WHEN IS UNBOUNDED SEARCH NECESSARY?

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- For any pair of computable dense linear orders without endpoints there exists a computable isomorphism between them

$$a_0 <_A \dots\dots\dots <_A a_n <_A \mathbf{a}$$

$$b_0 <_B \dots\dots\dots <_B b_n \qquad \exists b\,(b_n <_B b)$$

unbounded search

- For any computable poset there exists a computable linear extension of it

$$x_0 <_L \cdots <_L x_n \qquad \mathbf{x}$$

- For any pair of computable dense linear orders without endpoints there exists a computable isomorphism between them

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unbounded search $\sim$ NON PRIM REC

- For any computable poset there exists a computable linear extension of it PRIM REC

$$x_0 <_L \cdots <_L x_n \qquad \mathbf{x}$$

Formal framework

- Symbols for any primitive recursive function -

PRA

$\mathsf{PA}^- + \mathrm{I}\Delta_0^0$

defining equations for any primitive recursive function

Formal framework

- Symbols for any primitive recursive function -
- Variables for numbers and for functions -

PRA^2

PRA

$\mathsf{PA}^- + \mathrm{I}\Delta^0_0$

defining equations for any primitive recursive function

defining equations for any primitive recursive functional

$$\forall f\ \forall n \quad \big(\mathsf{Add}(0, f, n) = f(n) \land \\ \mathsf{Add}(s(m), f, n) = s\,(m + f(n))\big)$$

Ulrich Kohlenbach, Things that can and things that can't be done in PRA, 2000

Models of PRA^2

$\mathcal{X} \subseteq \{f \mid f\colon M \to M\}$

closed under comp and prim recur

$M \vDash \mathsf{PRA}$

$$(M, \mathcal{X}, 0, s, <, \{f_n\})$$

prim recur functions

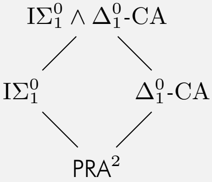
Comprehension, induction

$$\Sigma^0_n\text{-CA} \;\Big|\; \exists f\, \forall n\, ((f(n) = 1 \to \varphi(n)) \;\wedge\; (f(n) = 0 \to \neg\varphi(n))), \;\; \varphi \text{ is } \Sigma^0_n$$

$\mathsf{PRA}^2 \vdash \Delta^0_0\text{-CA} \wedge \mathrm{I}\Delta^0_0$

$\mathsf{PRA}^2 \nvdash \Delta^0_1\text{-CA}$

$\mathsf{PRA}^2 \nvdash \mathrm{I}\Delta^0_1$

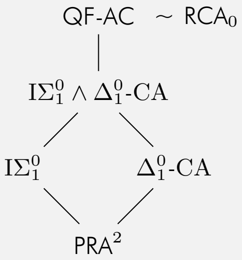


Choice function

QF-AC | $\forall n\, \exists m\, \theta(n, m) \rightarrow \exists f\, \forall n\, \theta(n, f(n)), \quad \theta \text{ is } \Delta^0_0$

$\text{QF-AC} \vdash \Delta^0_1\text{-CA} \wedge \mathrm{I}\Sigma^0_1$

$\Delta^0_1\text{-CA} \wedge \mathrm{I}\Sigma^0_1 \nvdash \text{QF-AC}$



• $\Delta^0_1\text{-CA} \wedge \mathrm{I}\Sigma^0_1 \nvdash$ QF-AC •

Proof's sketch.

Let $(\omega, \mathcal{X})$ such that

- $\mathcal{X}$ contains all 0-1-valued computable functions and
- $\mathcal{X}$ is closed under composition and primitive recursion.

Then $\mathcal{X}$ does not contain all computable functions.
While any model of QF-AC contains all computable functions, since QF-AC proves closure under minimisation. □

Δ^0_1-CA	$\exists f\, \forall n\, ((f(n) = 1 \to \varphi(n)) \wedge (f(n) = 0 \to \neg\varphi(n))),\ \varphi$ is Δ^0_1
QF-AC	$\forall n\, \exists m\, \theta(n, m) \to \exists f \forall n\, \theta(n, f(n)),\ \theta$ is Δ^0_0

• $\mathsf{PRA}^2 \vdash$ QF-AC $\leftrightarrow$ categoricity of dense linear orders without endpoints •

Proof's sketch, $\Rightarrow$.

Since it holds that $\forall \langle b_0, b_1 \rangle\, \exists \langle c, d, e \rangle\, (b_0 <_B b_1 \to c <_B b_0 <_B d <_B b_1 <_A e)$,
then by QF-AC there exists $f \colon \mathbb{N} \to \mathbb{N}$ such that
$\forall \langle b_0, b_1 \rangle\, (b_0 <_B b_1 \to \pi_1 f(\langle b_0, b_1 \rangle) <_B b_0 <_B \pi_2 f(\langle b_0, b_1 \rangle) <_B b_1 <_B \pi_3 f(\langle b_0, b_1 \rangle))$.

$$a_0 <_A \dots\dots\dots <_A a_n <_A \mathbf{a}$$

$$b_0 <_B \dots\dots\dots <_B b_n <_B \pi_3 f(\langle b_{n-1}, b_n \rangle)$$

□

Proof's sketch, $\Leftarrow$ (After Bazhenov Kalimullin 2021).

Fix θ Δ^0_0-formula such that $\forall x\, \exists y\, \theta(x, y)$.
Let $(A, <_A)$ be a primitive recursive dense linear order without endpoints.
Define $(B, <_B)$, dense linear order without endpoints, in such a way that, for any x and for each z

$$4x + 2 <_B z <_B 4x + 6 \Rightarrow \exists y < z\, \theta(x, y)$$

Let $h : B \to A$ be an isomorphism. Define $\xi(x) = h^{-1}(g_A(h(4x + 2), h(4x + 6)))$. Then

$$f(x) := \mu y \leq \xi(x)\ (\theta(x, y))$$

is a choice function for θ, i.e. $\forall x\, (\theta(x, f(x)))$

$$A \qquad h(4x+2) <_A \dots <_A a <_A \dots\dots <_A h(4x+6)$$

$$h \uparrow$$

$$B \qquad 4x+2 <_B \dots <_B \xi(x) <_B \dots\dots <_B 4x+6$$

$f(x)$

Theorem

Over PRA^2, the following are equivalent:

- QF-AC
- categoricity of dense linear orders without endpoints
- categoricity of random graphs
- categoricity of atomless Boolean algebras

Theorem

PRA^2 proves the followings:

- every poset can be linearised
- every complete consistent theory has a model
- every field can be embedded into its algebraic closure
- every vector space over a finite field has a basis