More Tight Monomials in Quantized Enveloping Algebras

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1 Introduction

Let $U = U_q(\mathbf{g})$ be the quantum group associated to a semisimple Lie algebra $\mathbf{g}$ of rank n by Drinfel'd and Jimbo (see [6] and [7]). The negative part U^- of U has a canonical basis $\mathbf{B}$ (see Kashiwara [8] and Lusztig [10, 14.4.6]) with some nice properties. For example, via action on highest weight vectors it induces bases for all of the finite-dimensional irreducible highest weight U-modules. To calculate the elements of $\mathbf{B}$ explicitly is a hard problem; it is solved completely only in types A_1 and A_2 (see [9, 3.4]). In [11], Lusztig describes a method to calculate certain elements of $\mathbf{B}$. A monomial

$$F_{i_1}^{(a_1)} F_{i_2}^{(a_2)} \cdots F_{i_m}^{(a_m)} \tag{1}$$

in U^- is said to be tight (respectively, semi-tight) if it belongs to $\mathbf{B}$ (respectively, is a nonnegative integer linear combination of elements of $\mathbf{B}$). Let W be the Weyl group corresponding to $\mathbf{g}$, with Coxeter generators $s_1, s_2, \ldots, s_n$, and let $w_0 = s_{i_1} s_{i_2} \cdots s_{i_m}$ be a reduced expression for the longest element in W. Fix an orientation of the Dynkin diagram of $\mathbf{g}$. Lusztig associates a quadratic form to these data, and shows that, given certain linear conditions on $a_1, a_2, \ldots, a_m \in \mathbb{N}$, the monomial (1) is tight (respectively semi-tight) provided this quadratic form satisfies a certain positivity condition (respectively, nonnegativity condition). Lusztig shows that the positivity condition (for tightness) always holds in type A_3 and asks when we have (semi-)tightness in type A_n. We show that, for a certain orientation of the Dynkin diagram, the positivity condition is always satisfied in type A_4 (as opposed to the claim in [11]). We also demonstrate Lusztig's comment that things get more complicated in cases with higher rank in type A. In particular, we exhibit a reduced expression for w_0 in type A_r (for any $r \geq 6$) with a quadratic form that does not even satisfy the condition for semi-tightness, for any orientation of the Dynkin diagram.

In Section 2, we describe the situation we are working in, and in Section 3 we summarize the results about tight monomials from [11] that we shall need. In Section 4 we show how they can be applied to the type A_4 case, and in Section 5 we explain how some techniques of linear programming providing spanning sets (in a certain sense) for cones can be applied to understanding the quadratic form. These techniques are applied to a tractable reduced expression in type A_5. In Section 6 we give the counter-examples mentioned above, and in Section 7 we give a general description (in type A_n) of the spanning vectors used in Section 5, showing how they can be parametrized in a natural way by cycles in symmetric groups. These vectors seem to be interesting combinatorially and also should be of use in further understanding the quadratic form.

2 Preliminaries

We use the treatment in [10, §§1-3]. Let $\mathbf{g}$ be a semisimple Lie algebra, with root system Φ, simple roots $\alpha_1, \alpha_2, \ldots, \alpha_n$, and Killing form $(\ ,\ )$. Let $h_1, h_2, \ldots, h_n$ be a basis for a Cartan subalgebra $\mathbf{h}$ of $\mathbf{g}$, satisfying $(h_i, h) = \alpha_i^*(h)$ for all h in $\mathbf{h}$ and all $i \in I = \{1, 2, \ldots, n\}$. Let Y be the $\mathbb{Z}$-lattice spanned by $h_1, h_2, \ldots, h_n$. Let $\omega_1, \omega_2, \ldots, \omega_n$ be the fundamental weights of $\mathbf{g}$, defined by $\omega_i(h_j) = \delta_{ij}$, and let X be the $\mathbb{Z}$-lattice spanned by them (the weight lattice). Let d be the minimal positive integer so that $d(\alpha_i, \alpha_i)$ is always even. (Note that then $d(\alpha_i, \alpha_j)$ is always an integer, as (α_i, α_j) is always rational and $2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$ is always an integer.) If the highest common factor of the $\frac{1}{2}d(\alpha_i, \alpha_i)$ is not 1, then replace d by d divided by this highest common factor. We then define $i \cdot j$ to be $d(\alpha_i, \alpha_j)$ for each $i, j \in I$, so $(I, \cdot)$ is a Cartan datum as in [10, 1.1.1]. For $\mu \in Y$ and $\lambda \in X$, define $\langle \mu, \lambda \rangle$ to be $\lambda(\mu)$. Define an embedding of I into Y by $i \mapsto h_i$ and into X by $i \mapsto \alpha_i$ for all $i \in I$. We then have a root datum of type $(I, \cdot)$ as in [10, 2.2.1], with $\langle h_i, \alpha_j \rangle = \alpha_j(h_i) = A_{ij}$ the corresponding symmetrizable Cartan matrix. For each $i \in I$, we define d_i to be the integer $\frac{1}{2}d(\alpha_i, \alpha_i)$. Then $d_i A_{ij} = \frac{1}{2}d(\alpha_i, \alpha_i)\left(\frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}\right) = d(\alpha_i, \alpha_j)$ for each $i, j \in I$, and is thus a symmetric matrix over $\mathbb{Z}$. We use the same numbering as [2, Planches 1 to IX].

Let $\mathbb{Q}(v)$ be the field of rational functions in an indeterminate v, and $\mathcal{A} \subseteq \mathbb{Q}(v)$ the ring $\mathbb{Z}[v, v^{-1}]$. For $N, M \in \mathbb{N}$ and $i \in I$ we put $v_i = v^{d_i}$ and define the following (which all lie in $\mathcal{A}$):

$$[N]_i = \frac{v_i^N - v_i^{-N}}{v_i - v_i^{-1}}, \quad [N]_i^! = [N]_i[N-1]_i \cdots [1]_i, \quad \begin{bmatrix} M \\ N \end{bmatrix}_i = \frac{[M]_i^!}{[N]_i^![M-N]_i^!}.$$

We define the quantized enveloping algebra U corresponding to the above data (as in [10, 3.1.1 & 33.1.5]) to be the $\mathbb{Q}(v)$-algebra U with generators $1, E_1, E_2, \ldots, E_n$, $F_1, F_2, \ldots, F_n$, and K_μ for

$\mu \in Y$, subject to the relations: (for each $i, j \in I$ and $\mu, \mu' \in Y$)

$$K_0 = 1,$$

$$K_\mu K_{\mu'} = K_{\mu+\mu'},$$

$$K_\mu E_i = v^{\alpha_i(\mu)} E_i K_\mu,$$

$$K_\mu F_i = v^{-\alpha_i(\mu)} F_i K_\mu,$$

$$E_i F_i - F_i E_i = \frac{\tilde{K}_i - \tilde{K}_i^{-1}}{v_i - v_i^{-1}},$$

$$E_i F_j - F_j E_i = 0, \quad i \neq j,$$

$$\sum_{p+p'=1-A_{ij}} (-1)^{p'} \begin{bmatrix} 1-A_{ij} \\ p' \end{bmatrix}_i E_i^p E_j E_i^{p'} = 0, \quad i \neq j,$$

$$\sum_{p+p'=1-A_{ij}} (-1)^{p'} \begin{bmatrix} 1-A_{ij} \\ p' \end{bmatrix}_i F_i^p F_j F_i^{p'} = 0, \quad i \neq j,$$

(where, for $i \in I$, we put $\tilde{K}_i = K_{d_i h_i}$ and $\tilde{K}_i^{-1} = K_{-d_i h_i}$). In the last two summations, p and p' are restricted to the nonnegative integers.

We make the following definitions (see [10, 3.1.1 & 3.1.13]). For $M \in \mathbb{N}$, and $i \in I$, we put $E_i^{(M)} = E_i^M/[M]_i^!$, and $F_i^{(M)} = F_i^M/[M]_i^!$, which are called *divided powers*. Let U^+ be the $\mathbb{Q}(v)$-subalgebra of U generated by the E_i, $i \in I$. Let U^- be the $\mathbb{Q}(v)$-subalgebra of U generated by the F_i, $i \in I$. Let W be the Weyl group of $\mathbf{g}$. So W is the group:

$$W = \langle s_1, s_2, \ldots, s_n \mid s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1 \ (i \neq j) \rangle$$

where $m_{ij} = 2, 3, 4, 6$ if $A_{ij}A_{ji} = 0, 1, 2, 3$, respectively.

3 Tight Monomials

We start by describing the quadratic form mentioned in the introduction, and summarizing some results from Lusztig's paper [11] for the situation we are interested in. Let D be the Dynkin graph of the semisimple Lie algebra $\mathbf{g}$. So D has vertices I and a set of edges Ω. We take an orientation of D; i.e. two maps $h \mapsto h'$ and $h \mapsto h''$ from Ω to I such that the ends of h are h', h''. Let $\mathbf{i} = (i_1, i_2, \dots, i_m)$ be a sequence in I. We consider the monomial $u := F_{i_1}^{(a_1)} F_{i_2}^{(a_2)} \cdots F_{i_m}^{(a_m)} \in U^-$. Let $\mathbf{B} \subseteq \mathbf{U}^-$ be the canonical basis of U^- (see Kashiwara [8] or Lusztig [10, 14.4.6]). Following Lusztig, we say u is *tight* (respectively, *semi-tight*) if $u \in \mathbf{B}$ (respectively, is an integer linear combination of elements of $\mathbf{B}$). We now describe a quadratic form defined in terms of D and $\mathbf{i}$, and a linear form $L_{\mathbf{a}}$ defined in terms of D, $\mathbf{i}$ and $\mathbf{a}$, and give sufficient conditions in terms of these for u to be tight or semi-tight.

For $i \in I$ let $Z(i)$ be the set $\{j \in [1, m] : \ i_j = i\}$. Let P' be the rational vector space with coordinate functions z_i^{pq} indexed by triples (i, p, q) such that $i \in I$ and $p, q \in Z(i)$ satisfy $p \neq q$. Define z_i^{pq} to be identically zero whenever $i \in I$ and $p, q \in [1, m]$ are not both contained in $Z(i)$, or if $p = q \in Z(i)$. Let P be the subspace of P' defined by the relations:

$$\sum_{r \leq p < s} z_i^{rs} = \sum_{r \leq p < s} z_i^{sr} \tag{2}$$

for all $p \in [1, m]$. Let P^+ be the set of all $z = (z_i^{pq}) \in P$ such that $z_i^{pq} \geq 0$ for all (i, p, q), $p \neq q$. Let $P_{\mathbb{Z}}^+$ be the set of all $z = (z_i^{pq}) \in P$ such that $z_i^{pq} \in \mathbb{N}$ for all (i, p, q), $p \neq q$. Define the quadratic form $Q : P' \mapsto \mathbb{Q}$ by:

$$\begin{aligned} Q = &\sum_{i \in I} \sum_{r \leq p < s \leq q} z_i^{pq} z_i^{rs} + \sum_{i \in I} \sum_{q < s \leq p < r} z_i^{pq} z_i^{rs} \\ &- \sum_{h \in \Omega} \sum_{r \leq p < s \leq q} z_{h'}^{pq} z_{h''}^{rs} - \sum_{h \in \Omega} \sum_{q < s \leq p < r} z_{h'}^{pq} z_{h''}^{rs}. \end{aligned} \tag{3}$$

We fix $\mathbf{a} = (a_1, a_2, \dots, a_m) \in \mathbb{N}^m$. For any two indices $s < s'$ in $[1, m]$ such that $s, s' \in Z(i)$, with $p \notin Z(i)$ whenever $s < p < s'$, set

$$N(s, s') = a_s + a_{s'} + \sum_{i' \in I \setminus \{i\}} \sum_{\substack{p \in Z(i') \\ s < p < s'}} e_{ii'} a_p,$$

where $e_{ii'}$ is the number of (unoriented) edges joining i with i'. More generally, given $s < s'$ in $[1, m]$ with $s, s' \in Z(i)$, set $N(s, s') = N(s_0, s_1) + N(s_1, s_2) + \cdots + N(s_{k-1}, s_k)$ where $\{s = s_0 < s_1 < \cdots < s_k = s'\}$ are the elements of $Z(i) \cap [s, s']$ in increasing order. This is compatible with the previous definition. Consider the linear form

$$L_{\mathbf{a}} := \sum_i \sum_{s < r} N(s, r) z_i^{rs}$$

on P'. The following is a subcase of the corollary in Section 10 of [11].

Theorem 3.1 *(1) If the non-homogeneous quadratic form $Q + L_{\mathbf{a}}$ takes only positive values on $P_{\mathbb{Z}}^{+} \setminus \{0\}$, then the monomial (1) is tight.*

(2) If $Q + L_{\mathbf{a}}$ takes only nonnegative values on $P_{\mathbb{Z}}^{+}$, then (1) is semi-tight.

Remark: The forms Q and $L_{\mathbf{a}}$ arise in [11] in the context of the perverse sheaf approach to the canonical basis. Given a monomial u as above, set, for each $i \in I$, $\mathbf{V}_i$ to be a vector space of dimension $\nu(i) := \sum_{p \in Z(i)} a_p$. Let $\mathbf{V} := \oplus_{i \in I} \mathbf{V}_i$, and let E be the algebraic variety $\oplus_{h \in \Omega} \mathrm{Hom}(\mathbf{V}_{h'}, \mathbf{V}_{h''})$, depending only on ν. Let F be the variety of all flags $\phi = (0 = \mathbf{V}^m \subseteq \mathbf{V}^{m-1} \subseteq \cdots \subseteq \mathbf{V}^1 \subseteq \mathbf{V}^0 = \mathbf{V})$, where each $\mathbf{V}^p$ is an I-graded subspace of $\mathbf{V}$ such that for each p, the I-graded vector space $\mathbf{V}^{p-1}/\mathbf{V}^p$ is non-zero only in dimension i_p, where it has dimension a_p. Let $\widetilde{F}$ be the variety of all pairs (x, ϕ), where $x = (x_h) \in E$ and $\phi \in F$ as above are such that $x_h(\mathbf{V}^p_{h'}) \subseteq \mathbf{V}^p_{h''}$ for all p, h; let $\pi_1 : \widetilde{F} \to E$ be the first projection. We have a natural diagonal embedding $\widetilde{F} \subseteq \widetilde{F} \times_E \widetilde{F}$; let $\overline{F} := \pi_1(\widetilde{F})$, and let $\pi : \widetilde{F} \to \overline{F}$ be the restriction of π_1. There is a partition

$$\widetilde{F} \times_E \widetilde{F} = \sqcup_y (\widetilde{F} \times_E \widetilde{F})_y,$$

based on the relative position of two flags in F. Here y varies over a certain subset of $\mathbb{N}^c$ for some c. Lusztig defines a quadratic form $Q'(y) := \dim(\widetilde{F}) - \dim(\widetilde{F} \times_E \widetilde{F})_y$, and shows that the smallness of π (respectively, the semi-smallness of π), in the sense of Goresky and Macpherson, follows from a positivity condition (respectively, nonnegativity condition) on Q'. He then shows, using the perverse sheaf characterisation of the canonical basis, that u is tight (respectively, semi-tight), if π is small (respectively, semi-small), and then deduces Theorem 3.1 from the above by a change of variables in Q'.

We write the conditions in Theorem 3.1 in terms of weak positivity and weak nonnegativity. Let l be a positive integer and $S \subseteq \mathbb{Q}^l$. A quadratic form f on $\mathbb{Q}^l$ is said to be *weakly nonnegative* (respectively, *weakly positive*) on S if for all $x = (x_1, x_2, \ldots, x_l) \in S$, with each $x_i \geq 0$ (respectively, each $x_i \geq 0$ and $x \neq 0$), we have $f(x) \geq 0$ (respectively, $f(x) > 0$). If S is the whole of $\mathbb{Q}^l$ we say f is weakly nonnegative (respectively, weakly positive). We have:

Corollary 3.2 *Suppose that* $\mathbf{a} \in \mathbb{N}^m$ *and* $L_{\mathbf{a}}(z) \geq 0$ *for all* $z \in P_{\mathbb{Z}}^+$.

(a) If Q *weakly positive on* P *then the monomial (1) is tight.*

(b) If Q *is weakly nonnegative on* P *then the monomial (1) is semi-tight.*

Proof: The conditions in the corollary easily imply the conditions in the theorem. □

Remark: Note that Q takes only positive values on $P^+ \setminus \{0\}$ (i.e. is weakly positive on P) if and only if it takes only positive values on $P_{\mathbb{Z}}^+ \setminus \{0\}$. Clearly the latter is necessary for Q to be weakly positive on P, so suppose that Q takes only positive values on $P_{\mathbb{Z}}^+ \setminus \{0\}$, and let $z \in P^+ \setminus \{0\}$. For some positive integer t, $tz \in P_{\mathbb{Z}}^+ \setminus \{0\}$, so $Q(tz) > 0$, whence $Q(z) = 1/(t^2)Q(tz) > 0$, so Q is weakly positive on P. Similarly, Q takes only nonnegative values on P^+ if and only if it takes only nonnegative values on $P_{\mathbb{Z}}^+$. Note also that in case (b), if $L_{\mathbf{a}}(z) > 0$ for all $z \in P_{\mathbb{Z}}^+ \setminus \{0\}$, then the monomial (1) is actually tight. Thus it doesn't matter if we work over $\mathbb{Z}$ or over $\mathbb{Q}$.

In [11], Lusztig shows the following:

Proposition 3.3 *Suppose* $\mathbf{g}$ *is of type* A_n *with* $n \leq 3$, *and* $\mathbf{i} = (i_1, i_2, \ldots, i_m)$ *corresponds to a reduced expression for the longest word in the Weyl group. Then there exists an orientation of the Dynkin diagram for which the corresponding quadratic form* Q *is weakly positive on* P. *Thus if* $\mathbf{a} \in \mathbb{N}^m$ *and* $L_{\mathbf{a}}(z) \geq 0$ *for all* $z \in P_{\mathbb{Z}}^+$, *the monomial (1) is tight.* □

Remark: Suppose that for every pair $i_s, i_{s'}$, with $s, s' \in \{1, 2, \ldots, m\}$, $i_s = i_{s'}$ and such that $i_p \neq i_s$ whenever $s < p < s'$, we have that

$$\sum_p a_p \geq a_s + a_{s'},$$

where the sum is over all p with $s < p < s'$ which are joined with i in the Dynkin diagram. Lusztig remarks in [11, §16] that if this is so, then $L_{\mathbf{a}}(z) \geq 0$ for all $z \in P_{\mathbb{Z}}^+$. He asks under what circumstances the corresponding monomial (1) is tight or semi-tight. In the next sections we'll provide an answer in type A_4 and investigate what happens in higher cases.

4 Some Tight Monomials in Type A_4

We now assume $\mathbf{g}$ is of type A_4, and we fix an orientation of D where the edge h joining vertices i and $i+1$ has $h' = i$ and $h'' = i+1$ for $i = 1, 2, 3$. We also assume that we have $\mathbf{a} \in \mathbb{N}^m$ satisfying $L_{\mathbf{a}}(z) \geq 0$ for all $z \in P_{\mathbb{Z}}^{+}$. We then test condition (a) of Corollary 3.2. A simple approach to this is to take the quadratic form Q on P' and eliminate z_i^{pq}'s using the relations (2) (that is, for each relation, eliminate one z_i^{pq}). One thus obtains a new quadratic form Q_1 in some subset of the variables z_i^{pq} (denote this subset by S). Note that Q_1 depends on a choice of the variables eliminated. Suppose that Q_1 is found to be weakly positive (as a function of the variables in S). It is then clear that condition (a) is satisfied. This is the approach used for some cases in [11]. However, we should note that it is not necessarily the case that Q_1 is weakly positive if condition (a) is satisfied — when testing the weak positivity of Q_1 we don't assume that the eliminated variables are nonnegative.

Proposition 4.1 *Suppose* $\mathbf{g}$ *is of type* A_4*, the orientation of* D *is as above, and* $\mathbf{i} = (i_1, i_2, \ldots, i_m)$ *is a sequence in* I *such that* $s_{i_1} s_{i_2} \cdots s_{i_m}$ *is a reduced expression for the longest word in the Weyl group. Suppose* $\mathbf{a} \in \mathbb{N}^m$ *is such that* $L_{\mathbf{a}}(z) \geq 0$ *for all* $z \in P_{\mathbb{Z}}^{+}$*. Then condition (a) is satisfied for the quadratic form* Q *corresponding to* $\mathbf{i}$*, and we can conclude that the corresponding monomial (1) is tight.*

Proof: First a simple program in Maple [14] was written which would produce a list of the reduced expressions for the longest word in the Weyl group. Let $\sim$ be the equivalence relation on the set of reduced expressions for the longest word defined by setting $w \sim w'$ if there exists a sequence of commutations taking w to w'. (A commutation is a relation of the form $s_i s_j = s_j s_i$ where i and j are not connected in the Dynkin diagram). We only have to take a list of equivalence class representatives for $\sim$ and test condition (a) for each element in this list, since it is clear if condition (a) is satisfied for one element in an equivalence class, it is satisfied for all. The idea was to use a theorem of Matsumoto (see [5, 64.20]):

Theorem 4.2 *Suppose that $s_{i_1}s_{i_2}\cdots s_{i_t} = s_{j_1}s_{j_2}\cdots s_{j_t}$ are two reduced expressions for an element $w \in W$, where W is a Weyl group with Coxeter generators s_i. Then there exists a finite sequence of braid relations which, when applied to the first expression, in order, gives the second.* $\square$

The algorithm for the program was as follows:

(1) Input: a list L consisting of the single element $s_1s_3s_2s_4s_1s_3s_2s_4s_1s_3$, a reduced expression for the longest word in the Weyl group W of type A_4.

(2) For each element $s_{i_1}s_{i_2}\cdots s_{i_m}$ of L find all subsequences i_a, i_b, i_c of $i_1, i_2, \ldots, i_m$ such that commutations can be applied to $s_{i_1}s_{i_2}\cdots s_{i_m}$ to produce a word containing the sequence $s_{i_a}s_{i_b}s_{i_c}$, and such that $i_a = i_c$ and i_a is joined to i_b in the Dynkin diagram. For each of these, apply the corresponding long braid relation $s_{i_a}s_{i_b}s_{i_a} = s_{i_b}s_{i_a}s_{i_b}$ to produce a new reduced expression for the longest word; call this new set of words M.

(3) Now pass through the words in M one by one. For each one, test for equivalence with each element of L; if it is not equivalent to any element of L, add it to L.

(4) If no new element was added to L in step (3), we are done; the algorithm ends and L is the list we want. Otherwise return to (2).

Then a Maple program was written that would calculate for each reduced expression in our list, the quadratic form Q. The method described at the start of this section was then applied. For each i, p, we used the relation

$$\sum_{r \le p < s} z_i^{rs} = \sum_{r \le p < s} z_i^{sr}$$

to eliminate z_i^{tu}, where t was the smallest element of $Z(i)$ and u was the smallest element of $Z(i)$ satisfying $p < u$. Note that as p varies over $[1, m]$ the same relation will appear several times; we of course ignored repetitions. Thus we obtained a quadratic form Q_1. It turned out that in each case, the quadratic form Q_1 was a *unit form,* i.e. for each variable z_i^{rs} not eliminated, the term in

$(z_i^{rs})^2$ in Q_1 was exactly $(z_i^{rs})^2$. For such a form, there is an algorithm to check for weak positivity — see [1], which is implemented in Crep [4], which runs via Maple. This algorithm was used to show that in each case, Q_1 is weakly positive. □

Note that to fix the orientation of the Dynkin diagram D is not a severe restriction, as we have a result independent of the orientation (i.e. the tightness of the corresponding monomial).

5 Type A_5

The same method applied to some reduced expressions for the longest word in case A_5 did not work — the quadratic form Q_1 was not weakly positive. Various other substitutions were attempted, without success. Note that such a result tells us nothing about the weak positivity of Q on P. A method is required which will give us an answer in these cases. Dr. H. von Höhne suggested using linear programming techniques to deduce from Q a quadratic form Q' which is weakly positive (respectively, weakly nonnegative) if and only if Q is weakly positive (respectively, weakly nonnegative) on P. These techniques were found in [12]. The author would like to thank Dr. von Höhne for some useful discussions on the problem. It will be seen that for the reduced expression $w_0 = s_1s_3s_5s_2s_4s_1s_3s_5s_2s_4s_1s_3s_5s_2s_4$ for the longest word in the Weyl group of type A_5 this corresponding quadratic form is weakly nonnegative for the orientation of D given by $h' = i$ and $h'' = i+1$ if h is the edge joining vertices i and $i+1$ and that for any orientation of D the quadratic form is not weakly positive. In the sequel we start in a more general setup, with $\mathbf{g}$ of type A_n, take the orientation of D just mentioned and $\mathbf{i} = (i_1, i_2, \ldots, i_m)$ a sequence in I.

We have the quadratic form Q corresponding to $\mathbf{i}$, defined on $P' = \mathbb{Q}^l$ for some l, and the subspace P of $\mathbb{Q}^l$ defined by certain relations on the coefficients. We want to look at the values of Q on the set of points of P where all of the coefficients are nonnegative, so we must understand this set. Let $z \in P'$; then the relations can be written in the form $Az = 0$, where A is an integer matrix. We wish to understand the set $P^+ = \{z \in P' : \ z \geq 0, Az = 0\}$ (where $z \geq 0$ means each component of z is nonnegative).

Using linear programming techniques, we can find a finite set of vectors $v_1, v_2, \ldots v_s$ in P^+ so that

$$P^+ = \{\sum_{j=1}^{s} \lambda_j v_j \ : \ \lambda_j \in \mathbb{Q}, \lambda_j \geq 0\}.$$

We summarize how this works.

To simplify things a bit, we fix $i \in I$ and consider only the subspace P'_i of P' given by the relations $z_j^{rs} = 0$ for $j \neq i$. This has coordinate functions z_i^{rs} for $r, s \in Z(i)$, $r \neq s$. We impose the relations involving the z_i^{rs} and call this subspace P_i. We denote a point in P'_i or P_i by the vector $z = (z_i^{rs})$, where r,s vary over $Z(i)$, with $r \neq s$. We investigate the set $P_i^+ := \{z \in P'_i \ : \ z \geq 0, \ A_i z = 0\}$, where A_i is the matrix of the relations involving only the z_i^{rs}. For ease of notation, we relabel $Z(i) = \{1, 2, \ldots, k\}$. This makes no difference to the resulting calculations. We order the $k(k-1)$ coordinates thus: $z_i^{12}, z_i^{13}, \ldots, z_i^{1k}, z_i^{23}, z_i^{24}, \ldots, z_i^{2k}, \ldots, z_i^{k-1,k}$, $z_i^{21}, z_i^{31}, \ldots, z_i^{k1}, z_i^{32}, z_i^{42}, \ldots, z_i^{k2}, \ldots, z_i^{k,k-1}$. With this ordering, the matrix $A_i = (C_i \ \ -C_i)$, where C_i is the $k-1$ by $\frac{1}{2}k(k-1)$ matrix given by:

$$C_i = \begin{pmatrix} 1 & 1 & \ldots & 1 & & & & & & & & \\ & 1 & \ldots & 1 & & & & & & & & \\ & & \vdots & & & \ldots & 1 & 1 & 1 & & & \\ & & & & & & & 1 & 1 & 1 & 1 & \\ & & & 1 & & & & & 1 & & 1 & 1 \end{pmatrix},$$

where an entry which doesn't appear is taken to be zero. We see that A_i is a $k-1$ by $k(k-1)$ matrix, of rank $k-1$. Add a row to the bottom of A_i consisting entirely of $1's$, to give a matrix B_i of rank k. An *extreme homogeneous solution* of our problem is one obtained in the following manner. Take k columns of B_i such that the corresponding $k \times k$ matrix X is invertible. Set $z_i^{rs} = 0$ if it does not correspond to one of these columns and solve the equation $B_i z = (0, 0, \ldots, 0, 1)^t$ with this restriction — it has a unique solution by the invertibility of X. This is said to be an extreme homogeneous solution if each $z_i^{rs} \geq 0$. Then we have:

Lemma 5.1 *Either $P_i^+ = \{0\}$ (in the case there are no extreme homogeneous solutions) or P_i^+ is the set of nonnegative (rational) linear combinations of the extreme homogeneous solutions.*

Proof: see [12, Lemma 1, P116].

We have thus solved our problem for P_i^+. We consider some small cases. If $k = 1$, there are no coordinates z_i^{rs}. If $k = 2$, $A_i = (1 \ \ -1)$ and we get the single vector given by $z_i^{12} = z_i^{21} = \frac{1}{2}$. If $k = 3$ we obtain the vectors which are the transposes of: $(\frac{1}{2}, 0, 0, \frac{1}{2}, 0, 0), (0, \frac{1}{2}, 0, 0, \frac{1}{2}, 0), (0, 0, \frac{1}{2}, 0, 0, \frac{1}{2})$, $(\frac{1}{3}, 0, \frac{1}{3}, 0, \frac{1}{3}, 0)$ and $(0, \frac{1}{3}, 0, \frac{1}{3}, 0, \frac{1}{3})$. This is easy to see merely by considering the order-3 subsets of the set of columns of A_i. In fact the vectors obtained have a nice description in the general case — see §7. By multiplication by a suitable scalar (in this case 2 or 3) we make all of these vectors integer vectors; note that this does not affect their nonnegative rational linear span.

So for each i we have k_i vectors $v_{i,1}, v_{i,2}, \ldots, v_{i,k_i}$ such that $P_i^+ = \{\lambda_1 v_{i,1} + \lambda_2 v_{i,2} + \cdots + \lambda_{k_i} v_{i,k_i} \ : \ \lambda_j \in \mathbb{Q}, \lambda_j \geq 0\}$. It is clear that we have $P^+ = \{\sum_{i \in I} \sum_{j=1}^{k_i} \lambda_{ij} v_{ij} \ : \ \lambda_{ij} \in \mathbb{Q}, \lambda_{ij} \geq 0\}$. For each $\lambda = (\lambda_{ij})$, with each $\lambda_{ij} \in \mathbb{Q}$ (where for each i, j runs from 1 to k_i), define $Q'(\lambda) = Q(\sum_{i \in I} \sum_{j=1}^{k_i} \lambda_{ij} v_{ij})$. Then we have:

Lemma 5.2 *Q is weakly nonnegative (respectively, weakly positive) on P if and only if Q' is weakly nonnegative (respectively, weakly positive).* $\square$

We now set $\mathbf{g}$ to be of type A_5 and fix $\mathbf{i} = (1, 3, 5, 2, 4, 1, 3, 5, 2, 4, 1, 3, 5, 2, 4)$. Note that here $|Z(i)| = 3$ for all i, so we only need the case $k = 3$. Using the ordering of vectors given above, the 25×25-matrix of the quadratic form Q' was calculated (with Maple), and found to be as follows:

$$X := \begin{pmatrix} A & -B & 0 & 0 & 0 \\ -B^t & A & -B^t & 0 & 0 \\ 0 & -B & A & -B & 0 \\ 0 & 0 & -B^t & A & -B^t \\ 0 & 0 & 0 & -B & A \end{pmatrix},$$

where the zeros stand for 5×5 zero matrices,

$$A = \begin{pmatrix} 2 & 1 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 & 2 \\ 2 & 2 & 2 & 4 & 2 \\ 2 & 2 & 2 & 2 & 4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix}.$$

We first note that Q is not weakly positive on P, since if $z_1^{16} = z_1^{61} = 1$, $z_2^{49} = z_2^{94} = 1$, $z_2^{4,14} = z_2^{14,4} = 1$, $z_3^{27} = z_3^{72} = 2$, $z_4^{5,10} = z_4^{10,5} = 1$, $z_4^{5,15} = z_4^{15,5} = 1$, $z_5^{38} = z_5^{83} = 1$, and all other coordinates are set to zero, then $z \in P^+$ and $Q(z) = 0$. This was found with the help of Maple and Crep. Note that this z is invariant under the map taking all z_i^{rs} to z_i^{sr}. It is clear from the definition of the quadratic form that if we used any orientation of the Dynkin diagram, the value of the quadratic form on such a z would be the same. Thus for any orientation of D the corresponding quadratic form is not weakly positive. We can represent any such $z \in P^+$ in pictorial form in the following manner:

Draw a graph containing $|I|$ rows of vertices, such that for each $i \in I$ the ith row contains one vertex corresponding to each element of $Z(i)$. The vertices are arranged in such a manner so that if $r \in Z(i)$ and $s \in Z(i+1)$ then the vertex corresponding to the smaller of r and s appears to the left of the vertex corresponding to the larger of the two. If $z_i^{rs} = z_i^{sr}$ is non-zero then an edge is drawn between the vertices (in row i) corresponding to r and s, and it is labelled with the value of z_i^{rs} (we omit the label if $z_i^{rs} = 1$).

Such a representation makes it easy to calculate $Q(z)$ (for our fixed orientation). Pairs r, s in $[1, m]$ should be thought of as intervals; contributions to $Q(z)$ come from certain types of intersections of these intervals (thanks are due to Dr. von Höhne for the idea of thinking in terms of intervals). For example, to calculate the contribution to Q from the first of the four terms in the definition, we should calculate for each i the product of the labels on edges corresponding to pairs of intervals $([p, q], [r, s])$ appearing in the ith row of the pictorial representation which intersect in such a way that we have $r \leq p < s \leq q$. This includes identical pairs of intervals (which don't appear in any of the other terms). Then we should add up the results coming from each $i \in I$. The values of the other three terms can be calculated in a similar manner. In this example the picture is as in Figure 1.

Note the diagram indicates there is an edge from 4 to 14. (We won't draw any z's where there are edges corresponding to intervals $[p, q]$ and $[r, s]$ with $q = r$.) Now we calculate the value of the first term of Q. We get $6 \times 1^2 + 2^2 = 10$ from the identical pairs of intervals. The interval $[4, 9]$

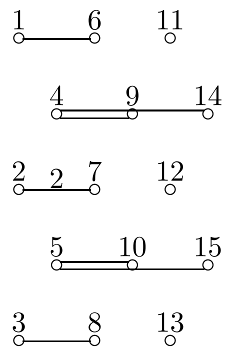

Figure 1

intersects the interval $[4, 14]$ (in the correct way), and the interval $[5, 10]$ intersects the interval $[5, 15]$. Thus we get a total of $10 + 2 = 12$ for the first term. The contribution from the second term is 0 — no pairs of intervals corresponding to a fixed i intersect in the correct way. Since there are no loops in the Dynkin diagram, the third and fourth terms of Q can be written as

$$-\sum_{h\in\Omega}\sum_{r<p<s<q} z^{pq}_{h'} z^{rs}_{h''} - \sum_{h\in\Omega}\sum_{q<s<p<r} z^{pq}_{h'} z^{rs}_{h''}. \tag{4}$$

We thus get a contribution to this for every pair of intervals in adjacent rows which intersect such that neither completely contains the other. (Note that we cannot have s=p as the sets $Z(i)$ are disjoint). In our example we have a contribution of -1 from each of 12 intersections, a total of -12. Thus we have $Q(z) = 12 - 12 = 0$.

So, Q is not weakly positive on P, but the question remains as to whether it is weakly nonnegative on P. We need a criterion for checking that a quadratic form is weakly nonnegative, so we can check if this condition holds for Q' (and thus for Q on P). We use the following:

Theorem 5.3 *Suppose f is a quadratic form and $M = M^t$ is the matrix of f. Then f is weakly nonnegative if and only if for every principal submatrix D of M with* $\det(D) < 0$ *there is a cofactor of the last column of D which is negative.*

Proof: This is due to E. Keller and cited without proof as Theorem 4.2 in [3]; see [15, 4.7] for a proof. Note that copositivity as referred to in these papers is the same as weak nonnegativity. □

Let M be the matrix of Q'; the rank of M is 18 (using Maple). Therefore we need only check principal submatrices of order ≤ 18. Also note that the following vectors are in the kernel of M (again from Maple):

$$(1,1,1,-1,-1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0),$$

$$(0,0,0,0,0,1,1,1,-1,-1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0),$$

$$(0,0,0,0,0,0,0,0,0,0,1,1,1,-1,-1,0,0,0,0,0,0,0,0,0,0),$$

$$(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,1,-1,-1,0,0,0,0,0),$$

$$(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,1,-1,-1),$$

$$(1,0,-1,0,0,0,1,-1,0,0,0,0,0,0,0,0,-1,1,0,0,-1,0,1,0,0),$$

$$(0,-1,1,0,0,1,-1,0,0,0,0,0,0,0,0,-1,1,0,0,0,0,1,-1,0,0).$$

Therefore, since the first vector lies in the kernel, any principal submatrix corresponding to a subset of $\{1,2,\ldots,25\}$ which contains $\{1,2,3,4,5\}$ will have determinant zero, and needn't be checked. Similarly with the other vectors. A computer program in the C programming language was written to test the condition in the theorem for Q', using the short cuts described here. The algorithm used cycled through all subsets of $\{1,2,\ldots,25\}$ of size ≤ 18 and checked the corresponding submatrix if the subset didn't contain one of the subsets corresponding to the kernel vectors above. Determinants were calculated using row reduction. In the end Q' was found to be weakly nonnegative. We conclude:

Proposition 5.4 *Suppose that* $\mathbf{g}$ *is of type* A_5*, and* $\mathbf{i} = (1,3,5,2,4,1,3,5,2,4,1,3,5,2,4) \in \mathbb{N}^{15}$*. Suppose* $\mathbf{a} \in \mathbb{N}^{15}$ *is such that* $L_{\mathbf{a}}(z) \geq 0$ *for all* $z \in P^+$*. Then condition (2) in Theorem 3.1 is satisfied by* $Q + L_{\mathbf{a}}$*, so we can conclude that the corresponding monomial* $F_{i_1}^{(a_1)} F_{i_2}^{(a_2)} \cdots F_{i_m}^{(a_m)}$ *is semi-tight.* $\square$

Note: If we have $L_{\mathbf{a}}(z) > 0$ for all $z \in P^+$ then in fact condition (1) is satisfied by $Q + L_{\mathbf{a}}$ and we can conclude in this case that the corresponding monomial is tight.

We consider now the sequence $\mathbf{i} = (1,2,1,3,2,1,4,3,2,1,5,4,3,2,1)$ which corresponds to a reduced expression for the longest word in the Weyl group. Define $z \in P^+$ using the picture in Figure 2 as we did in the previous case.

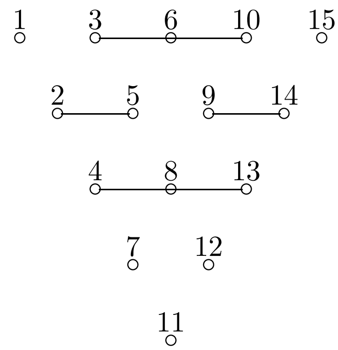

Figure 2

So $z_1^{3,10} = z_1^{10,3} = 1$, $z_2^{2,5} = z_2^{5,2} = 1$, $z_2^{9,14} = z_2^{14,9} = 1$, $z_3^{4,13} = z_3^{13,4} = 1$ and all other coordinates are set to zero. Then a calculation as before shows $Q(z) = 0$, so Q is at best weakly nonnegative on P. (Note that as above the fact that always $z_i^{rs} = z_i^{sr}$ ensures this result is independent of orientation). However to check if Q is weakly nonnegative or not on P using Theorem 5.3 as before would take too long because of the size of the matrix of the quadratic form involved.

6 Counter-examples

We now demonstrate that there are cases where Q is not even weakly nonnegative on P. We set $\mathbf{g}$ to be of type A_6 and take our usual orientation of the Dynkin diagram. We consider the sequence $\mathbf{i} = (1,2,1,3,2,1,4,3,2,1,5,4,3,2,1,6,5,4,3,2,1)$, which corresponds to a reduced expression for the longest element in the Weyl group. We look at the corresponding quadratic form Q. and define z using the picture in Figure 3.

So: $z_1^{3,6} = z_1^{6,3} = 1$, $z_1^{10,15} = z_1^{15,10} = 1$, $z_2^{5,14} = z_2^{14,5} = 2$, $z_3^{4,8} = z_3^{8,4} = 1$, $z_3^{13,19} = z_3^{19,13} = 1$, $z_4^{7,18} = z_4^{18,7} = 1$, and all other coordinates are set to zero. Then we have $z \in P^+$ and $Q(z) = -1$.

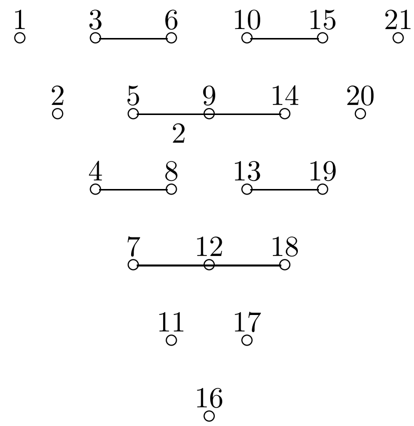

Figure 3

So Q is not weakly nonnegative on P. Now fix any $\mathbf{a} \in \mathbb{N}^m$, and let $L = L_{\mathbf{a}}(z)$. We have, for $t \in \mathbb{N}$, $L_{\mathbf{a}}(tz) + Q(tz) = tL - t^2$, which is negative for t large enough, so we see that in this case condition (2) of Theorem 3.1 fails. Note that again because this z is invariant under the map taking all z_i^{rs} to z_i^{sr}, for any orientation of D the corresponding quadratic form is not weakly nonnegative.

The above examples can clearly be generalized, with the help of the pictures. For example, suppose U is the quantum group of type A_n and $\mathbf{i} = (i_1, i_2, \dots, i_m)$ is a sequence (for example, corresponding to a reduced expression for the longest word in the Weyl group), such that the corresponding graph of the $Z(i)$'s admits such a construction as in the example above. Then the corresponding quadratic form (for any orientation of D) Q will not be weakly nonnegative on the subspace P given by the relations (2) (and condition (2) in Theorem 3.1 will fail). This includes, for example, in type A_n, $n \geq 6$, the sequence $\mathbf{i} = (1, 2, 1, 3, 2, 1, \dots, r, r-1, \dots, 2, 1)$ (which corresponds to a reduced expression for the longest word in the Weyl group). In this case the picture is a triangle with r points in the first row, $r - 1$ in the second, and so on.

This also includes, in type A_n, $n \geq 6$, the reduced expression for the longest word which is obtained in the following way. Let O be the sequence $(1, 3, 5, \dots, o)$, where o is the largest odd number smaller than or equal to n, and let E be the sequence $(2, 4, 6, \dots, e)$, where e is the largest even number smaller than or equal to n. Let OE be the concatenation of the two. If n is even, let $\mathbf{i}$ be the sequence obtained by writing OE $\frac{1}{2}n$ times followed by O, and if n is odd let $\mathbf{i}$ be the sequence obtained by writing OE $\frac{1}{2}(n+1)$ times. Then $\mathbf{i}$ corresponds to a reduced expression for the longest word in the Weyl group. Using the above construction, the corresponding quadratic form is not weakly nonnegative for any orientation of D. Thus we see that in higher cases, things do not work out nicely (although it may turn out that the corresponding monomials are still tight).

7 Cycles and Spanning Vectors

In this section we give a general description (for type A_n) of a set of vectors whose nonnegative rational linear span is P_i^+; existence is guaranteed by Lemma 5.1. Such a description should be helpful in further understanding the quadratic form Q and its application in the theory of Lusztig, as well as being interesting in its own right.

Recall that

$$P_i^+ := \{z \in P_i' \; : \; z \geq 0, A_i z = 0\},$$

where P_i' is the subspace of P' spanned by those z_j^{rs} for which $j = i$, and A_i is the matrix given in the previous section. Note that we simplify matters by relabelling $Z(i)$ as $\{1, 2, \ldots, k\}$. We start with the matrix B_i, which is a k by $k(k-1)$ matrix, and we must look at the $k \times k$ submatrices of B_i with non-zero determinant. We first need the following:

Definition 7.1 *Let M be a (not necessarily square) rational matrix. Then M is said to be* totally unimodular *if every subdeterminant of M is ± 1 or zero.*

Lemma 7.2 *The matrix A_i is totally unimodular.*

Proof: Note that each column of A_i consists of a string of 0's, followed by a string of 1's, followed by a string of 0's (where the strings can be empty). By Example 7 on page 279 of [13], we conclude that A_i is unimodular. $\square$

Proposition 7.3 *Let x be an extreme homogeneous solution of our problem. Then x is a positive multiple of a vector with entries in $\{0,1\}$. Furthermore, if S is the set of columns of A_i corresponding to the non-zero entries of x, then $\sum_{c\in S} c = 0$ (this is clear) and if $\sum_{c\in S'} c = 0$ for some non-empty subset $S' \subseteq S$, then $S' = S$.*

Proof: Let M be a $k \times k$ submatrix of B_i leading to an extreme homogeneous solution x; suppose that M is constructed from the columns $c_{p_1}, c_{p_2}, \ldots, c_{p_k}$ of A_i together with a bottom row of 1's. Let $y = (x_{p_1}, x_{p_2}, \ldots, x_{p_k})$. We know that $\det(M) \neq 0$ and that y is the unique vector satisfying $My = (0, 0, \ldots, 0, 1)^t$. Now let us use Cramer's Rule to calculate $y = (y_1, y_2, \ldots, y_k)$. To calculate y_j, we take the matrix M and replace the jth column by $(0, 0, \ldots, 0, 1)^t$. Then y_j is the determinant of this new matrix, divided by the determinant of M. Because A_i is totally unimodular, the new matrix has determinant in $\{0, \pm 1\}$ (as it is plus or minus the determinant of a square submatrix of A_i), whence each y_j is $\pm 1/\det(M)$ or is zero. But because x is an extreme homogeneous solution, each y_j must be nonnegative and must therefore be either zero or $1/|\det(M)|$. We also have $x_t = 0$ if $t \notin \{p_1, p_2, \ldots, p_k\}$, so we have proved the first part of the proposition.

Next, let S be the set of columns of A_i corresponding to the non-zero entries of x. Then it is clear that $\sum_{c\in S} c = 0$. Suppose $\phi \neq S' \subseteq S$, $S' \neq S$ and $\sum_{c\in S'} c = 0$. Then also $\sum_{c\in S\setminus S'} c = 0$. Now let's calculate the determinant of M by expanding along the bottom row (consisting entirely of 1's). We get that $\det(M)$ is the sum of (plus or minus) the determinants of the matrices obtained by taking various subsets (of size $k-1$) of the columns $\{c_{p_1}, c_{p_2}, \ldots, c_{p_k}\}$ of A_i. But each such subset must completely contain S or S', so each such determinant is zero. We conclude that $\det(M) = 0$, a contradiction. □

We would thus like to calculate the sets of up to k columns of A_i whose sum is zero, such that no non-empty proper subset has zero sum. Note that there is a correspondence between the positive columns of A_i and the intervals contained in $\{1, \ldots, k-1\}$, given by taking a column to the subset consisting of the numbers of the rows where 1's appear in the column. There is a similar correspondence for the negative columns. In the sequel, the identity permutation is considered not to be a cycle. The sets of columns we are interested in can be described in the following manner:

Proposition 7.4 *Let S_k be the symmetric group on $\{1, 2, ..., k\}$, and let π be a cycle in S_k. Let E be the set of all positive columns of A_i corresponding to the (integer) intervals $[i, \pi(i) - 1]$, for $i \in \{1, 2, ..., k\}$ with $\pi(i) > i$. Let F be the set of all negative columns of A_i corresponding to the (integer) intervals $[\pi(i), i - 1]$, for $i \in \{1, 2, ..., k\}$ with $\pi(i) < i$. Then $E \cup F$ is a set of at most k columns whose sum is zero, such that no proper subset has zero sum, and in fact every such set of columns arises in this way.*

Proof: Let S be a set of columns from the matrix with zero sum such that no proper subset has zero sum. Let E be the set of positive columns in S and F^- the set of negative columns in S. Let F be the set of positive columns corresponding to the negative columns in F^-. We will identify positive columns by their corresponding intervals (of 1's), and talk about sums of intervals to mean sums of the corresponding positive columns. The fact that no proper subset of S has zero sum is equivalent to:

$(*)$ we can find no pair of non-empty subsets $E_1 \subseteq E$ and $F_1 \subseteq F$ such that the sum of the intervals in E_1 is equal to the sum of the intervals in F_1, unless $E_1 = E$ and $F_1 = F$.

Lemma 7.5 *If $[i, j]$ is an interval in E (respectively, F) then no other interval in E (respectively, F) has starting point i or finishing point j. If $[i, j]$ is an interval in E (respectively, F) then no interval in F (respectively E) has starting point $j + 1$, or finishing point $i - 1$. That is, no pair of intervals in one of the sets can have a common starting point or end point, and it is not possible for a pair of intervals, one from E and one from F, to have an empty intersection but have their union equal to an interval.*

Proof: This lemma is the major step in proving the proposition. We will construct a 'loop' in S. First, we define a directed graph, G, with vertices V given by

$$\{(c, E) \,:\, c \in E\} \,\dot{\cup}\, \{(c, F) \,:\, c \in F\}$$

In the notation for vertices, E and F are treated merely as two distinct symbols. The edges are defined as follows:

(a) $([i,j],E) \longrightarrow ([j+1,k],E)$,

(b) $([j+1,k],F) \longrightarrow ([i,j],F)$,

(c) $([i,j],E) \longrightarrow ([k,j],F)$,

(d) $([i,k],F) \longrightarrow ([i,j],E)$,

in each case for all $i,j,k \in \{1,2,\dots,k-1\}$ such that the two vertices exist.

It will turn out that this graph is merely a single cycle on its vertices. Suppose first that G contains no (directed) cycles. The first claim is that if $([i,j],P)$ (where $P=E$ or F) is a vertex of the graph, there must be at least one vertex connected to it, with the arrow going from $([i,j],P)$ to the new vertex. Suppose no such new vertex exists and, firstly, that $P=E$. Then by (a) and (c) there is

(i) no interval $[j+1,k]$ in E, and

(ii) no interval in F of the form $[k,j]$.

We conclude that the $(j+1)$th entry in the sum of the columns in E is less than the jth entry by at least 1. This is because any interval in E containing $j+1$ must also contain j, by (i). We now consider what is happening in F at this point. It must be true that at least one interval finishes also at j, since the entries in the sum of the columns in F are the same as the entries in the sum of the columns in E. This is a contradiction, by (ii). The argument is entirely similar if $[i,j]$ is an interval in F. Thus we have an arrow leading from $([i,j],P)$ to another vertex.

But now we can start at $([i,j],P)$, find an edge $([i,j],P) \longrightarrow ([i_1,j_1],P_1)$, and repeat the process. But as we have assumed that G contains no cycles, we never repeat a vertex in this chain, a contradiction as S is finite.

We conclude G contains at least one cycle. Let $V' \subseteq V$ be the set of vertices in a minimal cycle in G, and G' the corresponding full subgraph. By minimality the graph G' must merely be a cycle on its vertices; there can be no arrows other than those in the cycle. Let $E' := \{c \in \mathrm{E} \colon (c,E) \in V'\}$ and let $F' := \{c \in \mathrm{F} \colon (c,F) \in V'\}$.

We now show that the lemma is true for the pair E', F'. For, suppose that $[i, j]$ and $[i, j']$ both occurred in E' (with j, j' distinct). Then since G' is just a cycle, there must be a unique $([a, b], P)$ in V' such that $([a, b], P) \longrightarrow ([i, j], E)$. But if $P = E$, then by definition of edges, $b = i - 1$ and $([a, b], E) \longrightarrow ([i, j'], E)$ is also an edge, a contradiction to the structure of G'. Similarly, if $P = F$, then by the definition of edges, $a = i$, and we also have the edge $([a, b], F) \longrightarrow ([i, j'], E)$, a contradiction to the structure of G'.

Suppose next that $[i, j]$ and $[i', j]$ both occurred in E' (j and j' distinct). Let $([a, b], P)$ be a vertex in V' so that $([i, j], E) \longrightarrow ([a, b], P)$ is an edge. If $P = E$, then $a = j + 1$ and $([i', j], E) \longrightarrow ([a, b], E)$ is also an edge. If $P = F$, then $b = j$ and $([i', j], E) \longrightarrow ([a, b], F)$ is also an edge. In both cases we get a contradiction to the structure of G'.

The arguments for the corresponding cases in F' are very similar. Next suppose that $[i, j]$ is an interval in E' and $[j + 1, k]$ is an interval in F'. Let $([a, b], P)$ be a vertex in V' such that $([j + 1, k], F) \longrightarrow ([a, b], P)$. If $P = E$ we must have $j + 1 = a$ and thus $([i, j], E) \longrightarrow ([a, b], E)$ is an edge. If $P = F$ we must have $b = j$ and thus $([i, j], E) \longrightarrow ([a, b], F)$ is an edge. In both cases we get a contradiction to the structure of G'.

Finally, suppose that $[i, j]$ is an interval in F' and $[j + 1, k]$ is an interval in E'. Let $([a, b], P)$ be a vertex in V' such that $([a, b], P) \longrightarrow ([j + 1, k], E)$. If $P = E$ then we have $b = j$ and thus $([a, b], E) \longrightarrow ([i, j], F)$ is an edge. If $P = F$ then we have $a = j+1$ and thus $([a, b], F) \longrightarrow ([i, j], F)$ is an edge. In both cases we get a contradiction to the structure of G'.

Next, the sum of the intervals in E' is equal to the sum of the intervals in F'. To see this, consider an edge $([i, j], E) \longrightarrow ([j + 1, k], E)$ in G'. We can replace this by a vertex $([i, k], E)$ (i.e. replace the vertices $([i, j], E)$ and $([j + 1, k], E)$ by $([i, j]), E)$ and replace the edge coming into $([i, j], E)$ with one going into $([i, k], E)$ from the same vertex and similarly replace the edge going out of $([j + 1, k], E)$ with an edge going out of $([i, k], E)$ and ending at the same vertex. We do this for each edge which is between two vertices of the form (c, E). We do exactly the same thing with edges in G' between vertices both of the form (c, F). Overall, we have just redrawn the graph with the same rules and the new reduced set of vertices. We are left with edges of type (c) and (d) only. Call the reduced sets of vertices V_1 and V_2. We still have a cycle. Thus if $([i,j],E)$ is now a vertex in V_1 there is in fact a unique vertex $([i,s],F)$ in V_2 and a unique vertex $([t,j],F)$ in V_2 (for some s and t). It is possible that $s=j$ and $t=i$, in the case when $V_1=\{([i,j],E)\}$ and $V_2=\{([i,j],F)\}$. It is now clear that the sum of the intervals in V_1 is the sum of the intervals in V_2: whenever an interval in V_1 starts, a unique interval in V_2 starts, and vice versa, and whenever an interval in V_1 finishes, a unique interval in V_2 finishes, and vice versa. But the sum of the intervals in V_1 is the sum of the intervals in E' and the sum of the intervals in V_2 is the sum of the intervals in F', by construction. By $(*)$, we conclude that $E'=E$ and $F'=F$. We have thus proved the lemma, since we know it already holds for E' and F'. □

We now finish the proof of the proposition. We construct a permutation π in S_k as follows. If $[i,j]$ is an interval in E $(=E')$, we define $\pi(i)$ to be $j+1$. If $[i,j]$ is an interval in F $(=F')$, we define $\pi(j+1)$ to be i. For other i we define $\pi(i)=i$. The lemma is exactly what is needed to ensure π is a well defined permutation. The proof of the lemma (the structure of the graph $G'=G$) shows that in fact π is a cycle. It is clear by construction that E and F arise from π in the same way as in the statement of the proposition.

We next show that if π is any cycle in S_k and E and F are the corresponding sets of intervals as defined in the statement of the proposition, then the sum of the intervals in E is equal to the sum of the intervals in F. Let l be the order of $\pi\in S_k$. Fix $j\in\{1,2,\dots,k-1\}$. We must check that

$$|i : i <= j \leq \pi(i)-1| = |i : \pi(i) <= j \leq i-1|. \tag{5}$$

The left hand side is the jth entry in the column sum from E, and the right hand side is the jth entry in the column sum from F. Firstly note the following:

$$\begin{aligned} i <= j \leq \pi(i)-1 \quad &\Leftrightarrow \quad i \leq j+\frac{1}{2} \leq \pi(i), \text{ and} \\ \pi(i) <= j \leq i-1 \quad &\Leftrightarrow \quad \pi(i) \leq j+\frac{1}{2} \leq i, \end{aligned}$$

since j and $\pi(i)$ are integers. Next, we draw a graph of π. A point is marked on the graph at $(t,\pi^t(1))$, for $t=0,1,\dots,l$. For $t=0,1,\dots,l-1$, a line is drawn from $(t,\pi^t(1))$ to $(t+1,\pi^{t+1}(1))$.

Now the line $y = j + \frac{1}{2}$ is drawn on the graph. Since $j + \frac{1}{2}$ is never an integer, the left hand side in (5) is the number of passes up through this line the graph makes, and the right hand side is the number of down passes. (Note that the graph has no turning points on the line $y = j + \frac{1}{2}$). It is clear that these two numbers are equal. For example, take the case $\pi = (13524)$, $j = 2$ and $k = 5$. The graph of π is as in Figure 4.

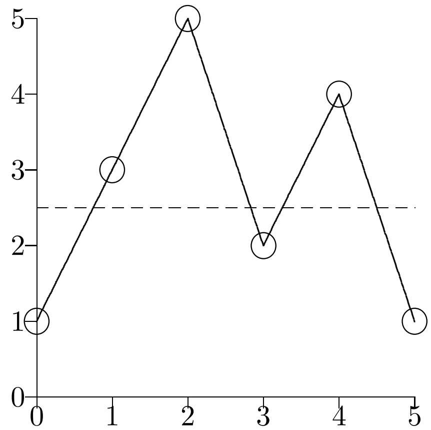

Figure 4

We see that the graph of π passes up through the line $y = 2\frac{1}{2}$ exactly twice, and down also exactly twice.

Thus, given a cycle in S_k we have two sets of intervals with the same sum, E and F. From the construction of E and F, it is clear that the graph in the lemma for this pair must already be a cycle (since π is a cycle). Suppose we had $\phi \neq E' \subseteq E$ and $\phi \neq F' \subseteq F$ satisfying $(*)$ such that the sum of the intervals in E was the same as the sum of the intervals in F. Then we could apply the lemma, and from its proof conclude that the corresponding graph (for E' and F') was a cycle. But the graph for E and F is already a cycle, so we must have $E = E'$ and $F = F'$.

So, given a cycle in S_k we have a pair E and F with equal sums satisfying $(*)$, and conversely. It is clear by construction that these two operations are inverse to each other, so the proposition is proved. □

Example

We take $k = 5$ and the cycle (13524). Basically the graph of this cycle above shows us what is happening. We have $E = \{[1,2],[3,4],[2,3]\}$ and $F = \{[2,4],[1,3]\}$, and the graph in the lemma consists of the cycle

$$([1,2],E) \longrightarrow ([3,4],E) \longrightarrow ([2,4],F) \longrightarrow ([2,3],E) \longrightarrow ([1,3],F) \longrightarrow ([1,2],E).$$

Corollary 7.6 *The vectors, provided by Lemma 5.1, whose nonnegative rational span is P_i^+, can all be taken to be of the form $x = (x_1, x_2, \dots x_{k(k-1)})$ with each $x_j \in \{0,1\}$, where the set of columns of A_i corresponding to the non-zero x_j correspond to a cycle in S_k as in Proposition 7.4.*

Remark: It seems reasonable to suggest that every vector corresponding to a cycle is an extreme homogeneous solution. The remarks after Lemma 5.1 show that this is true for $k = 2, 3$, and a calculation shows it to be true also for $k = 4$. Of course, if we are searching for a set of vectors whose nonnegative rational span is P_i^+, we can include all of the vectors corresponding to cycles in S_k as in Proposition 7.4 (and possibly have some redundant vectors).

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