}] = \overline{X}$$

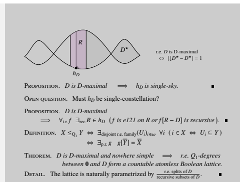
**Theorem.** *$D$ is D-maximal and nowhere simple $\implies$ r.e. $Q_1$-degrees between $\mathbf{0}$ and $D$ form a countable atomless Boolean lattice.*

**Detail.** The lattice is naturally parametrized by $\dfrac{\text{r.e. splits of } D}{\text{recursive subsets of } D}$. ∎

Applications of single sky to r.e. sets

D-maximal sets

D-maximality and single skies



- Finally, we consider *D-maximal* r.e. sets. A typical D-maximal set looks like this: All but exactly one point in the downward closure of the picture of $D$ already lie in the picture of $D$ itself. That point is, unsurprisingly, called the *heel* of $D$.
- Well, that heel is also single-sky. Curiously, both the arguments for hyperhypersimple and r-maximal sets adapt to the D-maxiaml case, so there are at least two proofs. In particular, the index set of the heel is $\Delta^0_3$.
- You would think that there should be enough similarity between maximal and D-maximal sets for the heel to be single-constellation. I have thus far failed to establish that. So the single constellation property for D-maximal heels remains an open question.
- We have however a consolation prize: Any total recursive $f$ must restrict, on some recursive set $R$ around the heel, to an effectively one-to-one function *or* the $f$-image of the difference between $R$ and $D$ is recursive. (The difference is the lightly coloured region.) This uses single sky together with some specifics of D-maximal sets. Had we had single constellation, we could replace the recursivity of the image by 'constant on $R$'.
- For an application of this proposition, we recall a particular reducibility between sets of natural numbers called $Q_1$ (or *injective quasi-reducibility*): $X$ is $Q_1$-reducible to $Y$ if there is a uniformly recursively enumerable disjoint family $U$ such that a number $i$ belongs to $X$ if and only if the $i$th element of the family is included in $Y$. This is exactly like Shoenfield's Q-reducibility with the extra requirement that the r.e. family be pairwise disjoint. Disjointness is responsible for the subscript 1: any witness can testify at most once.
- That same subscript allows a perhaps simpler view of the reducibility from the other side: $X$ reduces to $Y$ when there is a partial recursive mapping of the complement of $Y$ onto the complement of $X$. This should hint at the relevance of the preceding proposition to $Q_1$.
- We have a theorem: If $D$ is a nowhere simple $D$-maximal set, then the *recursively enumerable* $Q_1$-degrees between the degree of recursive sets and that of $D$ are a countable atomless Boolean algebra.
- *Nowhere simplicity* is a technical condition that excludes some degenerate cases such as maximal sets and maximal subsets of recursive sets, so that the heel is not a topologically isolated point of the complement.
- The r.e. $Q_1$-degrees in the interval between $\mathbf{0}$ and $D$ turn out to correspond naturally to the algebra of r.e. splits of $D$ quotiented by recursive subsets of $D$.

# Thank You!



Thank You!

- That's it for today.
- Thank you very much for your attention.