Some Ramsey-theoretic principles in reverse mathematics over a weak base theory

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Plan of the talk

1. Some background:
    - Reverse Mathematics
    - a weaker base theory RCA_0
    - Ramsey's theorem
2. Computational strength of RT^n_k and induction axioms
3. First-order consequences of RT^n_k over RCA_0
4. Principles weaker than RT^2_2
5. Cohesiveness principle

Reverse Mathematics

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Reverse Mathematics

- What are the weakest axioms needed to prove a theorem T? Can you show they are equivalent to T over a weak base theory?
- If T has the form $\forall X \exists Y \varphi(X, Y)$, how hard is it to compute Y from X? Can you use T to compute solutions to other principles?
- What else can you prove if you add T to your base theory? What consequences for finite objects does T have (= what are its first-order consequences)?

Second-order arithmetic

Language L_2: first-order variables: $x, y, z, \dots$, second-order variables: $X, Y, Z, \dots$, non-logical symbols: $0, 1, +, \cdot, <, \in$.

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Arithmetical hierarchy:

- Σ^0_n: formulas with n alternating blocks of first-order quantifiers starting with $\exists$, then only bounded quantifiers $\forall x \leq t$, $\exists x \leq t$,
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- Π^0_n: defined dually, formulas starting with $\forall$,
- classes Σ^0_n, Π^0_n allow set parameters, Σ_n, Π_n are purely first-order (in language L_1: $x, y, z, \dots; 0, 1, +, \cdot, <$).

Arithmetical axioms

Given a class of formulas Γ we define the following axiom schemes:

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- Γ-CA = for each $\varphi \in \Gamma$ the comprehension axiom:
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The traditional base theory is RCA$_0$:
Σ^0_1-CA + $\mathrm{I}\Sigma^0_1$ + basic properties of +, $\cdot$, <

RCA$_0$ is said to correspond to computable mathematics. Its provably recursive functions are precisely the primitive recursive ones. Its first-order part is $\mathrm{I}\Sigma_1$.

A weaker base theory

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- The rst-order part of RCA_0 is axiomatized by B_1 + exp.
- The provably recursive functions of RCA_0 are precisely the elementary recursive ones.
- Over RCA_0, RCA_0 is equivalent to the statement ,,Every in nite set has arbitrarily large nite subsets".

Failure of $^{0}_{1}$-induction

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We consider I together with the family of its coded subsets $\mathrm{Cod}(M/I)$ as a second-order structure.

$X \in \mathrm{Cod}(M/I)$ iff $\exists s \in M$ s.t. s is (a code for) a finite set and $s \cap I = X$.

$\mathrm{SSy}(M) := \mathrm{Cod}(M/\omega)$

Ramsey's theorem

For fixed $n, k \geq 2$, RT^n_k is the sentence „For every $c : [\mathbb{N}]^n \to k$ there exists an infinite set $H \subseteq \mathbb{N}$ such that $c \upharpoonright [H]^n$ is constant.".

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- $\mathsf{RCA}_0 \vdash \mathsf{RT}^3_2 \leftrightarrow \mathsf{ACA}_0$

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- What are the first-order consequences of RT^2_2?
- They follow from $I\Sigma_2$ (Cholak, Jockusch, Slaman) and their Π_3 part is $B\Sigma_2$ (Patey, Yokoyama).
- Open problem: Is RT^2_2 arithmetically conservative over $B\Sigma^0_2$?

Equivalence with relativization to a$^0_1$-de nable cut

Theorem

Let $n; k$ 2. If $(M; X)$ RCA_0 and I is a proper $^0_1$-de nable cut in M, then

$$(M; X) \quad RT^n_k \quad i \quad (I; Cod(M=I)) \quad RT^n_k:$$

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Lemma (Chong-Mourad 1990)

Let $(M; X) \quad RCA_0$ and I be a cut in M. If both $Y \quad I$ and $I \, n Y$ are $^0_1$-de nable in M, then $Y \; 2 \; Cod(M=I)$.

Equivalence with relativization to a Σ^0_1-definable cut

Theorem

Let $n, k \geq 2$. If $(M, X) \models RCA_0$ and I is a proper Σ^0_1-definable cut in M, then

$$(M, X) \models RT^n_k \leftrightarrow (I, Cod(M/I)) \models RT^n_k.$$

Lemma (Chong-Mourad 1990)

Let $(M, X) \models RCA_0$ and I be a cut in M. If both $Y \subseteq I$ and $I \setminus Y$ are Σ^0_1-definable in M, then $Y \in Cod(M/I)$.

Proof idea for theorem. ($\Rightarrow$) Given a colouring $c : [M]^n \to k$ we transfer it onto $[I]^n$ using the cofinal set $A = \{a_i\}_{i \in I}$: $\hat{c}(i_1, \ldots, i_n) := c(a_{i_1}, \ldots, a_{i_n})$. By the Chong-Mourad coding lemma $\hat{c} \in Cod(M/I)$. We use RT^n_k in $(I, Cod(M/I))$ and transfer the solution back to M using the set A.

The ($\Leftarrow$) direction is similar. □

Computational strength of RT^n_k in first-order arithmetic

For $\ell \geq 1; n, k \geq 2$, let $\ell\text{-}RT^n_k$ be the following first-order sentence: For every Σ_ℓ-definable k-colouring of $[\mathbb{N}]^n$, there is a Σ_ℓ-definable infinite homogeneous set.

Computational strength of RT^n_k in rst-order arithmetic

For ` 1; n; k 2, let ˋ-RT^n_k be the following rst-order sentence: For every ˋ-de nable k-colouring of $[N]^n$, there is a ˋ-de nable in nite homogeneous set.

Every ˋ-RT^n_k is false in the standard model but how much of mathematical induction is needed to disprove it?

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Lemma

(a) I $_1$ proves that there is a $_1$-de nable 2-colouring of $[N]^2$ with no $_1$-de nable in nite homogeneous set,

(b) 8` n 2, I ` proves that there is a ` $_{n+1}$ -de nable 2-colouring of $[N]^n$ with no in nite homogeneous ` set.

In particular, I ` ` : ` -$RT^n_k$, for ` 1.

Computational strength of RT^n_k in first-order arithmetic

For ℓ 1; n; k 2, let ℓ-RT^n_k be the following first-order sentence: For every ℓ-definable k-colouring of $[N]^n$, there is a ℓ-definable infinite homogeneous set

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Lemma

(a) I $_1$ proves that there is a $_1$-definable 2-colouring of $[N]^2$ with no $_1$-definable infinite homogeneous set,

(b) $\forall\ell$ n 2, I ℓ proves that there is a ℓ $_{n+1}$-definable 2-colouring of $[N]^n$ with no infinite homogeneous ℓ set.

In particular, I ℓ ℓ : ℓ-RT^n_k, for ℓ 1.

Proof idea. Repeat the usual argument of Jockusch '72. I ℓ is needed for the claim that any infinite ℓ set has arbitrarily large finite subsets. □

Computational strength of RT^n_k in first-order arithmetic

Lemma

Let $n \geq 3$ and $(M; X) \models RCA_0 + RT^n_2$. If $M \models I\Sigma_\ell$ then $0^{(\ell)} \in X$.
As a consequence, $\Delta_{\ell+1}$-Def$(M) \subseteq X$ and $M \models B\Sigma_{\ell+1}$.

Computational strength of RT^n_k in first-order arithmetic

Lemma

Let $n \quad 3$ and $(M; X) \models RCA_0 + RT^n_2$. If $M \models I \quad \ell$ then $0^{(\ell)} \in X$.
As a consequence, $_{\ell+1}$-Def$(M) \quad X$ and $M \models B \quad _{\ell+1}$.

Proof idea.

For $\ell = 1$ we take the usual computable colouring of triples whose solutions compute $0'$:

for $x < y < z$ let $c(x; y; z) = 0$ if there is a Turing machine with a code below x that halts below z but not below y.
To check whether $_e(e)\#$ take any $x; y \in H$ such that $e < x < y$ and execute the computation $_e$ on e till the step y.
Thus $0' \in X$ and also $_2$-Def$(M) \quad X$ and $M \models B \quad _2$.

For the other cases proceed by induction up to ℓ and relativize the case of $\ell = 1$. □

First-order consequences of RT^3_2

Let R^n denote the first-order consequences of RT^n_2. Then:

- R^n is axiomatized by $B\Sigma_1 + \exp$ and the set $\{ I\Sigma_\ell \Rightarrow B\Sigma_{\ell+1} \wedge (I\Sigma_{\ell+1} \vee \Sigma_{\ell+1}\text{-}RT^n_k) : \ell \geq 1\}$,

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- it is strictly contained in PA,
- its Π_3-part is $B\Sigma_1 + \exp$,
- for $\ell \geq 1$, its $\Pi_{\ell+3}$-part lies strictly in between $B\Sigma_1 + \exp + \bigwedge_{1 \leq j \leq \ell} (I\Sigma_j \Rightarrow B\Sigma_{j+1})$ and $B\Sigma_{\ell+1}$.

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From the proof of the last two items it follows that for each $\ell \geq 1$, $n, k \geq 2$, the theory $B\Sigma_\ell + \neg\,\ell\text{-}\mathsf{RT}^n_k$ is consistent and thus $I\Sigma_\ell$ is the optimal amount of induction to disprove $\ell\text{-}\mathsf{RT}^n_k$.

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Example of a model of $\mathrm{B}\Sigma_1 + \Sigma_1\text{-}\mathrm{RT}^2_2$: take a model $M \models \mathrm{B}\Sigma_1 + \exp$ with $\mathrm{SSy}(M) \models \mathrm{RT}^n_k$ and trim it to some M' in which ω is Σ_1-definable. Then $(M', \Sigma_1\text{-Def}(M)) \models \mathrm{B}\Sigma_1 + \mathrm{RT}^n_k$ and thus $M' \models \mathrm{B}\Sigma_1 + \Sigma_1\text{-}\mathrm{RT}^2_2$.

Results for RT^2_2 over RCA_0

Let R^2 denote the rst-order consequences of $RCA_0 + RT^2_2$. Then:

- R^2 follows from I $_2$ (By Cholak, Jockusch, Slaman).
- Over : I $_1$, R^2 is equivalent to $_1$-RT^2_2.
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- Over : I $_1$, R^2 is equivalent to $_1$-RT^2_2.
- The $_3$-part of R^2 is B $_1$ + exp.
- The $_4$-part of R^2 is strictly weaker then B $_2$ but does not follow from I $_1$:
- the $_2$ cardinality scheme, C $_2$, is a $_4$-sentence that follows from $RCA_0 + RT^2_2$ but I $_1$ 0 C $_2$ (an automorphism argument).

C $_2$ = ,,There is no $_2$-de nable injection with domain N and a bounded range."

Principles weaker than RT^2_2

CAC = For every partial order $(\mathbb{N}, \leq)$ there exists an infinite set $S \subseteq \mathbb{N}$ which is a $\leq$-chain or $\leq$-antichain.

ADS = For every linear order $(\mathbb{N}, \leq)$ there exists an infinite set $S \subseteq \mathbb{N}$ which is an $\leq$-ascending or $\leq$-descending sequence.

CRT^2_2 = for every $c : [\mathbb{N}]^2 \to 2$ there exists an infinite $S \subseteq \mathbb{N}$ such that $c \upharpoonright S$ is stable, i.e. for every $x \in S$ there exists $y \in S$ such that for all $z \in S$ if $z \geq y$, then $c(x, y) = c(x, z)$.

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$$RCA_0 \vdash RT^2_2 \Rightarrow CAC \Rightarrow ADS \Rightarrow CRT^2_2$$

Results on the weaker principles

- Proofs of the following implications go through easily in RCA_0: $\mathrm{RT}^2_2 \Rightarrow \mathrm{CAC} \Rightarrow \mathrm{ADS}$, $\mathrm{RT}^2_2 \Rightarrow \mathrm{CRT}^2_2$.

Results on the weaker principles

- Proofs of the following implications go through easily in RCA$_0$: RT$^2_2$ $\Rightarrow$ CAC $\Rightarrow$ ADS, RT$^2_2$ $\Rightarrow$ CRT$^2_2$.
- Each of CAC, ADS and CRT$^2_2$ holds in a model of RCA$_0$ + $\neg$I$^0_1$ iff it holds on a $^0_1$-definable cut.
- For each of CAC, ADS and CRT$^2_2$ there are models of RCA$_0$ in which it is computably true.
- CAC, ADS and CRT$^2_2$ are $^0_3$-conservative over RCA$_0$.

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- For each of CAC, ADS and CRT^2_2 there are models of RCA_0 in which it is computably true.
- CAC, ADS and CRT^2_2 are $^0_3$-conservative over RCA_0.
- CAC, ADS are not $_4$- and CRT^2_2 is not $_5$-conservative over RCA_0.

Results on the weaker principles

- Proofs of the following implications go through easily in RCA_0: RT^2_2) CAC) ADS, RT^2_2) CRT^2_2.
- Each of CAC, ADS and CRT^2_2 holds in a model of RCA_0 + : I $^0_1$ i it holds on a $^0_1$-de nable cut.
- For each of CAC, ADS and CRT^2_2 there are models of RCA_0 in which it is computably true.
- CAC, ADS and CRT^2_2 are $^0_3$-conservative over RCA_0.
- CAC, ADS are not $_4$- and CRT^2_2 is not $_5$-conservative over RCA_0.
- All of RT^2_2, CAC, ADS and CRT^2_2 can be distinguished by their rst-order consequences over RCA_0. (It follows from their separation over WKL_0, cf. Towsner 2020.)

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COH: For each sequence $(R_n)_{n \in \mathbb{N}}$ of subsets of $\mathbb{N}$, there exists an unbounded set C which is cohesive for $(R_n)_{n \in \mathbb{N}}$ (i.e. for every $i \in \mathbb{N}$ either $C \subseteq^* R_i$ or $C \subseteq^* \overline{R_i}$).

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- $\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_2 \vdash \mathrm{CRT}^2_2 \Leftrightarrow \mathrm{COH}$ (Hirschfeldt, Shore 2007)

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Given two Δ^0_2-sets A_0, A_1 such that $A_0 \cup A_1 = \mathbb{N}$ we look for a Δ^0_2-set B such that $B \subseteq A_0$ and $\overline{B} \subseteq A_1$.

One can define a computable function $f : \mathbb{N} \times \mathbb{N} \to 2$ such that

$$\{s : f(n, s) = i\} \text{ is unbounded} \Rightarrow n \in A_i.$$

Define a computable sequence of sets $R_n = \{s : f(n, s) = 0\}$ and let C be cohesive for this sequence. Put $n \in B \iff C \subseteq^* R_n$. □

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If (M; $_1$-Def(M)) RCA_0 then (M; $_1$-Def(M)) 2 COH.

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Since there are models of RCA$_0$ + RT$^2_2$ of the form (M ; $_1$-Def(M)) we obtain

Corollary

RCA$_0$ 0 RT$^2_2$) COH.

Questions

- Does RT^3_2 imply RT^4_2 over RCA_0?
- Is $\mathsf{RCA}_0 + \mathsf{RT}^2_2 + \mathsf{B}\Sigma_2$ arithmetically conservative over $\mathsf{B}\Sigma_2$?
- Does ADS or CAC imply CRT^2_2 over RCA_0?
- Does COH imply $\mathsf{I}\Sigma^0_1$ over RCA_0? Is COH Π^0_3-conservative over RCA_0?

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