Finitely α-generated Structures

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June 1, 2022

What's a Scott Sentence?

Everything in this talk is motivated by a theorem of Scott's:

Scott's Isomorphism Theorem

Every countable structure can be described up to isomorphism (among countable structures) by a sentence φ of $L_{\omega_1\omega}$.

Such a sentence is called a **Scott sentence** for A.

This is exactly the kind of categoricity result which is not possible in the finitary first-order context.

Complexity of formulas in $L_{\omega_1\omega}$

Every formula of $L_{\omega_1\omega}$ has a normal form, in the sense that it is equivalent to one that is either Σ_α or Π_α for some α. We define such formulas as follows:

- A $\Sigma_0 = \Pi_0$ formula is a finitary quantifier-free formula of L.
- A Σ_α formula is a formula of the form $\bigvee_{i\in\omega} \exists \bar{x}_i \phi_i(\bar{x}_i)$ where each ϕ_i is Π_β for $\beta < \alpha$.
- A Π_α formula is the negation of a Σ_α formula. Equivalently, a formula of the form $\bigwedge_{i\in\omega} \forall \bar{x}_i \phi_i(\bar{x}_i)$ where each ϕ_i is Σ_β for $\beta < \alpha$.

A d-Σ_α formula is a finite conjunction of Π_α and Σ_α formulas

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Disagreement in the Literature

Scott's isomorphism theorem is tied to an important invariant of a structure known as its Scott rank.

Unfortunately, many non-equivalent definitions of Scott Rank exist in the literature. Antonio Montalbán in "A Robuster Scott Rank" argued to standardize the following definition:

Definition (A. Montalbán)

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The List

Montalbán believed this notion was most robust, having many other conditions equivalent to it.

Theorem

The following are equivalent:

1. A has a $\Pi_{\alpha+1}$ Scott sentence.
2. The automorphism orbit of any tuple can be defined by a Σ_α formula (without parameters).
3. The set $Iso(A)$ of presentations of A is $\Pi_{\alpha+1}$ in the Borel hierarchy.
4. A is uniformly boldface Δ_α-categorical.
5. And so on...

In other words, Scott sentences are also related to notions in computability theory and descriptive set theory.

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Scott Complexity

Why just consider $\Pi_{\alpha+1}$ Scott sentences?

Fact: A structure has a $\Sigma_{\alpha+1}$ Scott sentence iff there is some finite tuple $\bar{c}$ such that $(A, \bar{c})$ has a Π_α Scott sentence.

Theorem (A. Miller)

For $\alpha \geq 1$, A has a Scott sentence that is d-Σ_α iff it has one that is $\Pi_{\alpha+1}$ and one that is $\Sigma_{\alpha+1}$.

Miller's result implies a unique least-complexity Scott sentence for the structure ($\Pi_\alpha, \Sigma_\alpha, d$-$\Sigma_\alpha$).

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Scott Complexity

Definition (R.A*, M. Harrison-Trainor, D. Turetsky, N. Greenberg)

The **Scott complexity** of a structure A is the least complexity of a Scott sentence for A.

Scott complexity is finer than Scott Rank, and just as robust.

Finitely α-generated Structures: Motivation

In previous work with Dino Rosseger, we calculated the Scott complexity of various scattered linear orders. In doing so, we noticed that a special tuple acted in many ways like the generating tuple of the structure, even though the structure was not finitely generated. This tuple is important and has several equivalent characterizations.

Observation

The following are equivalent:

- The structure $(A, \bar{c})$ has a $\Pi_{\alpha+1}$ Scott sentence.
- The tuple $\bar{c}$ is a tuple over which no other tuple is α-free.
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Finitely α-generated Structures

Definition

A tuple $\bar{c}$ is said to be an α-**generator** for a structure A if:

1. the automorphism orbit of each finite tuple of A is Σ_α-definable over $\bar{c}$.
2. The ordinal α is the least such that (1) holds.

A structure A with an α-generator is called an α-**generated structure.** These are exactly the structures with Scott complexity $\Sigma_{\alpha+2}$, d-$\Sigma_{\alpha+1}$, or $\Sigma_{\alpha+1}$ for limit α.

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Example: Finitely α-generated Structures

The structure $\mathbb{Z} + \mathbb{Z}$ is finitely 2-generated and has Scott complexity d-Σ_3.

It is not finitely generated in the language of linear orders, but is finitely generated in the language with the ordering, the predecessor, and the successor functions.

The generating tuples for $\mathbb{Z} + \mathbb{Z}$ in this expanded language are precisely the tuples which are 2-generators for $\mathbb{Z} + \mathbb{Z}$ as a linear order.

Every finitely generated structure is **almost rigid**, meaning that after fixing a finite tuple of parameters the structure has no nontrivial automorphisms.

In the case where A is almost rigid, being finitely α-generated and being finitely generated (after some alterations) coincide.

Lemma (R.A.*)

Suppose that A is finitely α-generated by $\bar{c}$ and almost rigid, witnessed by $\bar{d}$. Let $\{\phi_{\bar{a}}(\bar{x}, \bar{c}, \bar{d}) : \bar{a} \in A\}$ be the family of Σ_α-formulas defining the automorphism orbits of $(A, \bar{c}\bar{d})$. In the definitional expansion which includes a relation predicate for each $\phi_{\bar{a}}$, A is finitely generated by $\bar{c}\bar{d}$.

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Finitely α-generated Structures

In work with Julia Knight and Charlie McCoy, it was shown that a finitely generated group has a d-Σ_2 Scott sentence iff some generating tuple has a Π_1 automorphism orbit. This result generalizes to finitely α-generated structures.

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Matthew Harrison-Trainor and Turbo Ho showed that a finitely generated group has Scott complexity Σ_3 iff it contains a proper Σ_1 elementary substructure isomorphic to itself.

We say that A is a Σ_α-**elementary substructure** of B and write $A \preceq_{\Sigma_\alpha} B$ if A, B agree on all Σ_α formulas with parameters from A.

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Matthew Harrison-Trainor and Turbo Ho showed that a finitely generated group has Scott complexity Σ_3 iff it contains a proper Σ_1 elementary substructure isomorphic to itself. Towards a generalization, we have the following.

Theorem (R.A.*)

Suppose that A is finitely α-generated and contains a proper Σ_α elementary substructure isomorphic to itself whose image under automorphism is not all of A. Then the Scott complexity of A is $\Sigma_{\alpha+2}$.

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Open Question

Open Question: Suppose that A is finitely α-generated and has Scott complexity $\Sigma_{\alpha+2}$. Does A contain a Σ_α elementary substructure isomorphic to itself?

Avoiding Index Set Calculations

Historically, to calculate the Scott complexity of n structure, one gave a Scott sentence of a certain complexity and then calculatd the complexity of the structure's index set.

A structure is said to be **computable** if its atomic diagram, considered as a subset of ω via Gödel coding, is a computable set. For an X-computable structure A, the **index set** $I^X(A)$ is the set of all X-computable indices e for A, i.e., the set of all e such that the eth Turing machine with oracle X computes a copy of A.

This is possible via the following theorem:

Theorem (Folklore, Ash)

If A has Scott complexity Γ^X, then $I^X(A)$ has complexity Γ.

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However, Julia Knight and Charlie McCoy have shown that the reverse is not always true - i.e., the Scott complexity does not exactly match the complexity of the index set. We need additional means of calculating Scott complexity.

Our previous theorems about the Scott complexity of finitely α-generated structures are one example. Here is another.

Theorem (R.A.*)

Suppose $A \preceq_{\Sigma_\alpha} B$ for nonisomorphic A, B. Then B has no $\Pi_{\alpha+1}$ Scott sentence and A has no $\Sigma_{\alpha+1}$ Scott sentence.

For example, if I can show that A has a d-$\Sigma_{\alpha+1}$ Scott sentence and that $C \preceq_{\Sigma_\alpha} A \preceq_{\Sigma_\alpha} D$ then A has to have Scott complexity exactly d-$\Sigma_{\alpha+1}$.

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As an example, we show that

Theorem (R.A.*)

A divisible abelian group of nonzero finite rank has Scott complexity d-Σ_2.

An abelian group G is said to be **divisible** if for every $g \in G$ and every $n \in \omega$ there is an element $h \in G$ with $n \cdot h = g$.
As an example, the group $\mathbb{Q}$ is divisible. Another example of a divisible group is the following.

Definition

For p a prime, the Prüfer p-group $\mathbb{Z}_{p^\infty}$ is defined as

$$\left\{ \frac{\hat{a}}{p^i} \mid a \in \mathbb{Z}, i \geq 0 \right\} \subset \frac{\mathbb{Q}}{\mathbb{Z}}$$

where $\hat{a}$ is the image of a under the canonical homomorphism.

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Proof Sketch: To find an upper bound for the Scott complexity, we give a Scott sentence for the structure explicitly. For a lower bound, we use some group theory results to note that every such group is of the form $\oplus_{p\in P'\subseteq P}(\mathbb{Z}_{p^\infty})^{n_p} \oplus_{p\in P''\subseteq P} (\mathbb{Z}_{p^\infty})^{\omega} \oplus \mathbb{Q}^k$ for nonzero finite k, and that

$$\oplus_{p\in P'\subseteq P}(\mathbb{Z}_{p^\infty})^{n_p} \oplus_{p\in P''\subseteq P} (\mathbb{Z}_{p^\infty})^{\omega} \oplus \mathbb{Q}^{m-1}$$

is a Σ_1 elementary substructure of

$$\oplus_{p\in P'\subseteq P}(\mathbb{Z}_{p^\infty})^{n_p} \oplus_{p\in P''\subseteq P} (\mathbb{Z}_{p^\infty})^{\omega} \oplus \mathbb{Q}^{m}$$

for $0 < m \leq \omega$.

Thanks + References

Thank You!

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