Partition genericity and pigeonhole basis theorems

Ludovic PATEY
Joint work with Benoît Monin

Pigeonhole basis theorems

Pigeonhole basis theorems

Every k-coloring of $\mathbb{N}$ admits
a monochromatic subset of some kind

0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
15	16	17	18	19
20	21	22	23	24
25	26	27	28	....

Examples in Ramsey Theory

Theorem (Pigeonhole principle)

Every k-coloring of $\mathbb{N}$ admits an infinite monochromatic set.

Theorem (Van Der Waerden)

Every k-coloring of $\mathbb{N}$ admits monochromatic arithmetic progressions of arbitrary length.

Theorem (Hindman)

Every k-coloring of $\mathbb{N}$ admits an infinite set whose non-empty finite sums of distinct elements are monochromatic.

Examples in Computability Theory

Theorem (Jockusch and Dzhafarov)

If $C \not\leq_T \emptyset$, every k-coloring of $\mathbb{N}$ admits an infinite monochromatic subset which does not compute C.

Theorem (Liu)

Every k-coloring of $\mathbb{N}$ admits an infinite monochromatic subset of non-PA degree.

Theorem (Liu)

Every k-coloring of $\mathbb{N}$ admits an infinite monochromatic subset of non-random degree.

Examples in Computability Theory

Theorem (Wang)

If $C \notin \Sigma^0_1$, every k-coloring of $\mathbb{N}$ admits an infinite monochromatic subset H such that $C \notin \Sigma^0_1(H)$.

Theorem (Patey)

If f is hyperimmune, every k-coloring of $\mathbb{N}$ admits an infinite monochromatic subset H such that f is H-hyperimmune.

Theorem (Monin and Patey)

If $C \notin \Sigma^0_\alpha$, every k-coloring of $\mathbb{N}$ admits an infinite monochromatic subset H such that $C \notin \Sigma^0_\alpha(H)$.

Partition regularity

Definition

A class $\mathcal{P} \subseteq 2^{\mathbb{N}}$ is partition regular if

(1) it is non-empty : $\mathbb{N} \in \mathcal{P}$;

(2) it is closed under superset : $\forall X \in \mathcal{P}, \forall Y \supseteq X, Y \in \mathcal{P}$;

(3) $\forall X \in \mathcal{P}, \forall Z_0 \cup Z_1 \supseteq X, Z_0 \in \mathcal{P}$ or $Z_1 \in \mathcal{P}$.

For some largeness notion

- Prove that it is partition regular
- Prove that every large set admits a subset of some kind

Largeness in Ramsey's theory

Definition

A set $A \subseteq \mathbb{N}$ has positive upper density if

$$\limsup_n \frac{|A \cap \{0, \dots, n-1\}|}{n} > 0$$

- Positive upper density is partition regular

Theorem (Szemerdéri)

If A has positive upper density, it contains arithmetic progressions of arbitrary length.

Largeness in Ramsey's theory

Definition

A set $A \subseteq \mathbb{N}$ is thick if it contains arbitrarily large intervals.

Definition

A set $A \subseteq \mathbb{N}$ is syndetic if there is some k such that for every n,

$$A \cap [n, n+k] \neq \emptyset$$

- Thickness and syndeticity are not partition regular

Largeness in Ramsey's theory

Definition

A set $A \subseteq \mathbb{N}$ piecewise syndetic if it is the intersection of a thick set and a syndetic set.

- Piecewise syndeticity is partition regular

Theorem (Folklore)

If A is piecewise syndetic, it contains arithmetic progressions of arbitrary length.

Largeness in Computability Theory

Goal: find a notion of largeness $\mathcal{P} \subseteq 2^{\mathbb{N}}$ which is partition regular and which prove classical pigeonhole basis theorems in computability theory.

- closed under superset
- closed under partitionning

Typicality and basis theorems

Two notions of typicality: genericity and randomness.

Theorem (Folklore)

If $C \not\leq_T \emptyset$ and G is sufficiently generic, then $C \not\leq_T G$.

Theorem (Sacks)

If $C \not\leq_T \emptyset$ and Z is sufficiently random, then $C \not\leq_T Z$.

Typicality and basis theorems

Two notions of typicality: genericity and randomness.

Theorem (Demuth and Kučera)

No sufficiently generic set is of PA degree.

Theorem (Kučera)

No sufficiently random set is of PA degree.

Typicality and basis theorems

Two notions of typicality: genericity and randomness.

Theorem (Demuth and Kučera)

No sufficiently generic set is of random degree.

Theorem (Kjos-Hanssen and Liu)

Every sufficiently random set has an infinite subset of non-random degree.

Largeness in Computability Theory

Goal: find a notion of largeness $\mathcal{P} \subseteq 2^{\mathbb{N}}$ which is partition regular and which prove classical pigeonhole basis theorems in computability theory.

- closed under superset
- closed under partitionning
- such that typical sets are large

Partition genericity

Partition genericity : idea

Definition (Monin, P.)

A set $A \subseteq \mathbb{N}$ is sufficiently partition generic if it belongs to sufficiently many partition regular classes.

As for randomness and genericity, we can define levels of partition genericity by asking to intersect all partition regular classes of some complexity.

Partition genericity : intuition

- Every property on sets defines a class
- Partition regular classes play the role of dense sets
- Partition generic sets satisfy all properties that can be satisfied at any time

Partition regularity

Definition

A partition regular class is principal if it is of the form $\{X \in 2^{\mathbb{N}} : n \in X\}$ for some $n \in \mathbb{N}$.

Definition

A partition regular class is non-trivial if it contains only infinite sets.

A partition regular class is non-trivial iff it does not contain any principal partition regular sublcass.

Σ_2^0 partition regular classes

Lemma (Monin, P.)

The $\underset{\sim}{\Sigma}_2^0$ partition regular classes are trivial.

Fix $\mathcal{P}$ a $\Sigma_2^{0,Z}$ trivial partition regular class

- If a $\Sigma_1^{0,Z}$ class contains all the finite sets, then it contains all the Z-hyperimmune sets (Mileti)
- Thus $\mathcal{P}$ contains no Z-hyperimmune set
- Let X be a bi-Z-hyperimmune set
- Neither X nor $\overline{X}$ belongs to $\mathcal{P}$.

Π^0_2 partition regular classes

Lemma

For every infinite set *X*, the following class is partition regular

$$\mathcal{L}_X = \{Y \in 2^{\mathbb{N}} : |X \cap Y| = \infty\}$$

In particular, $\mathcal{L}_{\mathbb{N}}$ is a non-trivial Π^0_2 partition regular class

Π^0_2 partition regular classes

Lemma

If $\mathcal{P} \subseteq 2^{\mathbb{N}}$ contains a partition regular class, the following class is its largest partition regular subclass

$$\mathcal{L}(\mathcal{P}) = \{X : \forall k \, \forall Z_0 \cup \cdots \cup Z_{k-1} \supseteq X \, \exists i < k \, Z_i \in \mathcal{P}\}$$

Moreover, if $\mathcal{P}$ is Π^0_2, so is $\mathcal{L}(\mathcal{P})$.

Partition genericity

Definition (Monin, P.)

A set $A \subseteq \mathbb{N}$ is partition generic if it belongs to every non-trivial Π^0_2 partition regular classes.

- $\mathbb{N}$ is partition generic
- Closed under finite changes
- Closed under supersets

Computable partition generic sets

Lemma (Monin, P.)

The computable partition generic sets are the co-finite sets

- Suppose X is co-infinite and computable
- The class $\mathcal{L}_{\overline{X}}$ is Π^0_2, non-trivial and partition regular
- $X \notin \mathcal{L}_{\overline{X}}$

Partition genericity and typicality

Lemma (Monin and P.)

Let $\mathcal{L}$ be a non-trivial mesurable partition regular class.
Then $\mathcal{L}$ has measure 1.

A real is Kurtz random if it belongs to every Σ^0_1 class of measure 1

Corrolary

Every Kurtz random is partition generic.

Partition genericity and typicality

A set $A \subseteq \mathbb{N}$ is co-hyperimmune if for every c.e. array $(F_{f(n)} : n \in \mathbb{N})$, then $F_{f(n)} \subseteq A$ for some n.

Lemma (Monin and P.)

Every co-hyperimmune set is partition generic.

Pigeonhole basis theorems

Theorem (Monin and P.)

Suppose $C \not\leq_T \emptyset$ and A is partition generic. Then there is an infinite subset $H \subseteq A$ such that $C \not\leq_T H$.

Theorem (Monin and P.)

Suppose A is partition generic. Then there is an infinite subset $H \subseteq A$ of non-PA degree.

Theorem (Monin and P.)

Suppose A is partition generic. Then there is an infinite subset $H \subseteq A$ of non-random degree.

Pigeonhole basis theorems

Theorem (Monin and P.)

Suppose $C \notin \Sigma^0_1$ and A is partition generic relative to C. Then there is an infinite subset $H \subseteq A$ such that $C \notin \Sigma^{0,H}_1$.

Theorem (Monin and P.)

Suppose f is hyperimmune and A is partition generic relative to f. Then there is an infinite subset $H \subseteq A$ such that f is H-hyperimmune.

Largeness in Computability Theory

Goal: find a notion of largeness $\mathcal{P} \subseteq 2^{\mathbb{N}}$ which is partition regular and which prove classical pigeonhole basis theorems in computability theory.

- closed under superset ✓
- closed under partitionning ✗
- such that typical sets are large ✓

Local partition genericity

Definition (Monin, P.)

A set $A \subseteq \mathbb{N}$ is locally partition generic if there is a non-trivial Π^0_2 partition regular class $\mathcal{L}$ such that A belongs to every Π^0_2 partition regular subclass of $\mathcal{L}$.

- closed under superset ✓
- closed under partitionning ✓
- such that typical sets are large ✓

Lowness for partition genericity

A set X is low for partition genericity if every set which is partition generic is partition X-generic.

A set X is low for partition regularity if every non-trivial $\Pi^{0,X}_2$ partition regular class admits a Π^0_2 partition regular subclass.

Lemma (Monin, P.)

A set is low for partition genericiy iff it is low for partition regularity iff it is computable.

References

Benoit Monin and Ludovic Patey.
Partition genericity and pigeonhole basis theorems, 2022.
Available at https://arxiv.org/abs/2204.02705.