The Lusztig cones of a quantized enveloping algebra of type A

Robert Marsh

Department of Mathematics and Computer Science, University of Leicester, University Road, Leicester LE1 7RH, England
E-mail: R.Marsh@mcs.le.ac.uk
Fax: +44 (0) 116 252 3915

Abstract

We show that for each reduced expression for the longest word in the Weyl group of type A_n, the corresponding cone arising in Lusztig's description of the canonical basis in terms of tight monomials is simplicial, and construct explicit spanning vectors.
Keywords: quantum group, canonical basis, Lusztig cone, longest word.

1 Introduction

Let $U = U_q(sl_{n+1}(\mathbb{C}))$ be the quantum group associated to the simple Lie algebra $\mathfrak{g} = sl_{n+1}(\mathbb{C})$ of type A_n. The negative part U^- of U has a canonical basis with favourable properties (see Kashiwara [6] and Lusztig [9, §14.4.6]). For example, via action on highest weight vectors it gives rise to bases for all the finite-dimensional irreducible highest weight U-modules.

The Lusztig cones first appeared in a paper [10] of Lusztig, where he proved that, if $\mathbf{i} = (i_1, i_2, \ldots, i_k)$ is a reduced expression for the longest word w_0 in the Weyl group of $\mathfrak{g}$ (i.e. $s_{i_1} s_{i_2} \cdots s_{i_k}$ is reduced), and $\mathbf{a} = (a_1, a_2, \ldots, a_k)$ satisfies certain linear inequalities, then the monomial

$$F_{i_1}^{(a_1)} F_{i_2}^{(a_2)} \cdots F_{i_k}^{(a_k)}$$

lies in the canonical basis of U^-, provided $\mathfrak{g}$ is of type A_1, A_2 or A_3. We call the set of points associated to $\mathbf{i}$ satisfying these linear equalities the *Lusztig cone* associated to $\mathbf{i}$. This result was extended to type A_4 in [12], but has been shown to fail for larger n by Reineke [16] and Xi [17]. However, in the crystal limit of Kashiwara, this cone plays an important role. This role will be explored in [13], and the joint paper [5] with Roger Carter, where a link is made with the regions of linearity of Lusztig's reparametrization functions. Indeed, it may be that the Lusztig cones are regions of linearity for reparametrization functions arising from a change of reduced expression for w_0 in the string parametrization of the canonical basis.

The Lusztig cones have also arisen in other contexts. For example, let $G = SL_n(\mathbb{C})$, and let $x_1, x_2, \ldots, x_n$ be a choice of one-parameter simple root subgroups of G. Zelevinsky [18] shows that a generic element $x \in L^{e,w_0}$, a reduced real double Bruhat cell of G, has form $x = x_{i_1}(t_1)x_{i_2}(t_2)\cdots x_{i_k}(t_k)$, where $\mathbf{i}$ is any fixed reduced expression for w_0. The parameters $t_1, t_2, \ldots, t_k$ can be regarded as functions on L^{e,w_0}, and Zelevinsky shows that the Laurent monomial $t_1^{a_1}t_2^{a_2}\cdots t_k^{a_k}$ is regular if and only if $\mathbf{a} = (a_1, a_2, \ldots, a_k)$ satisfies the defining inequalities of the Lusztig cone corresponding to $\mathbf{i}$. Lusztig cones also play a role in the paper [14] investigating the string cones — cones which arise from the string parametrization of the canonical basis of U^- (which also depends on a choice of reduced expression $\mathbf{i}$ for w_0).

In this paper, we investigate the Lusztig cones and show that they possess a beautiful combinatorial structure. We prove that each Lusztig cone is the set of integral points of a simplicial cone and give an explicit description of spanning vectors for it, in terms of the corresponding reduced expression for w_0. Provided we label points in a Lusztig cone in the correct way, in this description there are n spanning vectors common to each Lusztig cone. The other vectors correspond to the bounded chambers of the 'chamber ansatz' for the corresponding reduced expression (see [2, §§1.4, 2.3]), and each vector depends only on the set of braids which pass below the corresponding chamber.

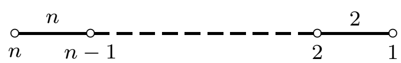
We give an explicit description of these spanning vectors, for an arbitrary reduced expression for w_0, using a relabelling of the bounded chambers with 'partial quivers' — certain graphs with a mixture of oriented and unoriented edges such that the subgraph of oriented edges is connected. These allow us to give a natural description of the spanning vectors (Theorem 3.8), and also enable us to use the quiver-compatible reduced expressions for w_0 as a way of approaching all reduced expressions.

In the case where $\mathbf{i}$ is quiver-compatible, for simply-laced type, these vectors have now also been studied by Bedard in [1]. Bedard describes these vectors using the Auslander-Reiten quiver of the quiver and homological algebra, showing they are closely connected to the representation theory of the quiver.

The description of the Lusztig cones given here should be of use in understanding the canonical basis. It is known that the Lusztig cone is contained in the corresponding string cone — this is shown by Premat [15] (a proof which holds for all symmetrizable Kac-Moody Lie algebras) and also in [13] for type A_n using the results derived here. This means that there is a subset of the canonical basis corresponding to each Lusztig cone and therefore a canonical basis element corresponding to each spanning vector described here. In types A_1, A_2 and A_3, the corresponding dual canonical basis element is known to be primitive — that is, it cannot be factorized as a product of dual canonical basis elements (this can be seen using [3]). These primitive elements, and whether they quasi-commute, play a key role in current understanding of the dual canonical basis (see [3], [4] and [11]), and also in the structure of the module categories of affine Hecke algebras (see [7]).

2 Lusztig cones are simplicial

For positive integers $p < q$ we denote by $[p, q]$ the set $\{p, p+1, \ldots, q\}$, and for a rational number x we denote by $\lceil x \rceil$ the smallest element of $\{y \in \mathbb{Z} : x \leq y\}$. Let $\mathfrak{g}$ be the simple Lie algebra $sl_{n+1}(\mathbb{C})$, with root system Φ, and simple roots $\alpha_1, \alpha_2, \ldots, \alpha_n$. We use the following numbering of the Dynkin diagram (and its edges):



Let W be the Weyl group of $\mathfrak{g}$. Recall that W can be presented with generators $s_1, s_2, \ldots, s_n$, subject to relations:

$$s_i^2 = e, \tag{1}$$

$$s_i s_j = s_j s_i \quad \text{if } |i-j| > 1, \tag{2}$$

$$s_i s_j s_i = s_j s_i s_j \quad \text{if } |i-j| = 1. \tag{3}$$

We recall also that relations of type (2) are called *short braid relations*, and relations of type (3) are called *long braid relations*. Two reduced expressions w_0 are said to be *commutation equivalent* if there exists a sequence of short braid relations taking the first to the second; the equivalence classes of this equivalence relation are called *commutation classes*. Let w_0 be the longest element in W. We shall identify a reduced expression for $s_{i_1} s_{i_2} \cdots s_{i_k}$ for w_0 with the corresponding k-tuple $\mathbf{i} = (i_1, i_2, \ldots, i_k)$. The *Lusztig cone*, $C_{\mathbf{i}}$, corresponding to $\mathbf{i}$ is defined to be the set of points $\mathbf{a} \in \mathbb{N}^k$ satisfying the following condition:
For every pair $i_s, i_{s'}$, with $s, s' \in [1, k]$, $i_s = i_{s'} = i$ and such that $i_p \neq i$ whenever $s < p < s'$, we have

$$(\sum_p a_p) - a_s - a_{s'} \geq 0, \tag{4}$$

where the sum is over all p with $s < p < s'$ such that i_p is joined to i in the Dynkin diagram.

In this section, we show that the Lusztig cone corresponding to any reduced expression for the longest word w_0 is simplicial. We can regard $C_{\mathbf{i}}$ as the subset of $\mathbb{Z}^k$ defined by the $k - n$

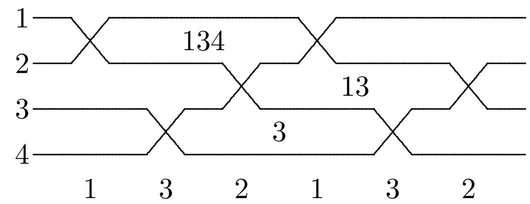


Figure 1: The Chamber Ansatz of $(1, 3, 2, 1, 3, 2)$.

inequalities (4) above, together with the k inequalities $a_j \geq 0$ for $j = 1, 2, \dots, k$. We will show that only n of the inequalities $a_j \geq 0$ are necessary to define $C_{\mathbf{i}}$.

The reduced expression $\mathbf{i}$ defines an ordering on the set Φ^+ of positive roots of the root system associated to W. If we write $\alpha^j = s_{i_1} s_{i_2} \cdots s_{i_{j-1}}(\alpha_{i_j})$ for $j = 1, 2, \dots, k$, then $\Phi^+ = \{\alpha^1, \alpha^2, \dots \alpha^k\}$ (and there are no repetitions in this list). For $\mathbf{a} = (a_1, a_2, \dots, a_k) \in \mathbb{Z}^k$, write $a_{\alpha^j} = a_j$. We shall use this relabelling of $\mathbf{a}$ throughout.

We shall need the chamber ansatz for $\mathbf{i}$ defined in [2, §§1.4, 2.3]. We take $n+1$ strings, numbered from top to bottom, and write $\mathbf{i}$ from left to right along the bottom of the diagram. Above a letter i_j in $\mathbf{i}$, the i_jth and $(i_j + 1)$st strings from the top above i_j cross. Thus, for example, in the case $n = 3$ with $\mathbf{i} = (1, 3, 2, 1, 3, 2)$, the chamber ansatz is shown in Figure 1. We denote the chamber ansatz of $\mathbf{i}$ by $\mathrm{CD}(\mathbf{i})$. A *chamber* will be regarded as a pair $(c, \mathbf{i})$, where c is a component of the complement of $\mathrm{CD}(\mathbf{i})$. We shall deal only with bounded chambers, so in the sequel, 'chamber' will always mean bounded chamber. Each chamber $(c, \mathbf{i})$ can be labelled with the numbers of the strings passing below it, denoted $l(c, \mathbf{i})$. Following [2], we call such a label a chamber set. For example, the chamber sets corresponding to the 3 chambers in Figure 1 are $134, 3, 13$. Note that the set of chamber sets of $\mathbf{i}$ is independent of its commutation class (so we can talk also of the chamber sets of such a commutation class). Every non-empty subset of $[1, n+1]$ can arise as a chamber set for some $\mathbf{i}$, except for subsets of the form $[1, j]$ and $[j, n+1]$ for $1 \leq j \leq n+1$; this is observed in the proof of [2, Theorem 2.7.1].

Since the crossings of strings correspond to letters in $\mathbf{i}$, and the horizontal height of a crossing is the value of the corresponding letter, the chambers $(c, \mathbf{i})$ correspond precisely to pairs of equal letters in $\mathbf{i}$ which have no further occurrence of that letter between them. We call such a pair *minimal*. Next, label the crossing point of strings i and j with the pair (i, j). If we identify (i, j) with the positive root $\alpha_{ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$ then the labels of the crossings will be precisely the positive roots in the ordering defined (above) by $\mathbf{i}$. We will write a_{ij} for $a_{\alpha_{ij}}$.

The key to understanding $C_{\mathbf{i}}$ is the following Lemma.

Lemma 2.1 *Suppose $\mathbf{a} \in C_{\mathbf{i}}$, and that α, β and $\alpha+\beta$ are all positive roots. Then $a_{\alpha+\beta} \geq a_\alpha + a_\beta$.*

Proof: We note that each chamber $(c, \mathbf{i})$ determines a minimal pair in $\mathbf{i}$ and thus a defining inequality of $C_{\mathbf{i}}$. Such an inequality can be written

$$(\sum a_\gamma) - a_{\gamma_1} - a_{\gamma_2} \geq 0$$

where γ_1 and γ_2 are the labels of the crossings at the left and right hand ends of c, respectively, and the sum is over the positive roots γ labelling the other crossings on the boundary of $(c, \mathbf{i})$.

Suppose that $\alpha = \alpha_{ij}$, and $\beta = \alpha_{jk}$ for some $1 \leq i < j < k \leq n+1$, so that $\alpha + \beta = \alpha_{ik}$. By inspection, it can be seen that the sum of the defining inequalities of $C_{\mathbf{i}}$ corresponding to chambers inside the triangle formed by the pseudolines i, j and k gives precisely the inequality $a_{ik} \geq a_{ij} + a_{jk}$. $\square$

It follows from Lemma 2.1 that $C_{\mathbf{i}}$ can be described as the set of points in $\mathbb{Z}^k$ satisfying the inequalities (4), together with the inequalities $a_\alpha \geq 0$ for α a simple root. This is a total of $k - n + n = k$ inequalities. We thus have:

Corollary 2.2 *Let $\mathbf{i}$ be a reduced expression for the longest word w_0 in type A_n. Then the corresponding Lusztig cone $C_{\mathbf{i}}$ is simplicial.*

It follows that there is a $k \times k$ matrix $P_{\mathbf{i}} \in M_k(\mathbb{Z})$ such that

$$C_{\mathbf{i}} = \{\mathbf{a} \in \mathbb{Z}^k \,:\, P_{\mathbf{i}}\mathbf{a} \geq 0\},$$

where for $\mathbf{z} \in \mathbb{Z}^k$, $\mathbf{z} \geq 0$ means that each entry in $\mathbf{z}$ is nonnegative. To write down $P_{\mathbf{i}}$ we fix an ordering on the set of k inequalities defining $C_{\mathbf{i}}$ — they are indexed by the simple roots and the chambers of $\mathbf{i}$. We do this for each $\mathbf{i}$. We also have:

Lemma 2.3 *The matrix $P_{\mathbf{i}} \in M_k(\mathbb{Z})$ has an inverse in $M_k(\mathbb{N})$.*

Proof: Let $\alpha = \alpha_{ij}$ be any positive root. By the proof of Lemma 2.1, we know that a sum of rows of $P_{\mathbf{i}}$ gives us a row with corresponding inequality $a_{ij} - a_{i,i+1} - a_{i+1,i+2} - \cdots - a_{j-1,j} \geq 0$. But we also have rows of $P_{\mathbf{i}}$ corresponding to $a_{i,i+1} \geq 0$, $a_{i+1,i+2} \geq 0, \ldots, a_{j-1,j} \geq 0$ (as $\alpha_{i,i+1}$, etc., are all simple roots). Adding together all of these rows thus gives us a row with corresponding inequality $a_{ij} \geq 0$, i.e. a row $(0, 0, \ldots, 0, 1, 0, 0, \ldots, 0)$, where the 1 appears in the position corresponding to a_{ij} (that is, in the lth position, where $\alpha^l = \alpha_{ij}$ in the ordering defined by $\mathbf{i}$). We can do this for any l, so if we write the coefficients of the lth linear combination as the lth row of a matrix $Q_{\mathbf{i}}$ then $Q_{\mathbf{i}}P_{\mathbf{i}}$ is the identity matrix and we are done. □

Now let $v_1, v_2, \ldots, v_k$ be the columns of $Q_{\mathbf{i}}$. Because $Q_{\mathbf{i}}$ and $P_{\mathbf{i}}$ are inverses we have that $\mathbf{a} \in \mathbb{Z}^k$ lies in $C_{\mathbf{i}}$ if and only if $P_{\mathbf{i}}\mathbf{a} \geq 0$, if and only if $\mathbf{a}$ is a nonnegative integer linear combination of $v_1, v_2, \ldots, v_k$. It is now clear that $C_{\mathbf{i}}$ is the set of integral points of the simplicial cone in $\mathbb{R}^k$ defined by the same inequalities as $C_{\mathbf{i}}$. Our next step in understanding $C_{\mathbf{i}}$ is to describe the 'spanning vectors' v_j, $j = 1, 2, \ldots, k$, for each $\mathbf{i}$. It will turn out that these spanning vectors can be neatly described in terms of 'partial quivers', which we define below, together with some vectors common to every $C_{\mathbf{i}}$. Firstly, we investigate chambers and chamber sets further.

Definitions 2.4

It is well-known that the graph with vertices χ_n, the set of reduced expressions for w_0, where $\mathbf{i}, \mathbf{i}' \in \chi_n$ are linked by an edge whenever there is a braid relation taking $\mathbf{i}$ to $\mathbf{i}'$, is connected. Note also that, applying a braid relation to a reduced expression $\mathbf{i}$ has a corresponding effect on the chamber ansatz for $\mathbf{i}$. If this is a short braid relation $s_is_j = s_js_i$ with $A_{ij} = 0$ then the only effect on the chamber ansatz for $\mathbf{i}$ is to reorder two crossings which do not interfere with each other. The chambers of $\mathbf{i}$ are clearly mapped bijectively onto the chambers of $\mathbf{i}'$ by this operation, and if $(c, \mathbf{i})$ is a chamber, with corresponding chamber $(c', \mathbf{i}')$ (under this mapping), we set $(c, \mathbf{i}) \sim (c', \mathbf{i}')$. If we have a long braid relation $B : s_is_js_i = s_js_is_j$ with $A_{ij} = A_{ji} = -1$, then this determines a chamber in $CD(\mathbf{i})$, with ends given by the crossings corresponding to the two s_i's in $s_is_js_i$. If $(c, \mathbf{i})$ is any other chamber of $\mathbf{i}$ then there is a corresponding pair of letters in $\mathbf{i}$ (from the crossings at either end of c) which is left unchanged by B and thus determines a chamber $(c', \mathbf{i}')$ in $\mathbf{i}'$. In this situation we also set $(c, \mathbf{i}) \sim (c', \mathbf{i}')$.

We extend this to a relation on chambers for arbitrary elements of χ_n by specifying that $(c, \mathbf{i}) \sim (c', \mathbf{i}')$ if there is a sequence

$$(c, \mathbf{i}) = (c_0, \mathbf{i}_0), (c_1, \mathbf{i}_1), \ldots, (c_m, \mathbf{i}_m) = (c', \mathbf{i}')$$

with $(c_j, \mathbf{i}_j) \sim (c_{j+1}, \mathbf{i}_{j+1})$ for $j = 0, 1, \ldots, m-1$. It is easy to see that $\sim$ is an equivalence relation. This relation has an alternative description:

Lemma 2.5 *Suppose $(c, \mathbf{i})$ and $(c', \mathbf{i}')$ are chambers. Then $(c, \mathbf{i}) \sim (c', \mathbf{i}')$ if and only if $l(c, \mathbf{i}) = l(c', \mathbf{i}')$. In other words, two chambers are in the same equivalence class if and only if they have the same chamber set.*

Proof: First, suppose that $(c, \mathbf{i}) \sim (c', \mathbf{i}')$, and that $\mathbf{i}$ and $\mathbf{i}'$ are related by a single braid relation. By the definition of $\sim$ and the structure of the chamber ansatz, it is clear that $l(c, \mathbf{i}) = l(c', \mathbf{i}')$. It follows that this is true for arbitrary $\mathbf{i}$ and $\mathbf{i}'$. The reverse implication is [2, Lemma 2.7.2]. □

3 Partial Quivers and Spanning Vectors

In order to describe the spanning vectors of the Lusztig cones (that is, the columns of the inverses of their defining matrices, $P_{\mathbf{i}}$), we use the quiver-compatible reduced expressions. These reduced expressions are spread sufficiently evenly throughout the set of all reduced expressions that we can use them to obtain properties for arbitrary reduced expressions.

This approach leads to an alternative way of labelling the chambers in a chamber ansatz, by 'partial quivers', which we shall now describe. We shall then use this together with a detailed analysis of the chamber ansatz of a quiver-compatible reduced expression in order to give an explicit description of the spanning vectors of an arbitrary Lusztig cone.

Thanks are due to R. W. Carter, who realised that it is often natural to label a spanning vector (or chamber) by the set of quiver-compatible reduced expressions for the longest word for which it appears; this led to the idea of a partial quiver given below.

A *partial quiver* P of type A_n is a copy of the Dynkin graph of type A_n with some of the edges directed (at least one, and possibly all), in such a way that the subgraph consisting of the directed edges and vertices incident with them is connected. We number the edges of a partial quiver from 2 to n, starting at the right hand end. We write P as a sequence of $n-1$ symbols, L, R or $-$, where L denotes a leftward arrow, R a rightward arrow, and $-$ an undirected edge. Thus any partial quiver will be of form $---***---$, where the $*$'s denote L's or R's.

If P, P' are partial quivers we write $P' \geq P$ (or $P \leq P'$) if every edge which is directed in P is directed in P' and is oriented in the same way, and say that P is a sub partial quiver of P'. For example, $---LRLL- \leq RLRLRLLL$.

It is known that if $\mathbf{i}$ is compatible with a quiver Q (in the sense of [8, §4.7]) then the set of reduced expressions for w_0 compatible with Q is precisely the commutation class of χ_n containing $\mathbf{i}$. We say that this commutation class is compatible with Q. Berenstein, Fomin and Zelevinsky (see [2, §4.4]) describe a method for constructing $\mathbf{i} \in \chi_{\mathbf{n}}$ compatible with any given quiver Q, as follows.

Suppose Q is a quiver of type A_n. Let $\Lambda \subseteq [2, n]$ be the set of all edges of Q pointing to the left.

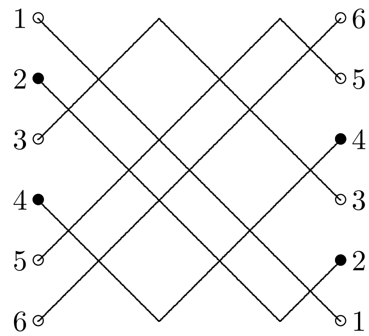


Figure 2: The arrangement Arr$\{2,4\}$.

Berenstein, Fomin and Zelevinsky construct an arrangement $\mathrm{Arr}(\Lambda)$. Consider a square in the plane, with horizontal and vertical sides. Put $n+1$ points onto the left-hand edge of the square, equally spaced, numbered 1 to $n+1$ from top to bottom, and do the same for the right-hand edge, but number the points from bottom to top. Line_h joins point h on the left with point h on the right. For $h = 1, n+1$, Line_h is a diagonal of the square. For $h \in [2,n]$, Line_h is a union of two line segments of slopes 1 and -1. There are precisely two possibilities for Line_h. If $h \in \Lambda$, the left segment has slope -1, while the right one has slope 1; for $h \in [2,n] \setminus \Lambda$, it goes the other way round. Berenstein, Fomin and Zelevinsky give the example of the arrangement for $n = 5$ and $\Lambda = \{2,4\}$, which we show in Figure 2 (points corresponding to elements of Λ are indicated by filled circles). They show that $\mathbf{i} \in \chi_{\mathbf{n}}$ is compatible with the quiver Q if and only if the chamber ansatz $\mathrm{CD}(\mathbf{i})$ is isotopic to $\mathrm{Arr}(\Lambda)$.

They also calculate the chamber sets of such a reduced expression [2, §4.4.5].

Proposition 3.1 *(Berenstein, Fomin and Zelevinsky.)*
Let $\mathbf{i}$ *be a reduced expression compatible with a quiver* Q*. Suppose that* $(c,\mathbf{i})$ *is a chamber, and that strings* i *and* j *(*$i<j$*) cross immediately above it in CD(*$\mathbf{i}$*). Then* $l(ij) := l(c,\mathbf{i})$ *is given in the following way: Write out* Q *and label the edges from 2 to* n*, starting at the right hand end. Define* X *to be the subset of* $[2,n]$ *corresponding to the* L*'s in* Q*. Then* $l(ij) = Y_1 \cup Y_2 \cup Y_3$*, where* $Y_1 = [i+1, j-1] \cap X$*,* $Y_2 = [1, i-1]$ *if* $i \in [2,n]$ *and* $i \notin X$*, and is otherwise empty, and* $Y_3 = [j+1, n+1]$ *if* $j \in [2,n]$ *and* $j \notin X$*, and is otherwise empty.* $\square$

Any given chamber set will arise in the chamber ansatz of a number of reduced expressions. We now give a description of the quiver-compatible reduced expressions whose chamber ansatz includes a fixed chamber set. Suppose $(c, \mathbf{i})$ is a chamber. We denote by $Q(c, \mathbf{i})$ the set of quivers Q for which $l(c, \mathbf{i})$ occurs as the chamber set of a chamber $(c', \mathbf{i}')$, where $\mathbf{i}'$ is compatible with Q. If P is a partial quiver, we define a subset $l(P)$ of $[1, n+1]$ in the following way. Let $l_1(P) = \{j \in [2, n] \, : \, \text{edge } j \text{ of } P \text{ is an } L\}$. If the rightmost directed edge of P is an R, and this is in position i, then let $l_2(P) = [1, i-1]$, otherwise the empty set. If the leftmost directed edge of P is an R, and this is in position j, then let $l_3(P) = [j+1, n+1]$, otherwise the empty set. Then put $l(P) = l_1(P) \cup l_2(P) \cup l_3(P)$. Note that for any partial quiver P, $l(P)$ is always a chamber set, that is, it is a subset of $[1, n+1]$ not of the form $[1, j]$ or $[j, n+1]$ for some $1 \leq j \leq n+1$. Thus l is a map from chamber sets to partial quivers.

Lemma 3.2 *The map l is a bijection.*

Proof: First of all, note that the number of partial quivers is the same as the number of chamber sets, i.e., $2^{n+1} - 2(n+1)$, so it is enough to show that l is surjective. Let S be a chamber set. If $1 \in S$, let a be the minimal element of $[1, n+1] \setminus S$; otherwise let a be the minimal element of S. If $n+1 \in S$, let b be the maximal element of $[1, n+1] \setminus S$; otherwise let b be the maximal element of S. Let P be a partial quiver with the edges j, $a \leq j \leq b$ directed and the others undirected. For $a \leq j \leq b$, let edge j of P be an L if $j \in S$ and let it be an R if not. Then it can be seen from the definition of the map l above that $l(P) = S$ and we are done. $\square$

Proposition 3.3 *Let $(c, \mathbf{i})$ be a chamber. Then there is a unique partial quiver $P = P(c, \mathbf{i})$ such that $Q(c, \mathbf{i}) = \{Q \, : \, Q \text{ is a quiver and } Q \geq P\}$. Furthermore, if $(c, \mathbf{i})$ and $(c', \mathbf{i}')$ are two chambers, then $l(c, \mathbf{i}) = l(c', \mathbf{i}')$ if and only if $P(c, \mathbf{i}) = P(c', \mathbf{i}')$. That is to say, their chamber sets are equal if and only if their partial quivers are equal. We have $l(P(c, \mathbf{i})) = l(c, \mathbf{i})$, and that every partial quiver arises as $P(c, \mathbf{i})$ for some chamber $(c, \mathbf{i})$. Thus the equivalence classes of chambers are in one-to-one correspondence with the partial quivers.*

Proof: We firstly claim that if we take a quiver Q, a reduced expression $\mathbf{i}$ compatible with Q and a partial quiver $P \leq Q$ then $l(P) = l(c, \mathbf{i})$ for some chamber $(c, \mathbf{i})$. Fix a quiver Q and let P be a partial quiver with $P \leq Q$. Suppose P is of the form $----Y*****X----$, where X, Y and the $*$'s are L's or R's and the X appears in position a and the Y appears in position b ($a < b$), with the usual numbering. If $X = L$ let i be the position of the first L in Q strictly to the right of X (or 1 if this doesn't exist). If $X = R$ let $i = a$. If $Y = L$ let j be the position of the first L in Q strictly to the left of Y (or $n+1$ if this doesn't exist). If $Y = R$ let

$j=b$. It is easy to check using Proposition 3.1 that $l(ij)=l(P)$; in fact the three components of $l(ij)$ and $l(P)$ coincide.

This deals with all of the cases except when P is of the form $---R---$, where edge i is an R. Here $l(P)=[1,i-1]\cup[i+1,n+1]$. But note that in the chamber ansatz for Q string i (corresponding to an R) goes first up to the top of the diagram and then downwards (by the description of Berenstein, Fomin and Zelevinsky). So at some point it bounces off the top of the diagram. Therefore the chamber where it bounces will have chamber set precisely $[1,i-1]\cup[i+1,n+1]$. Thus the claim is shown.

Since chambers of a reduced expression $\mathbf{i}$ correspond to minimal pairs of equal letters, the chamber ansatz for $\mathbf{i}$ must have precisely $\frac{1}{2}n(n-1)$ chambers. This is also the number of partial quivers $P\leq Q$. Thus the chamber sets of $\mathbf{i}$ are precisely the $l(P)$ for $P\leq Q$, P a partial quiver (since l is bijective).

Next, suppose that $(c,\mathbf{i})$ is a chamber, with chamber set $l=l(c,\mathbf{i})$. Let P be the partial quiver such that $l(P)=l$. Then l is a chamber set of a reduced expression compatible with a quiver Q if and only if $P\leq Q$. Thus $Q(c,\mathbf{i})=\{Q\,:\,Q \text{ is a quiver and } Q\geq P\}$, as required. It is clear that P is unique with this property. Also, we see that $l(P(c,\mathbf{i}))=l(c,\mathbf{i})$. Furthermore, if $l(c,\mathbf{i})=l(c',\mathbf{i}')$, then since $P(c,\mathbf{i})$ is defined in terms of $l(c,\mathbf{i})$ it is clear that $P(c,\mathbf{i})=P(c',\mathbf{i}')$. Conversely, if $P(c,\mathbf{i})=P(c',\mathbf{i}')=P$ then applying l to both sides gives $l(c,\mathbf{i})=l(c',\mathbf{i}')$, and we are done. □

Corollary 3.4 *Let Q be a quiver, and $\mathbf{i}$ a reduced expression compatible with Q. Then the labels $P(c,\mathbf{i})$, for $(c,\mathbf{i})$ a chamber for $\mathbf{i}$, are precisely the partial quivers $P\leq Q$.*

We next need to understand which crossings surround a chamber labelled by a given partial quiver. We start with the following Lemma, which is easily checked:

Lemma 3.5 *Suppose $X=(c,\mathbf{i})$ is a chamber in $CD(\mathbf{i})$. Then let α and β be the positive roots labelling the crossings at the left and right ends of X, and $\gamma_1,\gamma_2,\ldots,\gamma_t$ the positive roots labelling the crossings above and below X. Then $\alpha+\beta=\sum_{i=1}^{t}\gamma_i$.* □

In order to describe these crossings, we shall use the following definition.

Definition 3.6 Let $2\leq a\leq b\leq n$, and let Q be a quiver. We define the (a,b)-sub partial quiver of Q to be the partial quiver P with $P \leq Q$, leftmost directed edge numbered b and rightmost directed edge numbered a.

Lemma 3.7 *Let Q be a quiver and $\mathbf{i} \in \chi_n$ a reduced expression compatible with Q. Let $X = (c, \mathbf{i})$ be a chamber in CD$(\mathbf{i})$, with $P = P(c, \mathbf{i})$. Let a be the position of the rightmost directed edge of P, and b the position of the leftmost directed edge. We divide the possibilities for P into four cases, depending on the orientation of these edges of P:*

$$(1)\quad \begin{array}{cccc} q & r & s & p \\ LRR\ldots RL & \ldots & LRR\ldots RL & \\ & b & a & \end{array}$$

$$(2)\quad \begin{array}{cccc} q & r & p & s \\ LRR\ldots RL & \ldots & RLL\ldots LR & \\ & b & a & \end{array}$$

$$(3)\quad \begin{array}{cccc} r & q & s & p \\ RLL\ldots LR & \ldots & LRR\ldots RL & \\ & b & a & \end{array}$$

$$(4)\quad \begin{array}{cccc} r & q & p & s \\ RLL\ldots LR & \ldots & RLL\ldots LR & \\ & b & a & \end{array}$$

In each case we define 4 numbers, p, q, r, s, which number edges of Q, defined by the above diagrams. For example, in the first case, the leftmost and rightmost edges of P are both L's. In this case, we define $r = b$, $s = a$, q to be the position of the first L in Q appearing strictly to the left of the leftmost edge in P and p to be the position of the first L in Q appearing strictly to the right of the rightmost edge in P. The other cases are similar. We use the convention that Q has an edge in position 1 with the same orientation as its edge in position a, and that it has an edge in position $n+1$ with the same orientation as its edge in position b (thus p, q, r, s are always well-defined). Write $p(c, \mathbf{i}) = p$, $q(c, \mathbf{i}) = q$, $r(c, \mathbf{i}) = r$ and $s(c, \mathbf{i}) = s$.

Make the convention that a crossing involving two equal strings means no crossing. Then strings p and q cross immediately above X, strings q and s cross to the left of X, strings p and r cross to the right of X and strings r and s cross below X.

Proof: First of all, note that each chamber in CD$(\mathbf{i})$ has at most one crossing bounding it on each side, left, right, above or below. If $p \neq q$, the result is an easy consequence of Proposition 3.1 and the construction of Berenstein, Fomin and Zelevinsky of quiver compatible reduced expressions. If not, the result is easy to see, as the partial quiver is a single R in the ath position, so $\ell(P) = [1, n+1] \setminus \{a\}$. □

Now we shall see how partial quivers can be used to describe the spanning vectors of a Lusztig cone. Firstly, let us set up some notation. Fix $\mathbf{i} \in \chi_n$. Then we know that

$$C_{\mathbf{i}} = \{\mathbf{a} \in \mathbb{Z}^k \,:\, P_{\mathbf{i}}\mathbf{a} \geq 0\},$$

where $P_{\mathbf{i}}$ is a $k \times k$ matrix as defined after Lemma 2.1, with inverse $Q_{\mathbf{i}}$ (over $\mathbb{Z}$). Note that $n(n-1)/2$ of the rows of $P_{\mathbf{i}}$ correspond to inequalities arising from minimal pairs in $\mathbf{i}$. Each such minimal pair corresponds naturally to a chamber $(c, \mathbf{i})$. Thus, given a chamber $(c, \mathbf{i})$, there is a corresponding row of $P_{\mathbf{i}}$ and therefore a corresponding column of $Q_{\mathbf{i}}$, that is, a spanning vector of $C_{\mathbf{i}}$. We denote this spanning vector by $v(c, \mathbf{i})$. Similarly, the other n rows of $P_{\mathbf{i}}$ correspond to inequalities of the form $a_{\alpha_j} \geq 0$ for α_j a simple root, $1 \leq j \leq n$. We denote the corresponding spanning vectors by $v(j, \mathbf{i})$.

We shall usually regard elements v of $C_{\mathbf{i}}$ as being indexed by the positive roots (and thus by pairs of integers p, q with $1 \leq p < q \leq n+1$). We shall occasionally need to consider such vectors in the usual way, as elements of $\mathbb{N}^k$; we shall indicate this by writing $v_{\mathbf{i}}$.

Suppose that P is a quiver. We define a *sub partial quiver* Y of a quiver P to be a *component* of P if all of its edges are oriented the same way and Y is maximal in length with this property. We say Y has *type* L if its edges are oriented to the left, and *type* R if its edges are oriented to the right. For each component Y of P, let $a(Y)$ be the position of the rightmost directed edge of Y, and let $b(Y)$ be the position of the leftmost directed edge of Y. Let $v(Y)$ be the vector defined by setting, for $1 \leq p < q \leq n+1$, $v(Y)_{pq} = 1$ if $1 \leq p < a(Y) \leq b(Y) < q \leq n+1$ and $v(Y)_{pq} = 0$, otherwise.

Theorem 3.8
(a) Let $(c, \mathbf{i})$ be a chamber. The spanning vector $v(c, \mathbf{i})$ depends only upon $P(c, \mathbf{i})$ (equivalently, upon the chamber set, $\ell(c, \mathbf{i})$). For a partial quiver P we choose a chamber $(c, \mathbf{i})$ such that $P(c, \mathbf{i}) = P$ and write $v(P) = v(c, \mathbf{i})$.
(b) For $j = 1, 2, \ldots, n$, the spanning vector $v(j, \mathbf{i})$ depends only upon j; we denote it $v(j)$.
(c) For a partial quiver P, let

$$w(P) = \sum_{Y \text{ a component of } P} v(Y).$$

Then, for each $1 \leq p < q \leq n+1$, $v(P)_{pq} = \lceil \frac{1}{2} w(P)_{pq} \rceil$ (recall that $\lceil\ \rceil$ rounds to the nearest integer, rounding $\frac{1}{2}$ upwards). Furthermore, for $j = 1, 2, \ldots n$, $v(j)_{pq} = 1$ if $1 \leq p \leq j \leq j+1 \leq q \leq n+1$ and $v(j)_{pq} = 0$ otherwise.

Remark: Note that the theorem describes the columns of $P_{\mathbf{i}}^{-1}$, i.e. the spanning vectors of $C_{\mathbf{i}}$, which is the problem we have set out to solve.

Proof: We start by showing (a) and (b), i.e. that $v(c, \mathbf{i})$ (when its components are indexed by the positive roots) depends only on the equivalence class of the chamber $(c, \mathbf{i})$ for the relation $\sim$ defined in 2.4, and that $v(j, \mathbf{i})$ (when its components are indexed by the positive roots), depends only on j. This reduces the question to quiver-compatible reduced words, which are easier to deal with.

We start with two reduced expressions $\mathbf{i}, \mathbf{i}' \in \chi_n$, such that there is a short braid relation $(i, j) \to (j, i)$ taking $\mathbf{i}$ to $\mathbf{i}'$. Thus $\mathbf{i}'$ is obtained from $\mathbf{i}$ by exchanging two of its entries. From the definition of the Lusztig cone, it is clear that $P_{\mathbf{i}'}$ can be obtained from $P_{\mathbf{i}}$ by exchanging the corresponding columns. Thus $Q_{\mathbf{i}'}$ can be obtained from $Q_{\mathbf{i}}$ by exchanging the corresponding rows. This means that the spanning vectors for $\mathbf{i}'$ are obtained from those for $\mathbf{i}$ by exchanging the corresponding elements in the vectors, which means that the spanning vectors for $\mathbf{i}'$, when labelled by the positive roots, are the same as those for $\mathbf{i}$ (as the effect of the short braid relation on the ordering of the positive roots defined by $\mathbf{i}$ is to exchange the roots corresponding to the i and the j).

Now suppose that $\mathbf{i}'$ is obtained from $\mathbf{i}$ by applying a long braid relation $B: (i, j, i) \to (j, i, j)$. Since the pair of i's in $\mathbf{i}$ is minimal, there is a corresponding chamber $X = (c, \mathbf{i})$ in $\mathrm{CD}(\mathbf{i})$. Let $P'_{\mathbf{i}'}$ be the matrix obtained from $P_{\mathbf{i}'}$ by exchanging the columns corresponding to the pair of j's in the long braid relation in $\mathbf{i}'$. Recall (see Definitions 2.4), that the effect of the long braid relation on $\mathrm{CD}(\mathbf{i})$ is to map every chamber except X onto a chamber of $\mathrm{CD}(\mathbf{i}')$ with the same label; the chamber X corresponds to a chamber X' of $\mathrm{CD}(\mathbf{i}')$ with a label not equal to any of the labels of the chambers of $\mathrm{CD}(\mathbf{i})$. We thus have a correspondence between the chambers of $\mathrm{CD}(\mathbf{i})$ and the chambers of $\mathrm{CD}(\mathbf{i}')$. If $(c, \mathbf{i})$ is a chamber in $\mathrm{CD}(\mathbf{i})$ we denote the corresponding chamber of $\mathrm{CD}(\mathbf{i}')$ by $(c, \mathbf{i})'$. We denote the row of $P_{\mathbf{i}}$ corresponding to a chamber Y of $\mathrm{CD}(\mathbf{i})$ by r_Y, and the row of $P'_{\mathbf{i}'}$ corresponding to the corresponding chamber Y' of $\mathrm{CD}(\mathbf{i}')$ by $s_{Y'}$. We now prove the following claim.

Claim: $P'_{\mathbf{i}'}$ can be obtained from $P_{\mathbf{i}}$ by applying row operations which add or subtract r_X to rows other than r_X.

From the definition of the Lusztig cone we get that $s_{X'} = r_X$. Let $Y \neq X$ be a chamber of $\mathrm{CD}(\mathbf{i})$ with corresponding minimal pair of letters $p, \ldots, p$ in $\mathbf{i}$. If this pair of letters is to the left of the (i, j, i) of B, or to the right of B, then we have $s_{Y'} = r_Y$. The following can be checked easily. Suppose first that the (i, j, i) of B lies between the pair of p's. If $|i - p| > 1$ and

$|j-p|>1$, then $s_{Y'}=r_Y$. If $|i-p|=1$ and $|j-p|>1$ then $s_{Y'}=r_Y+r_X$. If $|i-p|>1$ and $|j-p|=1$ then $s_{Y'}=r_Y-r_X$. Finally, if $p=i$ then $s_{Y'}=r_Y-r_X$, and if $p=j$ then $s_{Y'}=r_Y+r_X$. Note also, that the rows corresponding to simple roots are the same in $P_{\mathbf{i}}$ and in $P'_{\mathbf{i}'}$, since the j in (i,j,i) can never correspond to a simple root in the ordering defined by $\mathbf{i}$. Thus we see that the claim is proved.

It follows that there is a column operation on $Q_{\mathbf{i}}$ which changes only the column corresponding to X and which takes $Q_{\mathbf{i}}$ to the inverse $Q'_{\mathbf{i}'}$ of $P'_{\mathbf{i}'}$. Now $Q'_{\mathbf{i}'}$ is the same matrix as $Q_{\mathbf{i}'}$ except that the rows corresponding to the i's in B have been exchanged. Since the effect of B on the ordering of the positive roots defined by $\mathbf{i}$ is to exchange the roots corresponding to the i's in B, it follows that if $Y\neq X$ is a chamber of CD($\mathbf{i}$) then the spanning vector corresponding to Y is the same as the spanning vector corresponding to Y' (if both vectors are labelled by the positive roots), and also, if $1\leq j\leq n$, then the spanning vector corresponding to $(j,\mathbf{i})$ is the same as that for $(j,\mathbf{i}')$.

Thus statements (a) and (b) are true. Note that every partial quiver P satisfies $P\leq Q$ for some quiver Q. So it is now enough to calculate the spanning vectors for a reduced word $\mathbf{i}$ compatible with a quiver Q. We fix such a Q and $\mathbf{i}$.

Fix $1\leq j\leq n$, and let v be the vector defined by $v_{pq}=1$ when $1\leq p\leq j\leq j+1\leq q\leq n+1$ and $v_{pq}=0$ otherwise. We shall show that $v(j)=v$. Let $P_{\mathbf{i}}$ be the defining matrix of $C_{\mathbf{i}}$ and let r_l, $1\leq l\leq n$, be the rows of $P_{\mathbf{i}}$ corresponding to the inequalities $v_{\alpha_l}\geq 0$ (where $\alpha_1,\alpha_2,\ldots,\alpha_l$ are the simple roots). For each chamber X in CD($\mathbf{i}$), let r_X be the corresponding row of $P_{\mathbf{i}}$.

It is clear that the product of v with r_l is δ_{jl} for $1\leq l\leq n$, because $v_{l,l+1}=\delta_{jl}$. Now let X be a chamber of CD($\mathbf{i}$). To calculate the product of v with r_X we have to calculate $\sum_{i=1}^t v_{\gamma_i}-v_\alpha-v_\beta$, where the γ_i are the positive roots labelling the crossings immediately above and below X, and α and β are the crossings at the right and left ends of X. But we know from Lemma 3.5 that $\sum_{i=1}^t\gamma_i=\alpha+\beta$ from which we get that the product of r_X with v is zero. We conclude that $v(j)=v$ as required (as the spanning vectors of $C_{\mathbf{i}}$ are the columns of the inverse of $P_{\mathbf{i}}$).

Suppose now that P is a partial quiver with $P\leq Q$. We know there is a chamber Y of CD($\mathbf{i}$) such that $P(Y)=P$. We have to show $v(P)=v$ where v is the vector as described in the Theorem. It is clear that $v_\alpha=0$ if α is simple, so we see that the product of r_l with v is zero for $1\leq l\leq n$. So let X be any chamber of CD($\mathbf{i}$). We need to calculate $f(X,Y):=\sum_{i=1}^t v_{\gamma_i}-v_\alpha-v_\beta$ (with γ_i, α and β as above), and show this is δ_{XY}. Let $P'=P(X)$ be the partial quiver label of X.

We have already calculated which crossings surround X, in Lemma 3.7. Let $p'=p(X)$, $q'=$

$q(X)$, $r' = r(X)$ and $s' = s(X)$. We must calculate $f(X,Y) = v_{p'q'} + v_{s'r'} - v_{s'q'} - v_{p'r'}$. Note that $v_{ij} = \lceil w_{ij}/2 \rceil$ where w_{ij} is the number of components of P entirely contained in the $(i+1, j-1)$-sub partial quiver of Q. Let a be the position of the rightmost directed edge of P, and let b be the position of the leftmost directed edge of P, so that P is the (a,b)-sub partial quiver of Q. Simlarly, let a' be the position of the rightmost directed edge of P', and let b' be the position of the leftmost directed edge of P', so that P' is the (a',b')-sub partial quiver of Q.

We consider the four cases of Lemma 3.7 for the partial quiver P'. We consider first case (1), where the leftmost and rightmost edges of P' are oriented to the left. The diagram in the Lemma indicates how to work out the values of p', q', r' and s'. Let us recall it here:

$$\begin{array}{llllll} & q' & \quad r' & & s' & \quad p' \\ (1) & LRR\ldots & RL & \ldots & LRR\ldots & RL \\ & & b' & & a' & \end{array}$$

The first and last sequence of R's may be empty, the other groupings not. (This may happen, for example, if the leftmost edge of P' and the edge immediately to the left of it are both oriented to the left).

We note that:

(a) If $a > a'$, then $w_{s'r'} = w_{p'r'}$ and $w_{s'q'} = w_{p'q'}$.
(b) If $b < b'$, then $w_{s'r'} = w_{s'q'}$ and $w_{p'r'} = w_{p'q'}$.

In these cases, $f(X,Y) = 0$ follows immediately from the definition, so we can assume that $a \le a'$ and $b \ge b'$. Let us now consider the case where both the first and last sequences of R's occur.

It can be seen from the picture and the definition of w_{ij} that:

(c) If $a = a'$, then $w_{s'r'} + 1 = w_{p'r'}$ and $w_{s'q'} + 1 = w_{p'q'}$.
(d) If $a < a'$, then $w_{s'r'} + 2 = w_{p'r'}$ and $w_{s'q'} + 2 = w_{p'q'}$.
(e) If $b > b'$, then $w_{s'r'} + 2 = w_{s'q'}$ and $w_{p'r'} + 2 = w_{p'q'}$.
(f) If $b = b'$, then $w_{s'r'} + 1 = w_{s'q'}$ and $w_{p'r'} + 1 = w_{p'q'}$.

We consider all possibilities, in each case computing $w_{p'q'}$, $w_{s'r'}$, $w_{s'q'}$ and $w_{p'r'}$ in terms of $w_{s'r'}$ and computing $f(X,Y) = \lceil w_{p'q'}/2 \rceil + \lceil w_{s'r'}/2 \rceil - \lceil w_{s'q'}/2 \rceil - \lceil w_{p'r'}/2 \rceil$. We use the fact that $w_{s'r'}$ is odd.

Case (I) $a = a'$ and $b = b'$: We have, by (c) and (f) above, that $w_{p'q'} = w_{s'r'}+2$, $w_{s'q'} = w_{s'r'}+1$

and $w_{p'r'} = w_{s'r'} + 1$. Thus $f(X,Y) = \lceil (w_{s'r'} + 2)/2 \rceil + \lceil w_{s'r'}/2 \rceil - \lceil (w_{s'r'} + 1)/2 \rceil - \lceil (w_{s'r'} + 1)/2 \rceil = 1$.

Case (II) $a < a'$ and $b = b'$: We have, by (d) and (f) above, that $w_{p'q'} = w_{s'r'}+3$, $w_{s'q'} = w_{s'r'}+1$ and $w_{p'r'} = w_{s'r'} + 2$. Thus $f(X,Y) = \lceil (w_{s'r'} + 3)/2 \rceil + \lceil w_{s'r'}/2 \rceil - \lceil (w_{s'r'} + 1)/2 \rceil - \lceil (w_{s'r'} + 2)/2 \rceil = 0$.

Case (III) $a = a'$ and $b > b'$: We have, by (c) and (e) above, that $w_{p'q'} = w_{s'r'}+3$, $w_{s'q'} = w_{s'r'}+2$ and $w_{p'r'} = w_{s'r'} + 1$. Thus $f(X,Y) = \lceil (w_{s'r'} + 3)/2 \rceil + \lceil w_{s'r'}/2 \rceil - \lceil (w_{s'r'} + 2)/2 \rceil - \lceil (w_{s'r'} + 1)/2 \rceil = 0$.

Case (IV) $a < a'$ and $b > b'$: We have, by (d) and (e) above, that $w_{p'q'} = w_{s'r'}+4$, $w_{s'q'} = w_{s'r'}+2$ and $w_{p'r'} = w_{s'r'} + 2$. Thus $f(X,Y) = \lceil (w_{s'r'} + 4)/2 \rceil + \lceil w_{s'r'}/2 \rceil - \lceil (w_{s'r'} + 2)/2 \rceil - \lceil (w_{s'r'} + 2)/2 \rceil = 0$.

We next consider the possibility that the leftmost and rightmost sequences of $R's$ do not occur, i.e. that $q' = r' + 1$ or $p' = s' - 1$. We note that:

(g) If $q' = r' + 1$ and $b = b'$ then (f) above still holds.
(h) If $q' = r' + 1$ and $b > b'$ then $w_{s'r'} = w_{s'q'}$ and $w_{p'r'} = w_{p'q'}$.
(i) If $p' = s' - 1$ and $a = a'$ then (c) above still holds.
(j) If $p' = s' - 1$ and $a < a'$ then $w_{s'r'} = w_{p'r'}$ and $w_{s'q'} = w_{p'q'}$.

As before, it follows immediately from the definition that $f(X,Y) = 0$ in cases (h) and (j).

Case (V) $q' = r' + 1$ and $p' \neq s' - 1$. The case $b > b'$ is dealt with by (h) above. If $b = b'$, then, by (g), the arguments in cases (I) and (II) can be used to show $f(X,Y) = \delta_{XY}$.

Case (VI) $q' \neq r' + 1$ and $p' = s' - 1$. The case $a < a'$ is dealt with by (j) above. If $a = a'$, then, by (i), the arguments in cases (I) and (III) can be used to show $f(X,Y) = \delta_{XY}$.

Case (VII) $q' = r' + 1$ and $p' = s' - 1$. If $a < a'$ or $b > b'$, then by (h) or (j) above, we can see that $f(X,Y) = \delta_{XY}$. We are left with the case where $a = a'$ and $b = b'$, in which, by (g) and (i), the argument in case (I) can be used to show $f(X,Y) = \delta_{XY}$.

We have thus covered all possibilities where the leftmost and rightmost edges of P' are oriented to the left — case (1) of Lemma 3.7. Computation of $f(X,Y)$ in case (2) of Lemma 3.7 can be obtained from that in case (1) by exchanging the roles of p' and s'. The arguments are exactly the same, except that in this case, $w_{p'r'}$ (playing the role of $w_{s'r'}$) is *even*; also, it is the first sequence of R's or the final sequence of L's which may or may not occur (i.e. we may have

$q' = r'+1$ or $s' = p'-1$). The argument for case (1) shows that $v_{s'q'}+v_{p'r'}-v_{p'q'}-v_{s'r'} = -\delta_{XY}$ (note that p and s are swapped). The minus sign arises because $w_{p'r'}$ is even. It follows that $f(X,Y) = v_{p'q'} + v_{s'r'} - v_{s'q'} - v_{p'r'} = \delta_{XY}$. Similarly, case (3) is obtained by swapping q and r in case (1), and case (4) is obtained by swapping both q and r and p and s (in this last case, the argument for case (1) shows directly that $f(X,Y) = \delta_{XY}$; here $w_{p'q'}$, playing the role of $w_{s'r'}$, is again odd).

Thus, in all cases, we have seen that $f(X,Y) = \delta_{XY}$. We see that this holds for any pair of chambers X and Y, and thus the Theorem is proved. □

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