Rectangle diagrams for the Lusztig cones of quantized enveloping algebras of type A[1]

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Abstract

Let U be the quantum group associated to a Lie algebra $\mathbf{g}$ of type A_n. The negative part U^- of U has a canonical basis $\mathbf{B}$ defined by Lusztig and Kashiwara, with favourable properties. We show how the spanning vectors of the cones defined by Lusztig [12], when regarded as monomials in Kashiwara's root operators, can be described using a remarkable rectangle combinatorics. We use this to calculate the Lusztig parameters of the corresponding canonical basis elements, conjecturing that translates of these vectors span the simplicial regions of linearity of Lusztig's piecewise-linear function [9, §2].
Keywords: quantum group, canonical basis, Lusztig cone, longest word, piecewise-linear combinatorics.

1 Introduction

Let $U = U_q(\mathbf{g})$ be the quantum group associated to a semisimple Lie algebra $\mathbf{g}$ of rank n. The negative part U^- of U has a canonical basis $\mathbf{B}$ with favourable properties (see Kashiwara [7] and Lusztig [11, §14.4.6]). For example, via action on highest weight vectors it gives rise to bases for all the finite-dimensional irreducible highest weight U-modules.

Let W be the Weyl group of $\mathbf{g}$, with Coxeter generators $s_1, s_2, \ldots, s_n$, and let w_0 be the element of maximal length in W. Let $\mathbf{i}$ be a reduced expression for w_0, i.e. $w_0 = s_{i_1} s_{i_2} \cdots s_{i_k}$ is reduced. Lusztig obtains a parametrization of the canonical basis $\mathbf{B}$ for each such reduced expression $\mathbf{i}$, via a correspondence between a basis of PBW-type associated to $\mathbf{i}$ and the canonical basis. This gives a bijection $\varphi_{\mathbf{i}} : \mathbf{B} \to \mathbb{N}^k$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$.

Kashiwara, in his approach to the canonical basis (which he calls the global crystal basis), defines certain root operators $\widetilde{F}_i$ on the canonical basis (see [7, §3.5]) which lead to a parametrization of the canonical basis for each reduced expression $\mathbf{i}$ by a certain subset $Y_{\mathbf{i}}$ of $\mathbb{N}^k$. This gives a bijection $\psi_{\mathbf{i}} : \mathbf{B} \to Y_{\mathbf{i}}$. The subset $Y_{\mathbf{i}}$ is called the string cone.

There is no simple way to express the elements of $\mathbf{B}$ in terms of the natural generators $F_1, F_2, \ldots, F_n$ of U^-. This has been done for all of $\mathbf{B}$ only in types A_1, A_2, A_3 and B_2 (see [9, §3.4], [12, §13], [20] and [19]) and appears to become arbitrarily complicated in general.

A monomial $F_{i_1}^{(a_1)} F_{i_2}^{(a_2)} \cdots F_{i_k}^{(a_k)}$, where $F_i^{(a)} = F_i^a/[a]!$, is said to be *tight* if it belongs to $\mathbf{B}$. Lusztig [12] described a method which in low rank cases leads to the construction of tight monomials. He defined, for each reduced expression $\mathbf{i}$ of w_0, a certain cone $C_{\mathbf{i}}$ in $\mathbb{N}^k$ which we shall call the Lusztig cone. Let

$$M_{\mathbf{i}} = \{F_{i_1}^{(a_1)} F_{i_2}^{(a_2)} \cdots F_{i_k}^{(a_k)} \; : \; \mathbf{a} \in C_{\mathbf{i}}\}$$

be the set of monomials obtained from elements of $C_{\mathbf{i}}$. Lusztig showed that, in types A_1, A_2 and A_3, $M_{\mathbf{i}} \subseteq \mathbf{B}$. The author [14] has extended this to type A_4. However, counter-examples of Xi [21] and Reineke [18] show that this fails for higher rank.

Instead, one can consider monomials in Kashiwara's operators $\widetilde{F}_i$. Define, for a reduced expression $\mathbf{i}$ for w_0,

$$\mathbf{B}_{\mathbf{i}} = \{b \in \mathbf{B} \; : \; b \equiv \widetilde{F}_{i_1}^{a_1} \widetilde{F}_{i_2}^{a_2} \cdots \widetilde{F}_{i_k}^{a_k} \cdot 1, \; \mathbf{a} \in C_{\mathbf{i}}\}.$$

It has been shown by Premat [16] (in a more general set-up) that $C_{\mathbf{i}} \subseteq Y_{\mathbf{i}}$, so we have $\mathbf{B}_{\mathbf{i}} = \psi_{\mathbf{i}}^{-1}(C_{\mathbf{i}})$. Then it is known that $\mathbf{B}_{\mathbf{i}} = M_{\mathbf{i}}$ in types A_1, A_2, A_3 (using Lusztig's work [12] and direct calculation); it will be seen in [6] that this is also true in type A_4. This breaks down in higher ranks because of the above counter-examples. Thus $B_{\mathbf{i}}$ can be regarded as a generalisation of $M_{\mathbf{i}}$ for higher ranks, and it is likely that canonical basis elements in $B_{\mathbf{i}}$ will have a common form.

Furthermore, the cones $C_{\mathbf{i}}$ play a role with respect to various reparametrization functions of the canonical basis. Let $\mathbf{i}, \mathbf{j}$ be reduced expressions for w_0. There is a reparametrization function, $R_{\mathbf{i}}^{\mathbf{j}} = \varphi_{\mathbf{j}} \varphi_{\mathbf{i}}^{-1} : \mathbb{N}^k \to \mathbb{N}^k$ (see [9, §2]); a reparametrization function $T_{\mathbf{i}}^{\mathbf{j}} = \psi_{\mathbf{j}} \psi_{\mathbf{i}}^{-1} : Y_{\mathbf{i}} \to Y_{\mathbf{j}}$ (see [3, §§2.6, 2.7] and also the remark at the end of Section 2 of [15]). We also define a reparametrization function $S_{\mathbf{i}}^{\mathbf{j}} = \varphi_{\mathbf{j}} \psi_{\mathbf{i}}^{-1} : Y_{\mathbf{i}} \to \mathbf{N}^k$ linking the approaches of Lusztig and Kashiwara. The functions $R_{\mathbf{i}}^{\mathbf{j}}$ and $T_{\mathbf{i}}^{\mathbf{j}}$ are known to be piecewise-linear, and it follows that $S_{\mathbf{i}}^{\mathbf{j}}$ is also piecewise-linear. These functions are very difficult to understand; they have recently been described by Zelevinsky in [4], but their regions of linearity are only known in types A_1–A_5, B_2, C_3, D_4 and G_2 (see [5] and references therein).

Suppose now that $\mathbf{g}$ is of type A_n. In [13], it was shown that $C_{\mathbf{i}}$ could be expressed as the set of nonnegative integral combinations of a certain set of k spanning vectors, $v_j(\mathbf{i})$, $j = 1, \ldots, n$, $v_P(\mathbf{i})$, P a partial quiver of type A_n associated to $\mathbf{i}$ (as described in [13]). A partial quiver of type A_n is a quiver of type A_n with some directed edges replaced by undirected edges in such a way that the subgraph of directed edges is connected.

If an element $\mathbf{a} = (a_1, a_2, \ldots, a_k) \in C_{\mathbf{i}}$ is written $\mathbf{a} = (a_\alpha)_{\alpha \in \Phi^+}$, where Φ^+ is the set of positive roots of $\mathbf{g}$ and $a_\alpha = a_j$ where α is the jth root in the ordering on Φ^+ induced by $\mathbf{i}$, then the $v_j(\mathbf{i})$ and the $v_P(\mathbf{i})$ do not depend on $\mathbf{i}$ and are described in [13]. These vectors have also been studied by Bédard [1] in terms of the representation theory of quivers. Here we show that if $\mathbf{a} = (a_1, a_2, \ldots, a_k)$ is one of these vectors then the corresponding monomial of Kashiwara's operators $\widetilde{F}_{i_1}^{a_1} \widetilde{F}_{i_2}^{a_2} \cdots \widetilde{F}_{i_k}^{a_k} \cdot 1$ can be described using a remarkable rectangle combinatorics.

For $j = 1, 2, \ldots, n$ and P a partial quiver associated to $\mathbf{i}$, let $b_j(\mathbf{i}) = \psi_{\mathbf{i}}^{-1}(v_j(\mathbf{i}))$ and $b_P(\mathbf{i}) = \psi_{\mathbf{i}}^{-1}(v_P(\mathbf{i}))$ be the corresponding canonical basis elements. We will show that these elements are independent of the reduced expression $\mathbf{i}$. Thus the canonical basis elements arising in this way for at least one reduced expression $\mathbf{i}$ can be labelled b_j, $j = 1, 2, \ldots, n$, b_P, P a partial quiver of type A_n.

The following conjecture, which will be shown in type A_4 in [6], suggests that these elements are important in understanding the canonical basis. For a reduced expression $\mathbf{j}$ for w_0, let $c_j(\mathbf{j}) = \phi_{\mathbf{j}}(b_j)$ and let $c_P(\mathbf{j}) = \phi_{\mathbf{j}}(b_P)$ be the parametrizations of these canonical basis elements arising from Lusztig's approach.

Conjecture 1.1 *(Carter and Marsh)*

Let $\mathbf{i}$ be a reduced expression for w_0 and let $\mathbf{k} = (1, 3, 5, \ldots 2, 4, 6, \ldots)$ and $\mathbf{k}' = (2, 4, 6, \ldots, 1, 3, 5, \ldots)$, the opposite reduced expressions for w_0 arising in [9]. Then the vectors $c_j(\mathbf{k})$, $j = 1, 2, \ldots, n$, $c_P(\mathbf{k})$, P a partial quiver associated with $\mathbf{i}$ as above form a simplicial region of linearity $X_{\mathbf{i}}$ of Lusztig's piecewise-linear function $R_{\mathbf{k}}^{\mathbf{k}'}$. Furthermore, all such simplicial regions arise in this way, and the map $\mathbf{i} \mapsto X_{\mathbf{i}}$ from the graph of commutation classes of reduced expressions for w_0

(with edges given by the long braid relation) to the graph of simplicial regions of linearity of R (with edges given by adjacency) is a graph isomorphism.

In this paper we shall use the above rectangle combinatorics to calculate the vectors $c_j(\mathbf{j})$, $c_P(\mathbf{j})$ for the reduced expression $\mathbf{j} = (n, n-1, n, n-2, n-1, n, \ldots, 1, 2, \ldots, n)$ for w_0. This will be used in a future paper of Carter in order to calculate the vectors $c_j(\mathbf{k})$, $c_P(\mathbf{k})$, and thus the conjectural simplicial regions of linearity of $R_{\mathbf{k}}^{\mathbf{k}'}$. It is also possible that the above conjecture is true for *any* pair of opposite reduced expressions $\mathbf{k}, \mathbf{k}'$ for w_0; in particular for $\mathbf{j}$ and its opposite $\mathbf{j}'$, i.e. that the vectors $c_j(\mathbf{j})$ and $c_P^{\mathbf{j}}$ span simplicial regions of linearity of $R_{\mathbf{j}}^{\mathbf{j}'}$.

2 Parametrizations of the canonical basis

For positive integers $p < q$ we denote by $[p, q]$ the set $\{p, p+1, \ldots, q\}$, and for a rational number x we denote by $\lceil x \rceil$ the smallest element of $\{y \in \mathbb{Z} : x \leq y\}$.

Let $\mathbf{g}$ be the simple Lie algebra over $\mathbb{C}$ of type A_n and U be the quantized enveloping algebra of $\mathbf{g}$. Then U is a $\mathbb{Q}(v)$-algebra generated by the elements E_i, F_i, K_μ, $i \in \{1, 2, \ldots, n\}$, $\mu \in Q$, the root lattice of $\mathbf{g}$. Let U^+ be the subalgebra generated by the E_i and U^- the subalgebra generated by the F_i.

Let W be the Weyl group of $\mathbf{g}$ with Coxeter generators $s_1, s_2, \ldots, s_n$. It has a unique element w_0 of maximal length k. We shall identify a reduced expression $s_{i_1} s_{i_2} \cdots s_{i_k}$ for w_0 with the k-tuple $\mathbf{i} = (i_1, i_2, \ldots, i_k)$. We shall denote by χ_n the set of all reduced expressions $\mathbf{i} = (i_1, i_2, \ldots, i_k)$ for w_0. Given $\mathbf{i}, \mathbf{i}' \in \chi_n$, we say that $\mathbf{i} \simeq \mathbf{i}'$ if there is a sequence of commutations (of the form $s_i s_j = s_j s_i$ with $|i-j| > 1$) which, when applied to $\mathbf{i}$, give $\mathbf{i}'$. This is an equivalence relation on χ_n, and the equivalence classes are called commutation classes.

For each $\mathbf{i} \in \chi_n$ there are two parametrizations of the canonical basis $\mathbf{B}$ for U^-. The first arises from Lusztig's approach to the canonical basis [11, §14.4.6], and the second arises from Kashiwara's approach [7].

Lusztig's Approach

There is a $\mathbb{Q}$-algebra automorphism of U which takes each E_i to F_i, F_i to E_i, K_μ to $K_{-\mu}$ and v to v^{-1}. We use this automorphism to transfer Lusztig's definition of the canonical basis in [9, §3] to U^-.

Let $T''_{i,-1}$ be the automorphism of U as in [11, §37.1.3]. For $\mathbf{c} \in \mathbb{N}^k$ and $\mathbf{i} \in \chi_n$, let

$$F_{\mathbf{i}}^{\mathbf{c}} := F_{i_1}^{(c_1)} T''_{i_1,-1}(F_{i_2}^{(c_2)}) \cdots T''_{i_1,-1} T''_{i_2,-1} \cdots T''_{i_{k-1},-1}(F_{i_k}^{(c_k)}),$$

and define $B_{\mathbf{i}} = \{F_{\mathbf{i}}^{\mathbf{c}} : \mathbf{c} \in \mathbb{N}^k\}$. Then $B_{\mathbf{i}}$ is the basis of PBW-type for U^- corresponding to $\mathbf{i}$. Let $\bar{\ }$ be the $\mathbb{Q}$-algebra automorphism of U taking E_i to E_i, F_i to F_i, and K_μ to $K_{-\mu}$, for each $i \in [1, n]$ and $\mu \in Q$, and v to v^{-1}. Lusztig proves the following result in [9, §§2.3, 3.2].

Theorem 2.1 *(Lusztig)*
The $\mathbb{Z}[v]$-span $\mathcal{L}$ of $B_{\mathbf{i}}$ is independent of $\mathbf{i}$. Let $\pi : \mathcal{L} \to \mathcal{L}/v\mathcal{L}$ be the natural projection. The image $\pi(B_{\mathbf{i}})$ is also independent of $\mathbf{i}$; we denote it by B. The restriction of π to $\mathcal{L} \cap \overline{\mathcal{L}}$ is an isomorphism of $\mathbb{Z}$-modules $\pi_1 : \mathcal{L} \cap \overline{\mathcal{L}} \to \mathcal{L}/v\mathcal{L}$, and $\mathbf{B} = \pi_1^{-1}(B)$ is a $\mathbb{Q}(v)$-basis of U^-, called the canonical basis.

Lusztig's theorem provides us with a parametrization of $\mathbf{B}$, dependent on $\mathbf{i}$. If $b \in \mathbf{B}$, we write $\varphi_{\mathbf{i}}(b) = \mathbf{c}$, where $\mathbf{c} \in \mathbb{N}^k$ satisfies $b \equiv F_{\mathbf{i}}^{\mathbf{c}} \mod v\mathcal{L}$. Note that $\varphi_{\mathbf{i}}$ is a bijection.

Given any pair $\mathbf{i}, \mathbf{j} \in \chi_n$, Lusztig defines in [9, §2.6] a function $R_{\mathbf{i}}^{\mathbf{j}} = \varphi_{\mathbf{j}} \varphi_{\mathbf{i}}^{-1} : \mathbb{N}^k \to \mathbb{N}^k$, which he shows is piecewise-linear. He further shows that its regions of linearity are significant for the canonical basis, in the sense that elements b of the canonical basis with $\varphi_{\mathbf{i}}(b)$ in the same region of linearity of $R_{\mathbf{i}}^{\mathbf{j}}$ often have similar form.

Kashiwara's approach

Let $\widetilde{E}_i$ and $\widetilde{F}_i$ be the Kashiwara operators on U^- as defined in [7, §3.5]. Let $\mathcal{A} \subseteq \mathbb{Q}(v)$ be the subring of elements regular at $v = 0$, and let $\mathcal{L}'$ be the $\mathcal{A}$-lattice spanned by the set S of arbitrary products $\widetilde{F}_{j_1} \widetilde{F}_{j_2} \cdots \widetilde{F}_{j_m} \cdot 1$ in U^-. The following results were proved by Kashiwara in [7].

Theorem 2.2 *(Kashiwara)*
(i) Let $\pi' : \mathcal{L}' \to \mathcal{L}'/v\mathcal{L}'$ be the natural projection, and let $B' = \pi'(S)$. Then B' is a $\mathbb{Q}$-basis of $\mathcal{L}'/v\mathcal{L}'$

(the crystal basis).

(ii) The operators $\widetilde{E}_i$ *and* $\widetilde{F}_i$ *each preserve* $\mathcal{L}'$ *and thus act on* $\mathcal{L}'/v\mathcal{L}'$. *They satisfy* $\widetilde{E}_i(B') \subseteq B' \cup \{0\}$ *and* $\widetilde{F}_i(B') \subseteq B'$. *For* $b, b' \in B'$ *we have* $\widetilde{F}_i b = b'$ *if and only if* $\widetilde{E}_i b' = b$.

(iii) For each $b \in B'$, *there is a unique element* $\tilde{b} \in \mathcal{L}' \cap \overline{\mathcal{L}'}$ *such that* $\pi'(\tilde{b}) = b$. *The set of elements* $\{\tilde{b} : b \in B'\}$ *forms a basis of* U^-, *the* global crystal basis *of* U^-.

It was shown by Lusztig [10, 2.3] that the global crystal basis of Kashiwara coincides with the canonical basis. There is a parametrization of $\mathbf{B}$ arising from Kashiwara's approach, again dependent on $\mathbf{i} \in \chi_n$. Let $\mathbf{i} = (i_1, i_2, \ldots, i_k)$ and $b \in B$. Let a_1 be maximal such that $\widetilde{E}_{i_1}^{a_1} b \not\equiv 0 \mod v\mathcal{L}'$; let a_2 be maximal such that $\widetilde{E}_{i_2}^{a_2} \widetilde{E}_{i_1}^{a_1} b \not\equiv 0 \mod v\mathcal{L}'$, and so on, so that a_k is maximal such that $\widetilde{E}_{i_k}^{a_k} \widetilde{E}_{i_{k-1}}^{a_{k-1}} \cdots \widetilde{E}_{i_2}^{a_2} \widetilde{E}_{i_1}^{a_1} b \not\equiv 0 \mod v\mathcal{L}'$. We write $\psi_{\mathbf{i}}(b) = \mathbf{a} = (a_1, a_2, \ldots, a_k)$ and we have $b \equiv \widetilde{F}_{i_1}^{a_1} \widetilde{F}_{i_2}^{a_2} \cdots \widetilde{F}_{i_k}^{a_k} \cdot 1 \mod v\mathcal{L}'$. This is the crystal string of b — see [3, §2], [15, §2] and [8]. The map $\psi_{\mathbf{i}}$ is injective (see [15, §2.5]). Its image is a cone, which we shall call the *string cone* associated to $\mathbf{i}$, $Y_{\mathbf{i}} = \psi_{\mathbf{i}}(\mathbf{B})$. We define a function which compares Kashiwara's approach with Lusztig's approach. For $\mathbf{i}, \mathbf{j} \in \chi_n$, let $S_{\mathbf{i}}^{\mathbf{j}} = \varphi_{\mathbf{j}} \psi_{\mathbf{i}}^{-1} : Y_{\mathbf{i}} \to \mathbb{N}^k$.

3 The Lusztig cones and their spanning vectors

Lusztig [12] introduced certain regions which, in low rank, give rise to canonical basis elements of a particularly simple form. The *Lusztig cone*, $C_{\mathbf{i}}$, corresponding to $\mathbf{i} \in \chi_n$, is defined to be the set of points $\mathbf{a} \in \mathbb{N}^k$ satisfying the following inequalities:

(*) For every pair $s, s' \in [1, k]$ with $s < s'$, $i_s = i_{s'} = i$ and $i_p \neq i$ whenever $s < p < s'$, we have

$$(\sum_p a_p) - a_s - a_{s'} \geq 0,$$

where the sum is over all p with $s < p < s'$ such that i_p is joined to i in the Dynkin diagram (we call such a pair s, s' a *minimal pair*).

It was shown by Lusztig [12] that if $\mathbf{a} \in C_{\mathbf{i}}$ then the monomial $F_{i_1}^{(a_1)} F_{i_2}^{(a_2)} \cdots F_{i_k}^{(a_k)}$ lies in the canonical basis $\mathbf{B}$, provided $n = 1, 2, 3$. The author [14] showed that this remains true if $n = 4$. Counter-examples of Xi [21] and Reineke [18] show that this is not true in general.

Remark: It can be shown that, for any $\mathbf{i} \in \chi_n$, we have $C_{\mathbf{i}} \subseteq Y_{\mathbf{i}}$, i.e., the Lusztig cone is contained in the Kashiwara cone. The methods of this paper could be used to do this in type A_n, but the author

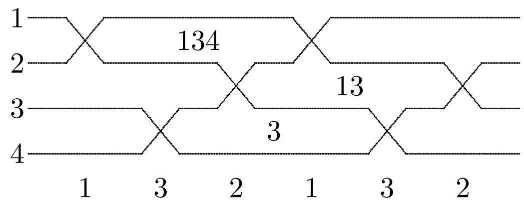

Figure 1: The Chamber Diagram of $(1,3,2,1,3,2)$.

has recently learnt that A. Premat [16] has a more succinct proof that applies to all symmetrizable Kac-Moody Lie algebras.

The reduced expression $\mathbf{i}$ defines an ordering on the set Φ^+ of positive roots of the root system associated to W. We write $\alpha^j = s_{i_1}s_{i_2}\cdots s_{i_{j-1}}(\alpha_{i_j})$ for $j = 1,2,\ldots,k$. Then $\Phi^+ = \{\alpha^1, \alpha^2, \ldots \alpha^k\}$. For $\mathbf{a} = (a_1, a_2, \ldots, a_k) \in \mathbb{Z}^k$, write $a_{\alpha^j} = a_j$. If $\alpha = \alpha_{ij} := \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$ with $i < j$, we also write a_{ij} for $a_{\alpha_{ij}}$.

We shall need the chamber diagram (chamber ansatz) for $\mathbf{i}$ defined in [2, §§1.4, 2.3]. We take $n+1$ strings, numbered from top to bottom, and write $\mathbf{i}$ from left to right along the bottom of the diagram. Above a letter i_j in $\mathbf{i}$, the i_jth and (i_j+1)st strings from the top above i_j cross. Thus, for example, in the case $n = 3$ with $\mathbf{i} = (1,3,2,1,3,2)$, the chamber diagram is shown in Figure 1.

We denote the chamber diagram of $\mathbf{i}$ by $\mathrm{CD}(\mathbf{i})$. A *chamber* will be defined as a pair $(c, \mathbf{i})$, where c is a bounded component of the complement of $\mathrm{CD}(\mathbf{i})$. Each chamber $(c, \mathbf{i})$ can be labelled with the numbers of the strings passing below it, denoted $l(c, \mathbf{i})$. Following [2], we call such a label a chamber set. For example, the chamber sets corresponding to the 3 bounded chambers in Figure 1 are 134, 3, 13. Note that the set of chamber sets of $\mathbf{i}$ is independent of its commutation class (so we can talk also of the chamber sets of such a commutation class). Every subset of $[1, n+1]$ can arise as a chamber set for some $\mathbf{i}$, except for subsets of the form $[1, j]$ and $[j, n+1]$ for $1 \leq j \leq n+1$; this is observed in the proof of [2, Theorem 2.7.1].

We now recall some of the results from [13] that we shall need. We have by [13, §4] that there is a matrix $P_{\mathbf{i}} \in GL_k(\mathbb{Z})$ such that $C_{\mathbf{i}} = \{\mathbf{a} \in \mathbb{Z}^k \ : \ P_{\mathbf{i}}\mathbf{a} \geq 0\}$, where, for $\mathbf{z} \in \mathbb{Z}^k$, $\mathbf{z} \geq 0$ means that each entry in $\mathbf{z}$ is nonnegative. It follows that $C_{\mathbf{i}}$ is the set of nonnegative linear combinations of the columns of $Q_{\mathbf{i}} = P_{\mathbf{i}}^{-1}$. These columns were described in [13], using the concept of a *partial quiver*.

A partial quiver P of type A_n is a quiver of type A_n which has some (or none) of its arrows replaced by undirected edges, in such a way that the subgraph obtained by deleting undirected edges and any vertex incident only with undirected edges is connected. We number the edges of a partial quiver from 2 to n, starting at the right hand end. We write P as a sequence of $n-1$ symbols, L, R or $-$, where L denotes a leftward arrow, R a rightward arrow, and $-$ an undirected edge. Thus any partial quiver will be of form $---***---$, where the $*$'s denote L's or R's.

If P, P' are partial quivers we write $P' \geq P$ (or $P \leq P'$) if every edge which is directed in P is directed in P' and is oriented in the same way, and say that P is a sub partial quiver of P'. For example, $---LRLL- \leq RLRLRLLL$.

It is known that if $\mathbf{i}$ is compatible with a quiver Q (in the sense of [9, §4.7]) then the set of reduced expressions for w_0 compatible with Q is precisely the commutation class of χ_n containing $\mathbf{i}$. We say that this commutation class is compatible with Q. Berenstein, Fomin and Zelevinsky (see [2, §4.4]) describe a method for constructing $\mathbf{i} \in \chi_\mathbf{n}$ compatible with any given quiver Q, as follows.

Suppose Q is a quiver of type A_n. Let $\Lambda \subseteq [2, n]$ be the set of all edges of Q pointing to the left. Berenstein, Fomin and Zelevinsky construct an arrangement $\text{Arr}(\Lambda)$. Consider a square in the plane, with horizontal and vertical sides. Put $n+1$ points onto the left-hand edge of the square, equally spaced, numbered 1 to $n+1$ from top to bottom, and do the same for the right-hand edge, but number the points from bottom to top. Line_h joins point h on the left with point h on the right. For $h = 1, n+1$, Line_h is a diagonal of the square. For $h \in [2, n]$, Line_h is a union of two line segments of slopes $\pi/4$ and $-\pi/4$. There are precisely two possibilities for Line_h. If $h \in \Lambda$, the left segment has slope $-\pi/4$, while the right one has slope $\pi/4$; for $h \in [2, n] \setminus \Lambda$, it goes the other way round. Berenstein, Fomin and Zelevinsky give the example of the arrangement for $n = 5$ and $\Lambda = \{2, 4\}$, which we show in Figure 2 (points corresponding to elements of Λ are indicated by filled circles). They show that $\mathbf{i} \in \chi_\mathbf{n}$ is compatible with the quiver Q if and only if the chamber diagram $\text{CD}(\mathbf{i})$ is isotopic to $\text{Arr}(\Lambda)$.

If P is a partial quiver, let $l(P)$ be the subset of $[1, n+1]$ defined in the following way. Put $l_1(P) = \{j \in [2, n] : \text{edge } j \text{ of } P \text{ is an } L\}$. If the rightmost directed edge of P is an R, and this is in position i, then let $l_2(P) = [1, i-1]$, otherwise the empty set. If the leftmost directed edge of P is an R, and this is in

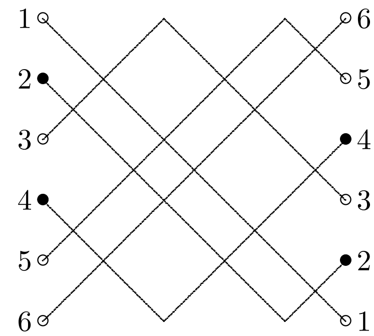

Figure 2: The arrangement Arr$\{2,4\}$.

position j, then let $l_3(P) = [j+1, n+1]$, otherwise the empty set. Then put $l(P) = l_1(P) \cup l_2(P) \cup l_3(P)$. We know from [13, §5.4] that l sets up a one-to-one correspondence between chamber sets and partial quivers; for a chamber $(c, \mathbf{i})$, we denote by $P(c, \mathbf{i})$ the partial quiver corresponding to $l(c, \mathbf{i})$. Also, if $\mathbf{i}$ is compatible with a quiver Q, then the partial quivers corresponding to its chambers are precisely those $P \leq Q$.

Each minimal pair in $\mathbf{i}$ corresponds naturally to a chamber $(c, \mathbf{i})$. Thus, given a chamber $(c, \mathbf{i})$, there is a corresponding row of $P_{\mathbf{i}}$ and therefore a corresponding column of $Q_{\mathbf{i}}$, that is, a spanning vector of $C_{\mathbf{i}}$. We denote this spanning vector by $a(c, \mathbf{i})$. Similarly, the other n rows of $P_{\mathbf{i}}$ correspond to inequalities of the form $a_{\alpha_j} \geq 0$ for α_j a simple root, $1 \leq j \leq n$. We denote the corresponding spanning vectors by $a(j, \mathbf{i})$.

We consider $a(c, \mathbf{i}) = (a_1, a_2, \ldots, a_k)$ as a multiset $M(c, \mathbf{i})$ of positive roots, where each positive root α_{ij} occurs with multiplicity, a_{ij}. Similarly we write $M(j, \mathbf{i})$ for the multiset corresponding to $a(j, \mathbf{i})$, for $j = 1, 2, \ldots, k$. These multisets have been described in [13].

Theorem 3.1 *(See [13, §5.12]).*
(a) The multiset spanning vector $M(c, \mathbf{i})$ depends only upon $P(c, \mathbf{i})$. For a partial quiver P we choose a chamber $(c, \mathbf{i})$ such that $P(c, \mathbf{i}) = P$ and write $M(P) = M(c, \mathbf{i})$.
(b) The multiset spanning vector $M(j, \mathbf{i})$ depends only upon j; we denote it by $M(j)$.
(c) For a partial quiver P, $M(P)$ is given by the following construction. We say that a sub partial quiver

Y of P is a component *of P if all of its edges are oriented in the same way and it is maximal in length with this property. We say Y has type L if its edges are oriented to the left, and type R if its edges are oriented to the right. For such a component Y of P, let $a(Y)$ be the number of the leftmost edge to the right of the component, and let $b(Y)$ be the number of the rightmost edge to the left of the component. Let $M(Y)$ be the set of positive roots α_{ij} satisfying $1 \leq i \leq a(Y) \leq b(Y) \leq j \leq n+1$. Then $M(P)$ is the multiset union (i.e. adding multiplicities) of the $M(Y)$, where Y varies over all components of P, with the multiplicity m_{ij} of α_{ij} replaced by $n_{ij} = \lceil \frac{1}{2} m_{ij} \rceil$. Furthermore, for $j = 1, 2, \ldots n$, $M(j)$ is the set of positive roots α_{pq} satisfying $1 \leq p \leq j \leq j+1 \leq q \leq n+1$.*

4 Canonical basis elements corresponding to spanning vectors

Recall that for $\mathbf{i}, \mathbf{i}' \in \chi_n$, the map $T_{\mathbf{i}}^{\mathbf{i}'}$ is defined by $T_{\mathbf{i}}^{\mathbf{i}'}(\mathbf{a}) = \psi_{\mathbf{i}'} \psi_{\mathbf{i}}^{-1}(\mathbf{a})$, for $\mathbf{a} \in Y_{\mathbf{i}}$. Define $T_2 : \mathbb{Z}^2 \to \mathbb{Z}^2$ by $T_2(a, b) := (b, a)$, and $T_3 : \mathbb{Z}^3 \to \mathbb{Z}^3$ by $T_3(a, b, c) := (\max(c, b-a), a+c, \min(a, b-c))$. The maps $T_{\mathbf{i}}^{\mathbf{i}'}$ satisfy the following:

Proposition 4.1 *(Berenstein and Zelevinsky)*
(a) Suppose $\mathbf{i}, \mathbf{i}' \in \chi_n$ and that $\mathbf{i} = \mathbf{i}'$ except that $(i_s, i_{s+1}) = (i, j)$ and $(i'_s, i'_{s+1}) = (j, i)$ for some index s and $i, j \in [1, n]$ with $|i - j| > 1$. Let $\mathbf{a} \in \mathbb{N}^k$ and write $\mathbf{a}' = T_{\mathbf{i}}^{\mathbf{i}'}(\mathbf{a})$. Then we have $a'_r = a_r$ for $r < s$ or $r > s+1$ and $(a'_s, a'_{s+1}) = T_2(a_s, a_{s+1})$.
(b) Suppose $\mathbf{i}, \mathbf{i}' \in \chi_n$, and that $\mathbf{i} = \mathbf{i}'$ except that $(i_s, i_{s+1}, i_{s+2}) = (i, j, i)$ and $(i'_s, i'_{s+1}, i'_{s+2}) = (j, i, j)$ for some index s and $i, j \in [1, n]$ with $|i - j| = 1$. Let $\mathbf{a} \in \mathbb{N}^k$ and write $\mathbf{a}' = T_{\mathbf{i}}^{\mathbf{i}'}(\mathbf{a})$. Then we have $a'_r = a_r$ for $r < s$ or $r > s+2$ and $(a'_s, a'_{s+1}, a'_{s+2}) = T_3(a_s, a_{s+1}, a_{s+2})$.

Proof: See [3, §§2.6, 2.7]. □

We need the following Lemma, which will help us to understand how spanning vectors of Lusztig cones are related.

Lemma 4.2 *(a) Suppose $\mathbf{i}, \mathbf{i}' \in \chi_n$ and that $\mathbf{i} = \mathbf{i}'$ except that $(i_s, i_{s+1}) = (i, j)$ and $(i'_s, i'_{s+1}) = (j, i)$ for some index s and $i, j \in [1, n]$ with $|i - j| > 1$. Let $\mathbf{a} = a(c, \mathbf{i})$, respectively $a(j, \mathbf{i})$, be a spanning vector of $C_{\mathbf{i}}$, and let $\mathbf{a}' = a(c', \mathbf{i}')$, respectively $(j, \mathbf{i}')$, be the corresponding spanning vector of $C_{\mathbf{i}'}$. Here*

$(c', \mathbf{i}')$ is the chamber that $(c, \mathbf{i})$ becomes when the commutation relation $(i, j) \to (j, i)$ is applied to $\mathbf{i}$. Then $a'_r = a_r$ for all $r < s$ or $r > s+1$, while $(a'_s, a'_{s+1}) = T_2(a_s, a_{s+1})$.

(b) Suppose $\mathbf{i}, \mathbf{i}' \in \chi_n$ and that $\mathbf{i} = \mathbf{i}'$ except that $(i_s, i_{s+1}, i_{s+2}) = (i, j, i)$ and $(i'_s, i'_{s+1}, i'_{s+2}) = (j, i, j)$ for some index s and $i, j \in [1, n]$ with $|i - j| = 1$. Let $\mathbf{a} = a(c, \mathbf{i})$, respectively $a(j, \mathbf{i})$, be a spanning vector of $C_{\mathbf{i}}$. In this case, we suppose further that $(c, \mathbf{i})$ is not the chamber of $CD(\mathbf{i})$ involving the substring (i, j, i), so that there is a chamber $(c', \mathbf{i}')$ with $P(c, \mathbf{i}) = P(c', \mathbf{i}')$. Let $\mathbf{a}' = a(c', \mathbf{i}')$, respectively $(j, \mathbf{i}')$. Then $a'_r = a_r$ for all $r < s$ or $r > s+1$, while $(a'_s, a'_{s+1}, a'_{s+2}) = T_3(a_s, a_{s+1}, a_{s+2})$.

Proof: Recall that the spanning vectors of $C_{\mathbf{i}}$ are obtained by inverting the defining matrix $P_{\mathbf{i}}$ of $C_{\mathbf{i}}$. Part (a) follows immediately from the definition of $P_{\mathbf{i}}$. Suppose we are in situation (b). Consider the case where $\mathbf{a} = a(c, \mathbf{i})$. Since $(c, \mathbf{i})$ is not the chamber corresponding to $(i, j, i) = (i_s, i_{s+1}, i_{s+2})$, we must have that $\mathbf{a}$ is perpendicular to the row of $P_{\mathbf{i}}$ corresponding to this chamber. In the case where $\mathbf{a} = a(j, \mathbf{i})$, this is also true. So, in either case, we see that $a_{s+1} = a_s + a_{s+2}$. It follows from the definition of T_3 that $T_3(a_s, a_{s+1}, a_{s+2}) = (a_{s+2}, a_{s+1}, a_s)$. Since the ordering on the positive roots induced by $\mathbf{i}$ agrees with that induced by $\mathbf{i}'$ except that the roots in positions s and $s+2$ are exchanged, the fact that $\mathbf{a}'$ is as claimed in the statement above follows from Theorem 3.1 (a) and (b). $\square$

Using Proposition 4.1 and Lemma 4.2 we have:

Corollary 4.3 *Let $\mathbf{i}, \mathbf{i}' \in \chi_n$, and let $(c, \mathbf{i}), (c', \mathbf{i}')$ be chambers with $P(c, \mathbf{i}) = P(c', \mathbf{i}')$. Then $T_{\mathbf{i}}^{\mathbf{i}'}(a(c, \mathbf{i})) = a(c', \mathbf{i}')$. Also, for $j \in [1, n]$, $T_{\mathbf{i}}^{\mathbf{i}'}(a(j, \mathbf{i})) = a(j, \mathbf{i}')$.*

We therefore have:

Theorem 4.4 *Let $\mathbf{i} \in \chi_n$. For $j = 1, 2, \ldots, n$ and P a partial quiver associated to $\mathbf{i}$, let $b_j(\mathbf{i}) = \psi_{\mathbf{i}}^{-1}(v_j(\mathbf{i}))$ and $b_P(\mathbf{i}) = \psi_{\mathbf{i}}^{-1}(v_P(\mathbf{i}))$ be the corresponding canonical basis elements. Then these elements are independent of the reduced expression $\mathbf{i}$. Thus the canonical basis elements arising in this way for at least one reduced expression $\mathbf{i}$ can be labelled b_j, $j = 1, 2, \ldots, n$, b_P, P a partial quiver of type A_n.*

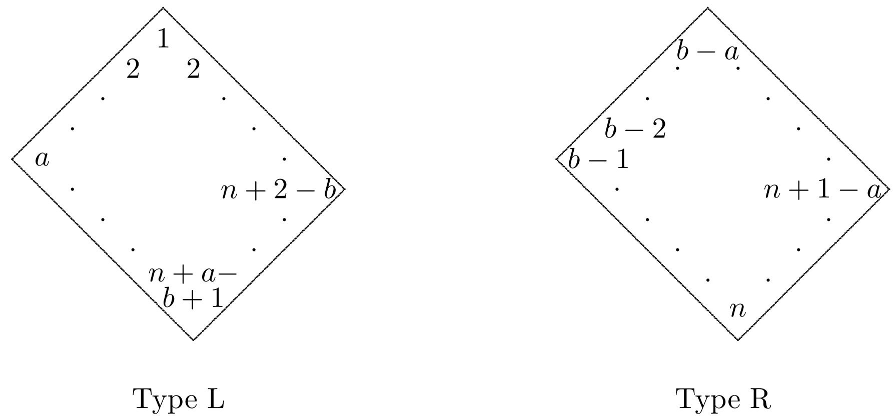

Figure 3: The rectangles $\rho(Y)$.

5 Rectangles

The above provides us with a description of the spanning vectors of a Lusztig cone as multisets of positive roots. We would also like a description of the monomial of Kashiwara operators corresponding to such a vector. We will use this in the next section to calculate Lusztig parametrizations $c_j(\mathbf{j})$, $c_P(\mathbf{j})$ of the canonical basis elements arising in Theorem 4.4. We shall show that these monomials can be described by a remarkable 'rectangle combinatorics' (invented by R. W. Carter) which we shall now describe.

Definition 5.1 Let P be a partial quiver, and Y a component of P. Let $a = a(Y)$ and let $b = b(Y)$ (see Theorem 3.1). We define the *rectangle* $\rho(Y)$ of Y, as follows. If Y is of type L, then $\rho(Y)$ is as in the left hand diagram in Figure 3; if Y is of type R then $\rho(Y)$ is as in the right hand diagram in Figure 3. We also define, for $j = 1, 2, \ldots, n$, $\rho(j)$ to be the type L rectangle with $a = j$ and $b = j + 1$ (for these values of a and b, both diagrams in Figure 3 coincide).

In each case, the rectangle is filled in with numbers in diagonal rows, with the numbers along a diagonal row from top right to bottom left increasing by 1 at each step and similarly from top left to bottom right. We define the diagram $D(P)$ of P in the following way. Go through the components Y of P one by one, from left to right. Fit the corresponding rectangles $\rho(Y)$ together, so that if a component of type L is followed by one of type R, the rectangles share leftmost corners, and if one of type R is followed by one of type L, they share rightmost corners. In each case, it is easy to see

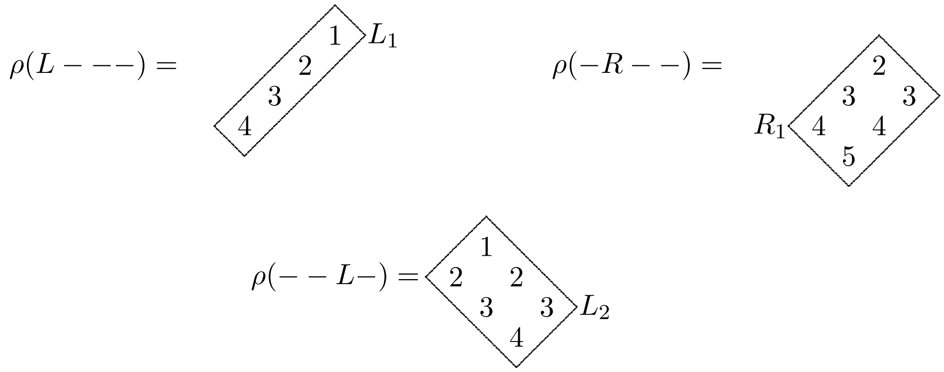

Figure 4: Examples of the rectangles $\rho(Y)$.

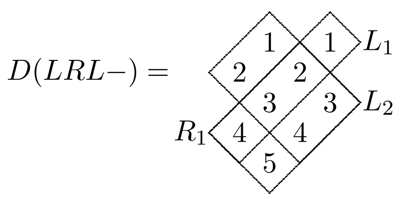

Figure 5: The diagram $D(LRL-)$.

that the overlapping numbers agree. We associate a multiplicity $m(s)$ to each position in $D(P)$ where a number s is placed; if t is the number of rectangles $\rho(Y)$ that s lies in, then $m(s)$ is defined to be $\lceil t/2 \rceil$. We label each rectangle with the corresponding component Y of P; on the left hand corner if Y is of type R, and on the right hand corner if Y is of type L. We define, for $j = 1, 2, \ldots, n$, $D(j)$ to be the diagram $\rho(j)$, with each position given multiplicity 1.

Example

We consider the case when $n = 5$ and $P = LRL-$. Then P has three components, $L_1 = L---$, $R_1 = -R--$ and $L_2 = --L-$. The diagrams $\rho(Y)$ are as in Figure 4.

We put them together, so that the 4 in $\rho(L---)$ matches the leftmost 4 in $\rho(-R--)$ and so that the rightmost 3 in $\rho(-R--)$ matches the rightmost 3 in $\rho(--L-)$, to get the complete diagram $D(LRL-)$; see Figure 5. The numbers 2 and 3 in the middle of the diagram each have multiplicity 2.

Definition 5.2 Suppose P is a partial quiver or $P = j$ for some $j \in [1, n]$. We define a *diagonal row* of $D(P)$ to be a diagonal row of numbers in $D(P)$ running from top left to bottom right. Such a diagram $D(P)$ defines a sequence $\mu(P)$ which is obtained by reading off the digits in the diagonal rows in the diagram, starting with the bottom left row. Each digit is repeated according to its multiplicity. In our example, $\mu(LRL-) = (4, 5, 2, 3, 3, 4, 1, 2, 2, 3, 1)$. There is a corresponding monomial $F(P)$ in U^-, obtained by replacing each digit, s, occurring with multiplicity $m(s)$, by the divided power $F_s^{(m(s))}$. In our example, $F(LRL-) = F_4F_5F_2F_3^{(2)}F_4F_1F_2^{(2)}F_3F_1$. Similarly, there is a corresponding monomial in the Kashiwara operators which we denote $\widetilde{F}(P)$. So in our example, $\widetilde{F}(LRL-) = \widetilde{F}_4\widetilde{F}_5\widetilde{F}_2\widetilde{F}_3^2\widetilde{F}_4\widetilde{F}_1\widetilde{F}_2^2\widetilde{F}_3\widetilde{F}_1 \cdot 1$.

The following theorem was conjectured by Carter:

Theorem 5.3 *Suppose* $\mathbf{i} \in \chi_\mathbf{n}$ *is compatible with a quiver* Q*, and let* $(c, \mathbf{i})$ *be a chamber in* $CD(\mathbf{i})$*. Let* $P = P(c, \mathbf{i})$ *be the corresponding partial quiver (so* $P \leq Q$*), with* $\mathbf{a} = a(c, \mathbf{i}) = (a_1, a_2, \ldots, a_k)$*. Alternatively, let* $P = j \in [1, n]$*. In either case,*

$$F_{i_1}^{(a_1)}F_{i_2}^{(a_2)}\cdots F_{i_k}^{(a_k)} = F(P).$$

In particular, the left hand side depends only upon P*.*

Proof: Suppose first that P is a partial quiver, and let Y be a component of P. Our first step will be to understand where the roots in $M(Y)$ occur in CD($\mathbf{i}$) (as labels of crossings). Let us suppose in the first case that Y is of type L, with $a = a(Y)$ and $b = b(Y)$. We draw the diagram for the arrangement corresponding to Q, defined by Berenstein, Fomin and Zelevinsky. We mark the crossing of strings i and j, $i \leq j$ by the root $\alpha_{ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$, and we write the quiver Q up the left hand side, so that edge j of Q appears next to the left hand end of string j, for each j. We write Q down the right hand side also.

Draw 4 dashed lines on the diagram. The first, A, should intersect the left vertical boundary between strings a and $a+1$, and go down and to the right at an angle of $-\pi/4$ (to the horizontal). The next, B, should intersect the left vertical boundary between strings b and $b-1$ and go up and to the right at an angle of $\pi/4$. Line C should intersect the right vertical boundary between strings a and $a+1$ and go up and to the left at an angle of $3\pi/4$. Finally, line D should intersect the right vertical boundary between strings b and $b-1$, and go down and to the left, at an angle of $-3\pi/4$. For an example, in type A_7, where $Q = RRLLRL$ and $Y = --LL--$, with $a(Y) = 3$ and $b(Y) = 6$, see Figure 6.

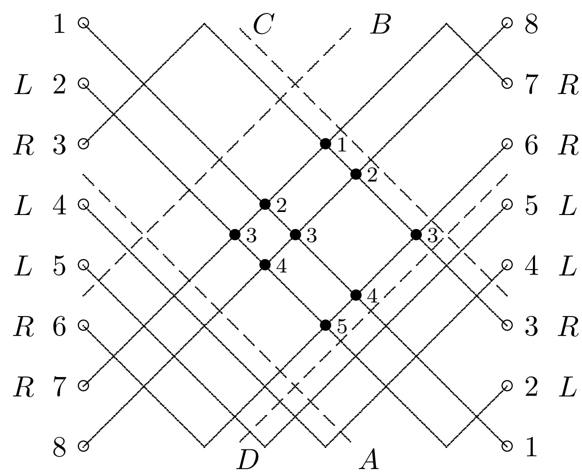

Figure 6: Example: $Q = RRLLRL, Y = --LL--$.

These four dashed lines bound a rectangle in the diagram; we shall see that the crossings inside it form a rectangle, ρ, and that their labels are precisely the positive roots in $M(Y)$. (In the example, these crossings are marked by filled circles.)

By construction, the lines crossing ρ parallel to A and C are parts of the strings c with $c \leq a$, and the lines crossing ρ parallel to B and D are parts of the strings d with $b \leq d$. Thus the crossings appearing inside ρ are those labelled with roots α_{cd} with $1 \leq c \leq a \leq b \leq d \leq n+1$, that is, the roots in $M(Y)$.

Let E be the line in the rectangle we are interested in which is parallel to C and closest to C, and suppose it is part of string x and that this corresponds to an R in Q. Let y be the first string that x intersects with after changing direction at the top of the diagram. By the construction of the diagram, the strings corresponding to the L's of Y are below the rectangle, so y cannot be one of these strings. Similarly, if y were a string corresponding to an L to the right of Y, it would be parallel to E initially (i.e. in the left part of the diagram) and then would only be perpendicular to E below ρ. Also, if y corresponded to an R to the right of Y in Q, x and y would intersect to the left of the point where x changed direction. We conclude that x and y intersect inside ρ.

Thus in the corresponding chamber diagram, the top crossing of the rectangle is on row 1 (so the corresponding simple reflection is s_1). Suppose next that x corresponds to an L in Q. Let y be the first string that x intersects. Since there can in this case be no R's to the right of Y in Q, x intersects every string corresponding to an arrow of Q to the right of Y below the rectangle ρ; similarly with the L's in Y. Thus y is to the left of P and x and y intersect in the rectangle.

Thus in both cases the top corner of the rectangle ρ is in row 1. Let us write the row of each crossing in ρ next to it. The number of crossings in a side of the rectangle parallel to A or C (equal to the number of crossings involving E inside ρ) is equal to $n+2-b$, and the number in a side parallel to B or D is a. Thus the labels of the crossings inside ρ are the same and in the same configuration as the numbers in $\rho(Y)$.

The case when Y is of type R is similar; in this case the bottom corner of the rectangle in the arrangement diagram is labelled n.

Now suppose that Y is of type L and immediately to the right of Y in P is a component Z of type R. By the construction of the rectangles, the leftmost crossings inside them coincide. Similarly, if Y is of type R and Z is of type L, the rightmost crossings in the rectangles coincide.

Thus the structure of the rectangles corresponding to components of P in the arrangement diagram, or the chamber diagram, corresponding to $\mathbf{i}$, is the same as in $D(P)$. The result follows by Theorem 3.1.

In the case where $P = j \in [1, n]$, let $Y = j$; here we consider Y to be the vertex j of Q, with no edges. We then argue as in the above proof for Y of type L. □

Corollary 5.4 *Suppose we are in the situation of the Theorem, and let $\widetilde{F}(P)$ be the monomial in the root operators as defined in 5.2. Then $\widetilde{F}_{i_1}^{a_1}\widetilde{F}_{i_2}^{a_2}\cdots\widetilde{F}_{i_k}^{a_k}\cdot 1 = \widetilde{F}(P)$.*

Proof: This follows immediately from the Theorem, since the only relations in U^- that are used are commutations, which the root operators $\widetilde{F}_i$ also satisfy. □

6 A Reparametrization

We will now use the rectangle combinatorics from the previous section to calculate the Lusztig parametrizations $c_j(\mathbf{j})$, $c_P(\mathbf{j})$ of the canonical basis elements arising in Theorem 4.4; here $\mathbf{j}$ is fixed as the reduced expression $(n, n-1, n, n-2, n-1, n, \ldots, 1, 2, \ldots, n)$ for w_0.

$D(LRL-) =$

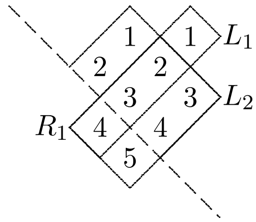

Figure 7: The central line in $D(LRL-)$.

Definition 6.1 Suppose P is a partial quiver and $D(P)$ is the diagram for P as defined above. The rectangles $\rho(Y)$ for Y a component of P divide $D(P)$ into smaller rectangles, which we shall call boxes. (In the example near the start of Section 5, there are 6 boxes). We regard these boxes as appearing in diagonal rows, from top left to bottom right As we work our way down the diagonal rows of boxes, starting from the top right of $D(P)$, the number of boxes in each diagonal is at first odd, and then even, or is at first even, and then odd. We draw a central line Z on $D(P)$ dividing these two sets of diagonal rows of boxes. In our example, the number of boxes in each row is 1,3,2, so the central line is indicated by the dashed line in Figure 7.

Let Y be a component of P, and define a set $S(Y)$ of positive roots as follows. The rectangle $\rho(Y)$ for Y is divided into two parts by the central line. Let $\rho^*(Y)$ be the part of $\rho(Y)$ which is labelled. The root α_{ij} lies in $S(Y)$ if and only if the intersection of a diagonal row of $D(P)$ (see Definition 5.2) with $\rho^*(Y)$ contains the numbers $i, i+1, \ldots, j-1$, in order, reading from top left to bottom right. Then $S(P)$ is defined to be the union of the sets $S(Y)$ for Y a component of P. In our example, $S(L_1) = \{\alpha_1, \alpha_2, \alpha_3\}$, $S(R_1) = \{\alpha_4 + \alpha_5\}$ and $S(L_2) = \{\alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3 + \alpha_4\}$, so $S(P) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4 + \alpha_5, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3 + \alpha_4\}$.

Let $v(P) \in \mathbb{N}^k$ be defined as follows. The reduced expression $\mathbf{j}$ defines an ordering on the positive roots; we write $\beta^l = s_{j_1} s_{j_2} \cdots s_{j_{l-1}}(\alpha_{j_l})$, for $l = 1, 2, \ldots, k$. Then let $v(P)_l$ be equal to 1 if $\beta^l \in S(P)$ and 0 otherwise. So in the example above, $v(P) = (0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 0, 1)$.

If $j \in [1, n]$, we define $S(j)$ by specifying that the root α_{ij} lies in $S(j)$ if and only if there is a diagonal row of $D(j)$ containing the numbers $i, i+1, \ldots, j-1$, in order, reading from top left to bottom right. We define $v(j)$ in the same way as $v(P)$ is defined using $S(P)$.

We know that, if $b \in \mathbf{B}$ and $i \in [1,n]$, then $\widetilde{F}_i b \equiv b' \pmod{vL'}$ for some unique $b' \in \mathbf{B}$. With respect to our fixed reduced expression $\mathbf{j}$, this defines an action on $\mathbb{N}^k$, defined by $\widetilde{F}_i(\varphi_{\mathbf{j}}(b)) = \varphi_{\mathbf{j}}(b')$. Using the identification in the last paragraph, this also induces an action on sets of positive roots, which we also denote by $\widetilde{F}_i$. Reineke, in [17, §1], describes the action of $\widetilde{F}_i$ on $\mathbb{N}^k$:

Proposition 6.2 *(Reineke)*

Suppose $\mathbf{v} = (v_1, v_2, \ldots, v_k) \in \mathbb{N}^k$. *Write, for* $1 \leq i < j \leq n+1$, $v_{ij} = v_l$, *where* $\beta^l = \alpha_{ij}$. *For such* $i < j$, *define*

$$f_{ij} = \sum_{l=j}^{n+1} v_{il} - \sum_{l=j+1}^{n+1} v_{i+1,l}.$$

Let j_0 *be minimal so that* $f_{ij_0} = \max_j f_{ij}$. *Then* $\widetilde{F}_i$ *increases* v_{ij_0} *by* 1, *decreases* v_{i+1,j_0} *by* 1 *(unless* $i+1 = j_0$, *when this latter effect does not occur), and leaves the other* v_{ij}*'s unchanged.*

Definition: Let $2 \leq a \leq b \leq n$, and let Q be a quiver. We define the (a,b)-*sub partial quiver* of Q to be the partial quiver P with $P \leq Q$, leftmost directed edge numbered b and rightmost directed edge numbered a.

As conjectured by Carter, we have:

Theorem 6.3 *Let* Q *be a quiver, with* $\mathbf{i} \in \chi_n$ *compatible with* Q. *Suppose that* $(c, \mathbf{i})$ *is a chamber of* Q, *with* $P = P(c, \mathbf{i})$. *Let* $\mathbf{j} = (n, n-1, n, n-2, n-1, n, \ldots, 1, 2, \ldots, n)$. *Let* b_P *be the canonical basis element arising in Theorem 4.4. Then the Lusztig parametrization* $c_P(\mathbf{j})$ *of* b_P *is given by* $v(P)$. *Similarly the Lusztig parametrization* $c_j(\mathbf{j})$ *of* b_j *is given by* $v(j)$.

Proof: We have to calculate $S_{\mathbf{i}}^{\mathbf{j}}(a(c,\mathbf{i}))$ or $S_{\mathbf{i}}^{\mathbf{j}}(a(j,\mathbf{i}))$, and thus we consider the corresponding monomial of root operators acting on 1, $\widetilde{F}(P) = \widetilde{F}_{i_1}^{a_1} \widetilde{F}_{i_2}^{a_2} \cdots \widetilde{F}_{i_k}^{a_k} \cdot 1$. Here $P = P(c, \mathbf{i})$ in the first case and j in the second. Theorem 5.3 gives a description of $\widetilde{F}(P)$ in terms of $D(P)$. Recall that $D(P)$ consists of a number of *diagonal rows* (see Definition 5.2), where diagonal row number 1 is the one which is at the top right of $D(P)$, diagonal row number 2 is below this (down and to the left), and so on. We now make a claim.

Let x be a position in $D(P)$ containing a number, i. Suppose x is in (diagonal) row a. Let r be the total number of rows in $D(P)$, and let $\widetilde{F}(s)$, $s = 1, 2, \ldots, r$, be the part of $\widetilde{F}(P)$ arising from the sth row. Thus $\widetilde{F}(P) = \widetilde{F}(r)\widetilde{F}(r-1)\cdots\widetilde{F}(1)\cdot 1$. For $s = 1, 2, \ldots, r$, let d_s be the number at the rightmost end of the sth row. Define $\widetilde{F}[x] := \widetilde{F}_i^{\lambda_i}\widetilde{F}_{i+1}^{\lambda_{i+1}}\cdots\widetilde{F}_{d_a}^{\lambda_{d_a}}\widetilde{F}(a-1)\cdots\widetilde{F}(2)\widetilde{F}(1)\cdot 1$, where, if j appears in row a, λ_j is the multiplicity associated to it.

So $\widetilde{F}[x]$ is a final segment of $\widetilde{F}(P)$, arising from the first $a-1$ rows of the diagram, together with the part of the ath row to the right of x, including x. Let $\mathbf{c} \in \mathbb{N}^k$ be defined by $\widetilde{F}[x] \equiv F_{\mathbf{i}}^{\mathbf{c}} \pmod{v\mathcal{L}'}$. Note that there is no problem working here modulo $v\mathcal{L}'$. We shall have $\widetilde{F}[x] \equiv b \pmod{v\mathcal{L}'}$ for a unique $b \in \mathbf{B}$ from Kashiwara's approach, and $b \equiv F_{\mathbf{i}}^{\mathbf{c}} \pmod{v\mathcal{L}'}$ for a unique $\mathbf{c} \in \mathbb{N}^k$ by Lusztig's approach and the comparison theorem [10, §2.3].

Let $S(x)$ be the corresponding multiset of positive roots, obtained by using the ordering on the set of positive roots induced by $\mathbf{j}$. We claim that $S(x)$ is given by the union of the following two sets: The first, $S_1(x)$ is defined to be those roots in $S(P)$ arising from rows $1, 2, \ldots, a-1$, together with those roots in $S(P)$ arising from row a of the form $\alpha_s + \alpha_{s+1} + \cdots + \alpha_t$ where $s \geq i$. The second, $S_2(x)$, is defined to be the set of roots of form $\alpha_i + \alpha_{i+1} + \cdots + \alpha_t$, where $\alpha_s + \alpha_{s+1} + \cdots + \alpha_t$ is a root in $S(P)$ arising from row a with $s < i$ (we call this procedure of passing from $\alpha_s + \alpha_{s+1} + \cdots + \alpha_t$ to $\alpha_i + \alpha_{i+1} + \cdots + \alpha_t$, 'cutting off').

We will prove the claim by induction on x. Let us assume that for a given x in $D(P)$, $S(x)$ is as described. Let $\mathbf{c}$ be as above, corresponding to x. Let y be one position to the left of x in the ath row of $D(P)$. We understand this to mean the rightmost element of the $(a+1)$st row of $D(P)$ if x is the leftmost element of the ath row. In this case for the purposes of the argument below we actually take x to be in the same row as y, immediately down and to the right of y, with corresponding (fictitious) label 1 more than that of y (i.e. actually just outside the diagram $D(P)$). Note that in this case, as we are assuming the claim to be true for x, we have $S_2(x) = \phi$. Suppose there is an i in position y. Then there is an $i+1$ in position x. Suppose that x and y both lie in the ath row of $D(P)$. We will show that the claim is also true for y. The base case, when x is taken to be one position to the right of (and below) the rightmost position of the first row, with $S(x) = \phi$, is trivially true. The Theorem will follow from the induction argument, as it is the claim for the case when x is the leftmost position of the last row of the diagram.

We must show that the set of positive roots (defined by $\mathbf{j}$) corresponding to $\mathbf{c}'$, where $\widetilde{F}[y] = \widetilde{F}_i^{\lambda_i}\widetilde{F}[x] \equiv F_{\mathbf{i}}^{\mathbf{c}'} \pmod{v\mathcal{L}'}$ is given by $S_1(y) \cup S_2(y)$, where i is the number labelling position y. In the case where y is the rightmost element of its row (numbered a), we take $\widetilde{F}[x]$ to be $\widetilde{F}(a-1)\cdots\widetilde{F}(2)\widetilde{F}(1)\cdot 1$. We define $\mathbf{c} \in \mathbb{N}^k$ by writing $c_{ij} = 1$ if $\alpha_{ij} \in S(x)$ and $c_{ij} = 0$ otherwise. We start with an important

Lemma 6.4 *If $l > i+1$ then we cannot have both $c_{il} = 1$ and $c_{i+1,l} = 1$.*

Proof: (of Lemma). First of all note that if $P = j \in [1,n]$, the Lemma is obvious, following easily from the structure of $D(P)$. So we can assume that P is a partial quiver. Suppose, for a contradiction, that $c_{il} = c_{i+1,l} = 1$. So $\beta_1 := \alpha_i + \alpha_{i+1} + \cdots + \alpha_{l-1}$ and $\beta_2 := \alpha_{i+1} + \alpha_{i+2} + \cdots + \alpha_{l-1}$ both lie in $S(x)$. We are assuming the claim to be true for x, so $S(x) = S_1(x) \cup S_2(x)$. We shall see that the structure of $D(P)$ leads to a contradiction; to do this we consider various cases:

Case (a) For $i = 1,2$, if $\beta_i \in S_2(x)$, let γ_i be the root in $S(P)$ which is 'cut off' to give β_i. Otherwise, let $\gamma_i = \beta_i$. Suppose that γ_1, γ_2 both come a rectangle for a component Y of P of type L which is the (a_1, b_1)-sub partial quiver of Q. The set of positive roots in $S(P)$ arising from Y is $\{\alpha_{1,n-b_1}, \alpha_{2,n+1-b_1}, \ldots, \alpha_{s,n+s-1+b_1}\}$ (where the central line passes through the rectangle corresponding to Y immediately below the row with leftmost number s). If $\alpha = \alpha_{pq}$ is any positive root, write $r(\alpha) = q-1$ and $l(\alpha) = p$. For each γ in this list, $r(\gamma)$ is distinct, and since 'cutting off' a root γ does not affect $r(\gamma)$, we conclude that this case cannot occur, since $r(\beta_1) = r(\beta_2)$.

Case (b) Suppose that β_1, β_2 both come from a component of P of type R, as in (a). We argue as in case (a) to see that this case cannot occur.

Case (c) Suppose that β_1 and β_2 arise from distinct components of P and that $\beta_1, \beta_2 \in S_1(x)$. Let h be the height of the roots in $S(P)$ arising from a component Y of P. Then the height of the roots in $S(P)$ arising from the component of P immediately to the right of Y (if such exists) is $h+t$, where t is the number of directed edges in Y. Thus, since $\mathrm{ht}(\beta_1) = 1 + \mathrm{ht}(\beta_2)$ (ht denotes height), for this case to occur, we must be in the situation where we have two components Y and Z of P, where Z immediately follows Y, and Y has precisely one directed edge. We assume that Y is of type L and Z is of type R (the argument in the other case is very similar). Suppose that Z is the (a_1, b_1)-sub partial quiver of Q. The two rectangles $\rho(Y)$ and $\rho(Z)$ will appear in $D(P)$ as in Figure 8, where as usual, the dashed line indicates the central line in $D(P)$. Then the roots for Y will be taken from above this line, and the roots for Z from below, such roots α will have differing values of $r(\alpha)$ so we see that this

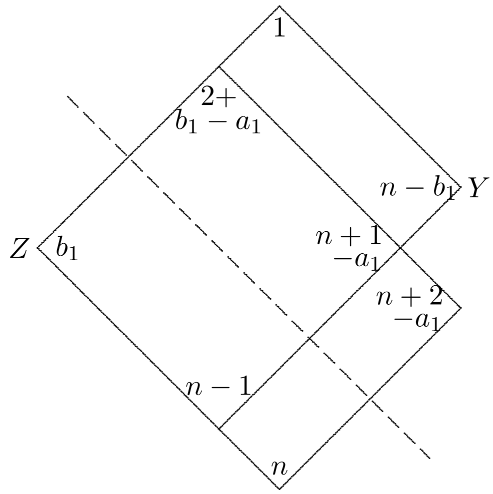

Figure 8: The rectangles $\rho(Y)$ and $\rho(Z)$.

case cannot occur.

Case (d) Suppose we have $\beta_1, \beta_2 \in S_2(x)$. Suppose first that x lies above the central line. We know that the lines in $D(P)$ which are the bottom right edges of rectangles $\rho(Y)$ where Y is a component of P of type L are all distinct and parallel. It follows that $r(\beta_1) \neq r(\beta_2)$, a contradiction. A similar argument holds if x lies below the central line.

Case (e) Suppose that $\beta_1 \in S_1(x)$ and $\beta_2 \in S_2(x)$. If β_1 arises from the ath row of $D(P)$, we can argue as in (d) (we will have again $r(\beta_1) \neq r(\beta_2)$, a contradiction). So assume that β_1 arises from the bth row, where $1 \leq b < a$. We know that β_2 arises via cutting off from a root $\gamma_2 \in S(P)$ which comes from the ath row of $D(P)$. Since $\mathrm{ht}(\beta_2) < \mathrm{ht}(\gamma_2)$ and $\mathrm{ht}(\beta_1) = 1 + \mathrm{ht}(\beta_2)$, we have $\mathrm{ht}(\beta_1) < \mathrm{ht}(\gamma_2)$. Therefore, arguing as in (c), we see that β_1 comes from a component X_1 occurring to the left of that giving rise to γ_2 (and thus β_2). We call the latter X_2. Suppose that X_1 does not occur immediately to the left of X_2, or that X_2 is of type R. Then by the structure of $D(P)$, the bottom right lines T_1, T_2 delimiting the rectangles $\rho(X_1)$ and $\rho(X_2)$ are distinct. Indeed, the line for $\rho(X_2)$ occurs below and to the right of $\rho(X_1)$.

We know that there is an $l-1$ in the intersection of the bth row with $\rho(X_1)$, immediately to the left and above the line T_1. Similarly, there is an $l-1$ in the intersection of the ath row with $\rho(X_2)$, immediately to the left and above the line T_2, by the construction of β_1, β_2. The relative position of lines T_1 and T_2 tells us that the former $l-1$ lies strictly to the left of the latter, while the fact that

$b < a$ tells us that the latter $l-1$ lies strictly to the left of the former, a contradiction. We are left with the case when X_1 immediately precedes X_2 and X_2 is of type L. In this case T_1 and T_2 coincide along part of their lengths. Now X_1 is of type R, so β_1 arises from a row below the central line, and X_2 is of type L, so β_2 arises from a row above the central line. It follows that $r(\beta_1) \neq r(\beta_2)$, a contradiction.
Case (f) Suppose that $\beta_1 \in S_2(x)$, $\beta_2 \in S_1(x)$, and β_1 and β_2 arise from different components. If β_2 arises from the ath row, we can argue as in (d). So assume β arises from the bth row, where $1 \leq b < a$. Then β_1 arises via cutting off from a root $\gamma_1 \in S(P)$ which comes from the ath row. Since $\mathrm{ht}(\beta_1) < \mathrm{ht}(\gamma_1)$ and $\mathrm{ht}(\beta_1) = 1 + \mathrm{ht}(\beta_2)$, we have $\mathrm{ht}(\beta_2) < \mathrm{ht}(\gamma_1)$, and we can argue as in (e).

We have now covered all possible cases for $\beta_1, \beta_2 \in S(x)$, and we see that the Lemma is proved. □

We now investigate further the structure of $\mathbf{c}$. By assumption, for each j, k, c_{jk} is equal to 0 or 1. The above shows that we cannot have both $c_{il} = c_{i+1,l} = 1$ for $l > i+1$. We shall now pin down further the possible values for c_{il} and $c_{i+1,l}$. We know that, for each $l > i+1$, $(c_{il}, c_{i+1,l}) = (0,0)$, $(0,1)$ or $(1,0)$. If we let $c_{i+1,i+1} := 1 - c_{i,i+1}$, then this is also true for $l = i+1$. For $m = 0, 1, 2, \ldots n-i$, write $a_m = c_{i,i+m+1}$ and $b_m = c_{i+1,i+m+1}$, so the pairs we are interested in are of form $\mathbf{x}_m = (a_m, b_m)$, $m = 0, 1, 2, \ldots n-i$. We first need a Lemma describing the action of $\widetilde{F}_i$ in terms of these pairs, using Proposition 6.2:

Lemma 6.5 *Suppose* $\mathbf{c}$ *is as in the above, and we have written the pairs* $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{n-i}$ *as above, with each* $\mathbf{x}_m$ *equal to* $(0,0)$*,* $(0,1)$ *or* $(1,0)$*. Rewrite this sequence, replacing pairs of these types with* 0*, '*$-$*' and '*$+$*', respectively. Cross out all* 0*'s, except initial* 0*'s. Cross out all non-initial pairs of symbols of form* $+-$*, and repeat until there are no such pairs of symbols in the sequence left. Use the following table of possible sequences to determine a symbol in the resulting sequence of symbols. This is defined to be the symbol that is indicated by a line above it. Every time a sequence of symbols appears in the list, as in* $-, -, \ldots, -$*, this means that the symbol concerned must occur at least once.*

(i) $0, -, -, \ldots, \overline{-}, +, +, \ldots, +,$
(ii) $0, -, -, \ldots, \overline{-},$
(iii) $\overline{0}, +, +, \ldots, +,$
(iv) $\overline{0},$
(v) $(+, -), -, -, \ldots, \overline{-}, +, +, \ldots, +,$

(vi) $(+,-),-,-,\ldots,\overline{-},$

(vii) $(\overline{+},-),+,+,\ldots,+,$

(viii) $\overline{+},-,$

(ix) $-,-,\ldots,\overline{-},+,+,\ldots,+,$

(x) $-,-,\ldots,\overline{-},$

(xi) $\overline{+},+,\ldots,+.$

In this way, a symbol in the original sequence is determined, and therefore an $\mathbf{x}_m$*. Then acting* $\widetilde{F}_i$ *has the following effect on* $\mathbf{c}$*. If* $\mathbf{x}_m = (0,1)$ *(i.e. a '−'), then* $c_{i+1,i+m+1} = 1$*. This is changed to zero (having no effect if* $m = 0$*), and* $c_{i,i+m+1}$ *is changed from zero to* 1*, thus changing the corresponding symbol in the sequence from a '−' to a '+'. If* $\mathbf{x}_m = (1,0)$ *or* $(0,0)$*, then we must have* $m = 0$ *(as no initial symbols are crossed out), and* $\widetilde{F}_i$ *has the effect of increasing* $c_{i,i+1}$ *by* 1*.*

Proof: We consider how the f_{ij}'s are related to the $\mathbf{x}_p$'s (see Proposition 6.2 for the definition of the f_{ij}'s). For each $0 \leq p \leq n-i$, write $\delta_p := f_{i,i+p+2} - f_{i,i+p+1}$. A simple calculation shows us that $\delta_p = 1$ if $(\mathbf{x}_p, \mathbf{x}_{p+1}) = (0,-)$ or $(-,-)$, that $\delta_p = -1$ if $(\mathbf{x}_p, \mathbf{x}_{p+1}) = (+,+)$ or $(+,0)$, and is zero in all other cases. Thus the sequence in the Lemma enables us to calculate the value of the f_{ij}'s, given the value of $f_{i,i+1}$ (for our fixed i). Recall that Proposition 6.2 tells us that, in order to see how $\widetilde{F}_i$ acts, we need to calculate the smallest j for which f_{ij} is maximal; write j_0 for this value of j. It is easy to check that the crossing out routines described above never cross out the symbol corresponding to $\mathbf{x}_p$ where $i+p+1 = j_0$. Thus recalculating this *after* doing the crossing out does not affect the final result. After crossing out is done, the only possible sequences are those as listed in the Lemma. It is then easy to check that j_0 is given by $j_0 = i+p+1$ where p is given by the rules in the Lemma in each case (i)-(xi). □

Example: To illustrate this Lemma we go back to our example (see Definition 5.1 and Definition 6.1). So $P = LRL-$ and $D(P)$ is as in Figure 7. Suppose x is the position of the rightmost 4 in $D(P)$, the last element in the third diagonal row of $D(P)$. Then y is the leftmost 3 in $D(P)$, the second element in the third diagonal row, so $i = 3$. The set $S_1(x)$ is by assumption $\{\alpha_1, \alpha_2, \alpha_1+\alpha_2+\alpha_3\}$, and $S_2(x)$ is $\{\alpha_4\}$. Thus $S(x)$ is the union of these two sets and $\mathbf{c} = (0,0,1,0,0,0,0,0,0,1,0,0,1,0,1)$. We can write $\mathbf{c}$ as in Figure 9.

$$
\begin{array}{ccccccccc}
 & & & & c_{16} & & & & \\
 & & & c_{15} & & c_{26} & & & \\
 & & c_{14} & & c_{25} & & c_{36} & & \\
 & c_{13} & & c_{24} & & c_{35} & & c_{46} & \\
c_{12} & & c_{23} & & c_{34} & & c_{45} & & c_{56}
\end{array}
\quad = \quad
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 1 & & 0 & & \underline{0} & & \\
 & 0 & & 0 & & \underline{0} & & \underline{0} & \\
1 & & 1 & & \underline{0} & & \underline{1} & & 0
\end{array}
$$

Figure 9: The vector $\mathbf{c}$.

We have $\mathbf{x}_0 = (c_{34}, c_{44}) = (0, 1)$, $\mathbf{x}_1 = (c_{35}, c_{45}) = (0, 1)$, and $\mathbf{x}_2 = (c_{36}, c_{46}) = (0, 0)$ — these are indicated by underlining in Figure 9 (note that c_{44} does not appear in the diagram). Thus the sequence of symbols as in Lemma 6.5 is $(-, -, 0)$. In the reduction step, we remove the final zero to get $(-, -)$ and see we are in case (x) of the Lemma, and that the final '$-$' is marked. This corresponds to $\mathbf{x}_1$ and the Lemma tells us that applying $\widetilde{F}_i$ changes c_{35} from 0 to 1 and c_{45} from 1 to 0. As a more complicated example, consider the sequence of symbols $(0, 0, +, +, -, -, +, +, +, 0, -, 0, -, -, +, -, -, -, -, 0, 0)$. We first of all remove non-initial zeros to get $(0, +, +, -, -, +, +, +, -, -, -, +, -, -, -, -)$ and then remove $+, -$ pairs to get $(0, -, -, -)$ — which is case (ii) of the Lemma. Thus the last '$-$' is indicated, corresponding to the last '$-$' of the original sequence.

We now begin the investigation as promised.

Lemma 6.6 *Suppose that $\mathbf{x}_j = (1, 0)$. Then $c_{i,i+j+1} = 1$, so that $\alpha_{i,i+j+1} \in S(x)$. Suppose that in fact $\alpha_{i,i+j+1} \in S_1(x)$. Then there is $m > j$ so that $\mathbf{x}_{j+1} = \mathbf{x}_{j+2} = \cdots = \mathbf{x}_{m-1} = (0, 0)$ and $\mathbf{x}_m = (0, 1)$. Thus $c_{i+1,i+m+1} = 1$, so $\alpha_{i+1,i+m+1} \in S(x)$; in fact $\alpha_{i+1,i+m+1} \in S_1(x)$.*

Proof: Let $l = i + j + 2$; we know that $\alpha_i + \alpha_{i+1} + \cdots + \alpha_l \in S_1(x)$. So $\alpha_i + \alpha_{i+1} + \cdots + \alpha_l \in S(P)$ and arises from the intersection of row b of $D(P)$ with a rectangle ρ, where $b < a$ (since α_i appears in it). Thus if $P = j \in [1, n]$, we have a contradiction by the structure of $D(P)$ (which consists of only one rectangle); i can appear at the start of one row in $D(P)$ only. So in this case the circumstances of the Lemma cannot occur. Therefore we can assume P is a partial quiver and we define X to be the component of P corresponding to ρ. Suppose that row b is not immediately above the central line in $D(P)$. Then row $b + 1$ of $D(P)$ intersects ρ, the rectangle for X. This is because i occurs in row a, in position y, where $a > b$. If row b were the bottom row in the rectangle for X, i would appear in the left hand corner of this rectangle, and therefore in no later row in $D(P)$ (by the structure of

$D(P)$), a contradiction. Thus the intersection of row $b+1$ with the rectangle for X gives a root $\alpha_{i+1}+\alpha_{i+2}+\cdots+\alpha_{l+1}\in S(P)$. Since $b+1\leq a$ and the label at position x in $D(P)$ is $i+1$, this root lies in $S_1(x)$ (i.e. is not cut off), as required (with $m=j+1$).

So we are left in the case where row b does lie immediately above the central line of $D(P)$. Then X must be of type L. Suppose first that X is the rightmost component of P. Then i appears in the leftmost column (where a column is perpendicular to what we are calling a row) of $D(P)$, and also is immediately above the central line. The central line coincides with the bottom left delimiting line of the rectangle for X. Consider now the rectangle for the component Y immediately to the left of X. This has bottom right delimiting line parallel to that for X, but length (from top left to bottom right) less than that for X. Thus its upper left delimiting line is down and to the right of that for X, and i is in the left hand corner of the rectangle for X. Then all of the upper left delimiting lines for components other than X appear down and to the right of that for X, so i cannot appear below the central line. But as $b<a$, x is below the central line, and therefore so is y, which is labelled i, and we have a contradiction. Thus there is a component immediately to the right of X in P; we call this Y. Let $\alpha_{i+1}+\alpha_{i+2}+\cdots+\alpha_t$ (some $t>l+1$) in $S(P)$ be the root arising from the intersection of the $(b+1)$st row of $D(P)$ with the rectangle for Y. Then as before $\alpha_{i+1}+\alpha_{i+2}+\cdots+\alpha_t\in S_1(x)$. Put $m=t-i$, and we have $(a_m,b_m)=(c_{i,i+m+1},c_{i+1,i+m+1})=(0,1)$, as required. We now have to show that $\mathbf{x}_{j+1}=\mathbf{x}_{j+2}=\cdots=\mathbf{x}_{m-1}=(0,0)$. That is, we have to show that
(a) $\alpha_i+\alpha_{i+1}+\cdots+\alpha_u\notin S(x)$ for $l+1\leq u<t$, and
(b) $\alpha_{i+1}+\alpha_{i+2}+\cdots+\alpha_u\notin S(x)$ for $l+1\leq u<t$.

We start with (a), and take $l+1\leq u<t$. Since all roots $\beta\in S_2(x)$ satisfy $l(\beta)=i+1$, we have $\alpha_i+\alpha_{i+1}+\cdots+\alpha_u\notin S_2(x)$. Therefore, it is enough to show that $\alpha_i+\alpha_{i+1}+\cdots+\alpha_u\notin S(P)$. But $\mathrm{ht}(\alpha_i+\alpha_{i+1}+\cdots+\alpha_u)$ is strictly between the heights of the roots in $S(P)$ arising from X and Y, and so by our description of the heights of the roots in $S(P)$ in case (c) of the proof of Lemma 6.4, we see that $\alpha_i+\alpha_{i+1}+\cdots+\alpha_u\notin S(P)$ and (a) holds.

For (b), we consider again $l+1\leq u<t$. If $u\neq l+1$, the argument used for (a) shows that $\alpha_{i+1}+\alpha_{i+2}+\cdots+\alpha_u\notin S(P)$. But if $u=l+1$, then this root has the same height as roots in $S(P)$ arising from X. So if it were in $S(P)$, it would have arisen from the intersection of the rectangle for X with row $b+1$ of $D(P)$. But this row is below the central line of $D(P)$, whence

$\alpha_{i+1}+\alpha_{i+2}+\cdots+\alpha_{l+1}$ does not arise in this way, and so does not lie in $S(P)$. Next, suppose that we had $\alpha_{i+1}+\alpha_{i+2}+\cdots+\alpha_u \in S_2(x)$. Then, since x must be below the central line, u must be immediately to the upper left of the bottom right line delimiting a rectangle corresponding to a component Z of P of type R. Since $\alpha_{i+1}+\alpha_{i+2}+\cdots+\alpha_u$ is assumed to be in $S_2(x)$, it must be the 'cut off' from a root $\gamma = \alpha_p + \alpha_{p+1} + \cdots + \alpha_u \in S(P)$, $p \leq i$, where γ arises from the rectangle corresponding to Z. Since $\mathrm{ht}(\gamma) > \mathrm{ht}(\alpha_{i+1}+\alpha_{i+2}+\cdots+\alpha_{l+1})$, Z must lie strictly to the right of X in P. We know that $\alpha_{i+1}+\alpha_{i+2}+\cdots+\alpha_t$ corresponds to the first line of the rectangle for Y below the central line in $D(P)$, so Z is strictly to the right of Y, as $u < t$. Thus the bottom right delimiting line for Z is strictly down to the right of that for Y. Thus this u lies to the right of the u corresponding to the α_u in $\alpha_{i+1}+\alpha_{i+2}+\cdots+\alpha_t$ (in the $(b+1)$st row of $D(P)$), and this u lies below the central line. This is impossible, as all u's in $D(P)$ to the right of the u in the $(b+1)$st row (which is immediately below the central line), lie *above* the central line. Thus $\alpha_{i+1}+\alpha_{i+2}+\cdots+\alpha_u \notin S_2(x)$ and so it is not in S for $l+1 \leq u < t$. So we see (b) also holds, and the Lemma is proved. □

Remark Note that all roots in $S_2(x)$ have a corresponding $\mathbf{x}_j$ of form $(0,1)$. All roots in $S_1(x)$ arising from the ath row of $D(P)$ will be of form $\alpha_p+\alpha_{p+1}+\cdots+\alpha_l$, for $l \geq p \geq i+1$, so only if $p = i+1$ will they contribute to an $\mathbf{x}_j$; in this case we shall get a corresponding $\mathbf{x}_j$ of form $(0,1)$.

We can now apply Lemma 6.5 in our situation. By Lemma 6.5, after the crossing out process, our sequence looks like $((+,-),-,-,\ldots,-)$, $((0),-,-,\ldots,-)$ or $((-),-,\ldots,-)$. By Lemma 6.5, applying $\widetilde{F}_i$ will change the last minus to a plus, and repeated applications will change the final λ_i minuses to pluses. By the proof of Lemma 6.6, it is clear that all of the $-$'s in the sequence, except for those in brackets, correspond to precisely those $\mathbf{x}_p$'s which correspond to roots of form $\alpha_{i+1}+\alpha_{i+2}+\cdots+\alpha_l \in S_2(x)$. Suppose first that there is no edge of a rectangle between x and y. By structure of $D(P)$, λ_i is the number of roots in $S_2(x)$. Applying $\widetilde{F}_i$ λ_i times thus replaces each root $\alpha_{i+1}+\alpha_{i+2}+\cdots+\alpha_l \in S_2(x)$ by $\alpha_i+\alpha_{i+1}+\cdots+\alpha_l$ (see Lemma 6.5). Next, suppose that there is an edge which is the bottom right edge of a rectangle between x and y. By the structure of $D(P)$, λ_i is 1 more than the number of roots in $S_2(x)$. Applying $\widetilde{F}_i$ $\lambda_i - 1$ times thus replaces each root $\alpha_{i+1}+\alpha_{i+2}+\cdots+\alpha_l \in S_2(x)$ by $\alpha_i+\alpha_{i+1}+\cdots+\alpha_l$ (see Lemma 6.5). We must have $c_{i,i+1} = 0$, so that the initial symbol in our sequence is a '$-$', and we must have a sequence of form $(-),-,-,\ldots,-$. From this we can conclude that applying $\widetilde{F}_i$ one more time changes this initial '$-$' (in brackets) into a '$+$', introducing the root α_i into $S(x)$. Finally, suppose there is an edge which is the top left edge of a rectangle between x and y. By the structure of $D(P)$, λ_i is 1 less than the number of roots in $S_2(x)$. Applying $\widetilde{F}_i$ λ_i times thus replaces each root $\alpha_{i+1} + \alpha_{i+2} + \cdots + \alpha_l \in S_2(x)$ by $\alpha_i + \alpha_{i+1} + \cdots + \alpha_l$ (see Lemma 6.5), except for the one with minimal value of l. Thus we see that in each case, applying $\widetilde{F}_i$ λ_i times replaces $S(x)$ with $S(y)$, proving that our claim is also true for y, and we see it is true for all points in the diagram. It follows that Theorem 6.3 is true. □

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