Computable Polish groups

Alexander Melnikov, Victoria University of Wellington

May 30, 2022

Leeds Computability Days 2022

Recall that a Polish(able) group is:

1. a Polish space (completely metrizable separable)
2. the group operations $\cdot$ and $^{-1}$ are continuous

Such groups play a central role in:

- topological group theory
- descriptive set theory
- abstract harmonic analysis, etc.

Recall that a Polish(able) group is:

1. a Polish space (completely metrizable separable)
2. the group operations $\cdot$ and $^{-1}$ are continuous

Such groups play a central role in:

- topological group theory
- descriptive set theory
- abstract harmonic analysis, etc.

Why study computable Polish groups?

In the classical countable case:

1. The isomorphism problem, the word problem, etc.
2. Computable structure theory, definability, classification problems

Can we attack similar problems for Polish(able) groups?

The is one obstacle...

Why study computable Polish groups?

In the classical countable case:

1. The isomorphism problem, the word problem, etc.
2. Computable structure theory, definability, classification problems

Can we attack similar problems for Polish(able) groups?

The is one obstacle...

Why study computable Polish groups?

In the classical countable case:

1. The isomorphism problem, the word problem, etc.
2. Computable structure theory, definability, classification problems

Can we attack similar problems for Polish(able) groups?

The is one obstacle...

Why study computable Polish groups?

In the classical countable case:

1. The isomorphism problem, the word problem, etc.
2. Computable structure theory, definability, classification problems

Can we attack similar problems for Polish(able) groups?

The is one obstacle...

What is a computable Polish group?

Polish groups are usually studied up to algebraic homeomorphism (aka topological isomorphism)

So these are *topological objects*

We need a notion that works up to homeomorphism

Polish groups are usually studied up to algebraic homeomorphism (aka topological isomorphism)

So these are *topological objects*

We need a notion that works up to homeomorphism

Polish groups are usually studied up to algebraic homeomorphism (aka topological isomorphism)

So these are *topological objects*

We need a notion that works up to homeomorphism

A computable topological space:

$$U_i \cap U_j = \bigcup_{k \in W_{f(i,j)}} U_k,$$

where f is computable, and (U_i) is *some* countable base.

Define effective continuity for a map h between such spaces:

$$h^{-1}(U_i) = \bigcup_{k \in W_{g(i)}} \tilde{U}_k, \quad \text{where } g \text{ is computable.}$$

A computable topological space:

$$U_i \cap U_j = \bigcup_{k \in W_{f(i,j)}} U_k,$$

where f is computable, and (U_i) is *some* countable base.

Define effective continuity for a map h between such spaces:

$$h^{-1}(U_i) = \bigcup_{k \in W_{g(i)}} \tilde{U}_k, \quad \text{where } g \text{ is computable.}$$

A computable topological space:

$$U_i \cap U_j = \bigcup_{k \in W_{f(i,j)}} U_k,$$

where f is computable, and (U_i) is *some* countable base.

Define effective continuity for a map h between such spaces:

$$h^{-1}(U_i) = \bigcup_{k \in W_{g(i)}} \tilde{U}_k, \quad \text{where } g \text{ is computable.}$$

Every time something works
for a computable topological space
with no additional assumptions

Every time something works
for a computable topological space
with no additional assumptions

you should be amazed

This seems extremely weak:

Definition

A Polish group G is computable topological if

– its domain is computable topological, and

– the operations are effectively continuous.

H is computably topologically presentable if

$H \cong G$ for some computable topological G

This seems extremely weak:

Definition

A Polish group G is computable topological if

– its domain is computable topological, and

– the operations are effectively continuous.

H is computably topologically presentable if

$$H \cong G \text{ for some computable topological } G$$

These seem very strong:

A group G is computably metrized if:

$$d(x_i, x_j) \text{ are uniformly computable for a dense } (x_i)_{i \in \omega}$$

and the operations are computable (as operators).

If $d(x_i, x_j)$ are merely uniformly right-c.e. reals, then G is

right-c.e. metrized

We usually want a complete d

These seem very strong:

A group G is computably metrized if:

$$d(x_i, x_j) \text{ are uniformly computable for a dense } (x_i)_{i\in\omega}$$

and the operations are computable (as operators).

If $d(x_i, x_j)$ are merely uniformly right-c.e. reals, then G is

right-c.e. metrized

We usually want a complete d

These seem very strong:

A group G is computably metrized if:

$$d(x_i, x_j) \text{ are uniformly computable for a dense } (x_i)_{i \in \omega}$$

and the operations are computable (as operators).

If $d(x_i, x_j)$ are merely uniformly right-c.e. reals, then G is

right-c.e. metrized

We usually want a complete d

Effective Birkhoff-Kakutani Theorem:

Theorem (Koh+M.+Ng)

For a Polish(able) group G, TFAE:

1. G is computable topological;
2. G admits a right-c.e. left-invariant metric.

The proof heavily uses the operations

Effective Birkhoff-Kakutani Theorem:

Theorem (Koh+M.+Ng)

For a Polish(able) group G, TFAE:

1. G is computable topological;
2. G admits a right-c.e. left-invariant metric.

The proof heavily uses the operations

The metric is complete in some important cases

For an abelian G

G is computably topologically presentable
iff
G is right-c.e. completely metrisable.

The metric is complete in some important cases

For an abelian G

G is computably topologically presentable
iff
G is right-c.e. completely metrisable.

We cannot improve ‘right-c.e.’ to ‘computable’:

Example (Koh+M.+Ng)

There is a right-c.e. metrized Polish abelian group

not topologically isomorphic

to any computably metrized group.

Proof.

Take a discrete Σ^0_1-presented abelian p-group A with no computable presentation. □

We cannot improve ‘right-c.e.’ to ‘computable’:

Example (Koh+M.+Ng)

There is a right-c.e. metrized Polish abelian group

not topologically isomorphic

to any computably metrized group.

Proof.

Take a discrete Σ^0_1-presented abelian p-group A with no computable presentation. □

An unexpected corollary

M. and Montalbán (2018):

Many results in computable structure theory are special cases of:

1. G is computable topological with c.e. strong inclusion;

2. G acts computably on a computable Polish metric space M.

So $g \in G$ are isomorphisms and $x \in M$ are 'structures'

C.e. strong inclusion gives you a dense set

M. and Montalbán (2018):

Many results in computable structure theory are special cases of:

1. G is computable topological with c.e. strong inclusion;

2. G acts computably on a computable Polish metric space M.

So $g \in G$ are isomorphisms and $x \in M$ are 'structures'

C.e. strong inclusion gives you a dense set

M. and Montalbán (2018):

Many results in computable structure theory are special cases of:

1. G is computable topological with c.e. strong inclusion;
2. G acts computably on a computable Polish metric space M.

So $g \in G$ are isomorphisms and $x \in M$ are 'structures'

C.e. strong inclusion gives you a dense set

M. and Montalbán (2018):

Many results in computable structure theory are special cases of:

1. G is computable topological with c.e. strong inclusion;
2. G acts computably on a computable Polish metric space M.

So $g \in G$ are isomorphisms and $x \in M$ are 'structures'

C.e. strong inclusion gives you a dense set

Because of the Effective Birkhoff-Kakutani Theorem

the results of M. a Montalbán (2018)

are really about right c.e. metrized groups!

So ‘right. c.e. metrized’ seems to be

the most general minimal computability notion

In the discrete case, ‘right-c.e. metrized’

corresponds to ‘Σ^0_1-presented’

and ‘computably metrized’ to ‘constructive’

So ‘right. c.e. metrized’ seems to be

the most general minimal computability notion

In the discrete case, ‘right-c.e. metrized’

corresponds to ‘Σ^0_1-presented’

and ‘computably metrized’ to ‘constructive’

It may seem that

‘computably completely metrizable’

is the right notion of computablity

Spoiler: Not really

It may seem that

'computably completely metrizable'

is the right notion of computablity

Spoiler: Not really

Case study: compact Polish groups

Totally disconnected compact groups are called profinite.

These are exactly the Galois groups of field extensions.

Profinite means it is the inverse limit of finite groups:

$$F_0 \leftarrow_{\phi_0} F_1 \leftarrow_{\phi_1} F_2 \leftarrow_{\phi_2} \cdots$$

In the separable (infinite) case, its domain is 2^ω.

So it is all about the operations.

Totally disconnected compact groups are called profinite.

These are exactly the Galois groups of field extensions.

Profinite means it is the inverse limit of finite groups:

$$F_0 \leftarrow_{\phi_0} F_1 \leftarrow_{\phi_1} F_2 \leftarrow_{\phi_2} \cdots$$

In the separable (infinite) case, its domain is 2^ω.

So it is all about the operations.

Totally disconnected compact groups are called profinite.

These are exactly the Galois groups of field extensions.

Profinite means it is the inverse limit of finite groups:

$$F_0 \leftarrow_{\phi_0} F_1 \leftarrow_{\phi_1} F_2 \leftarrow_{\phi_2} \dots$$

In the separable (infinite) case, its domain is 2^ω.

So it is all about the operations.

Totally disconnected compact groups are called profinite.

These are exactly the Galois groups of field extensions.

Profinite means it is the inverse limit of finite groups:

$$F_0 \leftarrow_{\phi_0} F_1 \leftarrow_{\phi_1} F_2 \leftarrow_{\phi_2} \dots$$

In the separable (infinite) case, its domain is 2^ω.

So it is all about the operations.

Totally disconnected compact groups are called profinite.

These are exactly the Galois groups of field extensions.

Profinite means it is the inverse limit of finite groups:

$$F_0 \leftarrow_{\phi_0} F_1 \leftarrow_{\phi_1} F_2 \leftarrow_{\phi_2} \dots$$

In the separable (infinite) case, its domain is 2^ω.

So it is all about the operations.

Definition

A computably completely metrized space is

effectively compact

if we can enumerate all its finite covers by rational balls.

A basic example: a *decidable* Π^0_1 class C in 2^ω.

This means we can *decide* whether a given string has an extension in the closed $C \subseteq 2^\omega$.

Definition

A computably completely metrized space is

effectively compact

if we can enumerate all its finite covers by rational balls.

A basic example: a *decidable* Π^0_1 class C in 2^ω.

This means we can *decide* whether a given string has an extension in the closed $C \subseteq 2^\omega$.

Definition

A computably completely metrized space is

effectively compact

if we can enumerate all its finite covers by rational balls.

A basic example: a *decidable* Π^0_1 class C in 2^ω.

This means we can *decide* whether a given string has an extension in the closed $C \subseteq 2^\omega$.

Computability for profinite groups is robust:

Theorem (Smith, LaRoche, Downey+M. 2022, M. 2018)

For a profinite G, TFAE:

1. G is the limit of a recursive system of finite groups
2. G admits an effectively compact presentation
3. $G \cong ([T], *, {}^{-1})$, where $T \subseteq 2^\omega$ is decidable closed and $*$, ${}^{-1}$ are computable operators on $[T]$
4. $G \cong Gal(F/E)$, where F/E is a computable field extension
5. [G abelian] The discrete Pontryagin dual of G is computable

Computability for profinite groups is robust:

Theorem (Smith, LaRoche, Downey+M. 2022, M. 2018)

For a profinite G, TFAE:

1. G is the limit of a recursive system of finite groups
2. G admits an effectively compact presentation
3. $G \cong ([T], *, ^{-1})$, where $T \subseteq 2^\omega$ is decidable closed and $*$, $^{-1}$ are computable operators on $[T]$
4. $G \cong Gal(F/E)$, where F/E is a computable field extension
5. [G abelian] The discrete Pontryagin dual of G is computable

Fact (essentially M2018, also DM2022)

For profinite groups

$$\text{computably completely metrizable} \not\Longrightarrow \text{eff. comp. presentable}$$

What about connected groups?

Fact (essentially M2018, also DM2022)

For profinite groups

$$\text{computably completely metrizable} \not\Longrightarrow \text{eff. comp. presentable}$$

What about connected groups?

A very powerful tool (M 2018, LMN 2021, and M 2021):

Theorem (Effective Pontryagin dualities)

For connected Polish abelian G:

$$G \text{ is eff. comp. presentable } \iff \widehat{G} \text{ is computably presentable}$$

and

$$G \text{ is computably metrizable } \iff \widehat{G} \text{ is } \Delta^0_2\text{-presentable}$$

for a p-divisible $\widehat{G}$.

Here $\widehat{G}$ is the discrete Pontryagin dual of G, all metrics are complete.

In the connected case:

$$\widehat{G} \cong H^1(G; \mathbb{Z}),$$

the first Čech cohomology group of the **space** of G.

We do not need the group-operations in G to get the dualities!

In contrast, we (heavily) used the operations in:

– the effective Birkhoff-Kakutani theorem,
– the profinite case (they all are 2^ω).

In the connected case:

$$\widehat{G} \cong H^1(G; \mathbb{Z}),$$

the first Čech cohomology group of the **space** of G.

We do not need the group-operations in G to get the dualities!

In contrast, we (heavily) used the operations in:

– the effective Birkhoff-Kakutani theorem,
– the profinite case (they all are 2^ω).

In the connected case:

$$\widehat{G} \cong H^1(G; \mathbb{Z}),$$

the first Čech cohomology group of the **space** of G.

We do not need the group-operations in G to get the dualities!

In contrast, we (heavily) used the operations in:

– the effective Birkhoff-Kakutani theorem,

– the profinite case (they all are 2^ω).

Our proofs of the effective Pontryagin dualities use:

1. Computable bases in TFAGs (Dobrica)
2. A new constructive version of Čech cohomology
3. Khisamiev's result about c.e. presented groups

In particular, we get:

Fact (LMN2021)

For connected compact Polish abelian groups,

$$\text{computably completely metrizable} \not\Longrightarrow \text{eff. comp. presentable}$$

It looks like effective compact presentations are the most natural ones

A few applications in computable topology

Theorem (NLM 2021)

There is a computably metrized connected space not homeomorphic to any effectively compact space.

This is because we do not need the group operation for the connected dualities.

Suppose we have an effectively compact M. Is it homeomorphic to the unit circle?

This is an example of a characterization problem.

Theorem (NLM 2021)

The characterization problem for solenoid spaces is arithmetical.

Proof.

The index set of completely decomposable groups is arithmetical [DM2014].

Now use effective Pontryagin duality and some other stuff! □

Suppose we have an effectively compact M. Is it homeomorphic to the unit circle?

This is an example of a characterization problem.

Theorem (NLM 2021)

The characterization problem for solenoid spaces is arithmetical.

Proof.

The index set of completely decomposable groups is arithmetical [DM2014].

Now use effective Pontryagin duality and some other stuff! □

Suppose we have an effectively compact M. Is it homeomorphic to the unit circle?

This is an example of a characterization problem.

Theorem (NLM 2021)

The characterization problem for solenoid spaces is arithmetical.

Proof.

The index set of completely decomposable groups is arithmetical [DM2014].

Now use effective Pontryagin duality and some other stuff! □

Suppose we have an effectively compact M. Is it homeomorphic to the unit circle?

This is an example of a characterization problem.

Theorem (NLM 2021)

The characterization problem for solenoid spaces is arithmetical.

Proof.

The index set of completely decomposable groups is arithmetical [DM2014].

Now use effective Pontryagin duality and some other stuff! □

There are lots of open problems.

Such studies are not restricted to compact groups.

There are also some really cool recent results about computable locally compact totally disconnected groups (with Nies).

But we have no time to discuss these results.

There are lots of open problems.

Such studies are not restricted to compact groups.

There are also some really cool recent results about computable locally compact totally disconnected groups (with Nies).

But we have no time to discuss these results.

Thanks